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AN EXPLICIT VISCOSITY ITERATIVE ALGORITHM FOR FINDING THE SOLUTIONS OF A GENERAL EQUILIBRIUM PROBLEM SYSTEMS

H. R. SAHEBI AND S. EBRAHIMI

Abstract. We suggest an explicit viscosity iterative algorithm for finding a common element of the set of solutions for an general equilibrium problem system (GEPS) involving a bifunction defined on a closed, convex subset and the set of fixed points of a nonexpansive semigroup on another one in Hilbert's spaces. Furthermore, we present some numerical examples(by using MATLAB software) to guarantee the main result of this paper.

1. Preliminaries

Let H be a real Hilbert space with norm $\|.\|$ and inner product $\langle .,.\rangle$ Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, forally \in C.$$

Such P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. The strongly (weak) convergent of $\{x_n\}$ to x is written by $x_n \to x$ ($x_n \rightharpoonup x$) as $n \to \infty$. Moreover, H satisfies the Opial's condition [15], if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$.
Recall that a mapping $T : C \to C$ is said to be nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. $F(T)$ denotes the set of fixed points of T. Let $\{T(s) : s \in [0,\infty)\}$ be a nonexpansive semigroup on a closed convex subset C, that is,

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(i) $T(0)x = x$, for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$, for all $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$, for all $x, y \in C$ and $s \geq 0$;

(iv) $s \mapsto T(s)x$ is continuous for all $x \in C$.

Denote by $F(S) = \bigcap_{s\geq 0} F(T(s))$. It is well known that $F(S)$ is closed and convex subset in H and $F(S) \neq \emptyset$ if C is bounded [1]. Recall that a self mapping $f : H \to H$ is a contraction if there exists $\rho \in (0,1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for each $x, y \in H$.
A mapping $B : C \to H$ is called $\alpha-$inverse strongly monotone [14, 20] if there exists a positive real number $\alpha > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2.$$

Shimizu and Takahashi [18] studied the strongly convergent of the sequence $\{x_n\}$ which is defined by:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad x \in C,$$

in a real Hilbert space, where $\{T(s) : s \in [0,\infty)\}$ is a strongly continuous semigroup of nonexpansive mappings on a closed convex subset C of a Hilbert space and $\lim_{n\to\infty} t_n = \infty$. Later, Plubtieng and Punpaeng [17] introduced the following iterative method:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds,$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $(0,1)$, $\{s_n\}$ is a positive real divergent sequence and $f : C \to C$ is a contraction. Under the conditions $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_n + \beta_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$, they proved the strong convergence of the sequence.

Also, Plubtieng and Punpaeng [16] introduced the following iterative scheme:
Let $S : C \to H$ be a nonexpansive mapping, defined sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0; \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Su_n, \forall y \in H. \end{cases}$$

They proved, under the certain appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \forall x \in F(S) \cap EP(F),$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mappings on a real Hilbert space H:

$$\min \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where A is strongly positive linear bounded operator and h is a potential function for γf, i.e., $h'(x) = \gamma f$, for all $x \in H$.

Let $A : H \to H$ be an inverse strongly monotone mapping and $F : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following GEPS

$$\text{Find } \tilde{x} \in C \text{ such that } F(\tilde{x}, y) + \langle Ax, y - x\rangle \geq 0, \text{ for all } y \in C. \tag{1.1}$$

The set of such $x \in C$ is denoted by $GEPS(F, A)$, i.e.,

$$GEPS(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}.$$

To study the generalized equilibrium problem (1.1), let F satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$ $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Recently, Kamraska and Wangkeeree [8] introduced a new iterative by viscosity approximation methods in a Hilbert space. To be more precisely, they proved the following result:

Theorem 1.1. *Let $S = (T(s))_{s\geq 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \to H$ be an α-contraction, $A : H \to H$ a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma}$. Let γ be a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $G : H \times H \to \mathbb{R}$ be a mapping satisfying hypotheses (A1)–(A4) and $\Psi : H \to H$ an inverse-strongly monotone mapping with coefficients $\delta > 0$ such that $F(S) \cap GEP(G, \Psi) \neq \emptyset$. Let the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1 \in H \ \ chosen \ \ arbitrary, \\ G(u_n, y) + \langle \Psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \\ y_n = \beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \forall n \geq 1. \end{cases}$$

Under the certain appropriate conditions, they proved that the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ is strongly convergent to z, which is a unique solution in $F(S) \cap GEP(G, \Psi)$ of the variational inequality

$$\langle(\gamma f - A)z, p - z\rangle \leq 0, \forall p \in F(S) \cap GEP(G, \Psi).$$

The problem studied in this paper is formulated as follows (By intuition from [6], [7], [8]): Let C_1 and C_2 be closed convex subsets in H. Suppose that $F(x, y)$ be a bifunction satisfy conditions $(A1) - (A4)$ with C replaced by C_1 and let $\{T(S) : s \in [0, \infty)\}$ be a nonexpansive semigroup on C_2. Find an element

$$x^* \in \bigcap_{i=1}^{k} GEPS(F_i, \Psi_i) \bigcap F(S),$$

where $GEPS(F_i, \Psi_i)$ and $F(S)$ the set of solutions of an general equilibrium problem system(GEPS) involving by a bifunction $F_k(u_n^{(i)}, y)$ on $C_1 \times C_1$ and the fixed point set of a nonexpansive semigroup $\{T(S) : s \in [0, \infty)\}$ on a closed convex subset C_2, respectively.

2. Preliminaries and Lemmas

The following lemmas will be useful for proving the main results of this article. Let A be a strongly positive linear bounded operator on H: that is, there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \text{ for all } x \in H.$$

Lemma 2.1 ([13])**.** *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq I - \rho\overline{\gamma}$.*

Lemma 2.2 ([2])**.** *Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for any $r > 0$ and $x \in H$ there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) *$F(T_r) = GEP(F)$;*

(d) $\|T_s x - T_r x\| \le \frac{s-r}{s}\|T_s x - x\|$;

(e) *$GEP(F)$ is closed and convex.*

Remark 2.3. It is clear that for any $x \in H$ and $r > 0$, by Lemma 2.2(a), there exists $z \in H$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \text{ for all } y \in H. \tag{2.1}$$

Replacing x with $x - r\Psi x$ in (2.1), we obtain

$$F(z, y) + \langle \Psi x, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \text{ for all } y \in H.$$

Lemma 2.4 ([19])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.5 ([18])**.** *Let C be a nonempty bounded closed convex subset of H and let $S = \{T(s) : s \in [0,\infty)\}$ be a nonexpansive semigroup on C. Then for any $h \in [0,\infty)$,*

$$\limsup_{t\to\infty}\ \sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(s)x\,ds - T(h)\Big(\frac{1}{t}\int_0^t T(s)x\,ds\Big)\right\| = 0,$$

for $x \in C$ and $t > 0$.

Lemma 2.6 ([23])**.** *Assume $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \delta_n,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in real number such that

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$;

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([4])**.** *If C is a closed convex subset of H, T is a nonexpansive mapping on C, $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x - Tx = 0$*

3. Explicit viscosity iterative algorithm

The viscosity method has been successfully applied to various problems coming from calculus of variations, minimal surface problems, plasticity theory and phase transition. It plays a central role too in the study of degenerated elliptic and parabolic second order equations [9], [11], [12]. First abstract formulation of the properties of the viscosity approximation have been given by Tykhonov [22] in 1963 when studying ill-posed problems (see [3] for details). The concept of viscosity solution for Hamilton-Jacobi equations, which plays a crucial role in control theory, game theory and partial differential equations has been introduced by Crandall and Lions [5]. In this section, we introduce a explicit viscosity iterative algorithm for finding a common element of the set of solution for an equilibrium problem involving a bifunction defined on a closed convex subset and the set of fixed points for a nonexpansive semigroup.
In this section, we introduce a new iterative for finding a common element of the set of solution for an equilibrium problem involving a bifunction defined on a closed convex subset and the set fixed points for a nonexpansive semigroup.

Theorem 3.1. *Let H be a real Hilbert space. Asuume that*

- *C_1, C_2 are two nonempty convex closed subsets H,*
- *$F_1, F_2, \dots, F_k$ be bifunctions from $C_1 \times C_1$ to $\mathbb{R}$ satisfying $(A1)-(A4)$,*
- *$\Psi_1, \Psi_2, \dots, \Psi_k$ is μ_i-inverse strongly monotone mapping on H,*
- *$f : H \to H$ is a $\rho-$contraction,*
- *A is a strongly positive linear bounded operator on H with coefficient λ and $0 < \gamma < \frac{\lambda}{\rho}$,*
- *$F(S) = \{T(s) : s \in [0,\infty)\}$ is a nonexpansive semigroup on C_2 such that $\bigcap_{i=1}^{k} F(S) \cap GEPS(F_i, \Psi_i) \neq \emptyset$,*
- *$\{x_n\}$ is a sequence generated in the following manner:*

$$\begin{cases} x_1 \in H, \text{ and } u_n^{(i)} \in C_1, \\ F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)} \rangle + \frac{1}{r_n} \langle y - u_n^{(1)}, u_n^{(1)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\ F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)} \rangle + \frac{1}{r_n} \langle y - u_n^{(2)}, u_n^{(2)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\ \vdots \\ F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)} \rangle + \frac{1}{r_n} \langle y - u_n^{(k)}, u_n^{(k)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\ \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_n^{(i)}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s) P_{C_2} \omega_n ds, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$ and $\{r_n\} \subset (0,\infty)$ is a real sequence which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $0 < b < r_n < a < 2\mu_i$ *for* $i \in \{1, 2, \dots, k\}$;

(C4) $\lim_{n\to\infty} t_n = \infty$, *and* $\sup |t_{n+1} - t_n|$ *is bounded.*

Then

(i) *the sequence* $\{x_n\}$ *is bounded;*

(ii) $\lim_{n\to\infty} \|\Psi_i x_n - \Psi_i x^*\| = 0$, *for* $i \in \{1, 2, \dots, k\}$, $x^* \in \bigcap_{i=1}^{k} GEPS(F_i, \Psi_i) \bigcap F(S)$;

(iii) $\lim_{n\to\infty} \left\| x_n - \frac{1}{t_n} \int_0^{t_n} T(s) P_{C_2} \omega_n ds \right\| = 0$ *and* $\lim_{n\to\infty} \left\| \omega_n - \frac{1}{t_n} \int_0^{t_n} T(s) P_{C_2} \omega_n ds \right\| = 0.$

Proof. **(i)** By the same argument in [7, 10],

$$\|(1-\beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n \lambda.$$

Let $q \in \bigcap_{i=1}^{k} F(S) \cap GEPS(F_i, \Psi_i)$. Observe that $I - r_n \Psi_i$ for any $i = 1, 2, \dots, k$ is a nonexpansive mapping. Indeed, for any $x, y \in H$,

$$\begin{aligned}
\|(I - r_n\Psi_i)x - (I - r_n\Psi_i)y\|^2 &= \|(x-y) - r_n(\Psi_i x - \Psi_i y)\|^2 \\
&= \|x-y\|^2 - 2r_n\langle x-y, \Psi_i x - \Psi_i y\rangle + r_n^2\|\Psi_i x - \Psi_i y\|^2 \\
&\leq \|x-y\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x - \Psi_i y\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

So

$$\|u_n^{(i)} - q\| \leq \|x_n - q\|, \tag{3.1}$$

and hence

$$\|\omega_n - q\| \leq \|x_n - q\|. \tag{3.2}$$

Thus

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds - q\| \\
&\leq \alpha_n \|\gamma f(x_n) - Aq\| + \beta_n \|x_n - q\| \\
&\quad + \|(1-\beta_n)I - \alpha_n A)\| \|\frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds - q\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\{\|\gamma f(x_n)-\gamma f(q)\|+\|\gamma f(q)-Aq\|\}+\beta_n\|x_n-q\|\\
&\quad +(1-\beta_n-\alpha_n\lambda)\frac{1}{t_n}\int_0^{t_n}\|T(s)P_{C_2}\omega_n-P_{C_2}q\|ds\\
&\leq \alpha_n\rho\gamma\|x_n-q\|+\alpha_n\|\gamma f(q)-Aq\|+\beta_n\|x_n-q\|\\
&\quad +(1-\beta_n-\alpha_n\lambda)\|\omega_n-q\|\\
&\leq \alpha_n\rho\gamma\|x_n-q\|+\alpha_n\|\gamma f(q)-Aq\|+\beta_n\|x_n-q\|\\
&\quad +(1-\beta_n-\alpha_n\lambda)\|x_n-q\|\\
&= (1-\alpha_n(\lambda-\gamma\rho))\|x_n-q\|+\alpha_n\|\gamma f(q)-Aq\|\\
&\leq \max\{\|x_n-q\|,\frac{\|\gamma f(q)-Aq\|}{\lambda-\gamma\rho}\}.
\end{aligned}
$$

By induction

$$\|x_n-q\|\leq \max\{\|x_1-q\|,\frac{\|\gamma f(q)-Aq\|}{\lambda-\gamma\rho}\}.$$

Therefore, the sequence $\{x_n\}$ is bounded and also $\{f(x_n)\},\{\omega_n\}$ and $\{\frac{1}{t_n}\int_0^{t_n}T(s)P_{C_2}\omega_n ds\}$ are bounded.

(ii) Note that $u_n^{(i)}$ can be written as $u_n^{(i)}=T_{r_n^{(i)}}(x_n-r_n\Psi_i x_n)$. By Lemma 2.2, for any $i=1,2,\dots,k$,

$$
\begin{aligned}
\|u_{n+1}^{(i)}-u_n^{(i)}\| &\leq \|T_{r_{n+1}^{(i)}}(I-r_{n+1}\Psi_i)x_{n+1}-T_{r_{n+1}^{(i)}}(I-r_n\Psi_i)x_n\|\\
&\quad +\|T_{r_{n+1}^{(i)}}(I-r_n\Psi_i)x_n-T_{r_n^{(i)}}(I-r_n\Psi_i)x_n\|\\
&\leq \|(I-r_{n+1}\Psi_i)x_{n+1}-(I-r_n\Psi_i)x_n\|\\
&\quad +\|T_{r_{n+1}^{(i)}}(I-r_n\Psi_i)x_n-T_{r_n^{(i)}}(I-r_n\Psi_i)x_n\|\\
&\leq \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|\Psi_i x_n\|\\
&\quad +\frac{r_{n+1}-r_n}{r_{n+1}}\|T_{r_{n+1}^{(i)}}(I-r_n\Psi_i)x_n-T_{r_n^{(i)}}(I-r_n\Psi_i)x_n\|.
\end{aligned}
$$

Then

$$\|u_{n+1}^{(i)}-u_n^{(i)}\|\leq \|x_{n+1}-x_n\|+2M_i|r_{n+1}-r_n|, \tag{3.3}$$

where $$M_i=\max\{\sup\{\frac{\|T_{r_{n+1}^{(i)}}(I-r_n\Psi_i)x_n-T_{r_n^{(i)}}(I-r_n\Psi_i)x_n\|}{r_{n+1}},\sup\{\|\Psi_i x_n\|\}\}\}.$$

Also

$$
\begin{aligned}
&\left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)P_{C_2}\omega_{n+1}ds-\frac{1}{t_n}\int_0^{t_n}T(s)P_{C_2}\omega_n ds\right\|\\
&=\left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}[T(s)\omega_{n+1}-T(s)\omega_n]ds+(\frac{1}{t_{n+1}}-\frac{1}{t_n})\int_0^{t_n}[T(s)\omega_n-T(s)q]ds\right.\\
&\quad\left.+\frac{1}{t_{n+1}}\int_{t_n}^{t_{n+1}}[T(s)\omega_n-T(s)q]ds\right\|
\end{aligned}
$$

$$\leq \|\omega_{n+1}-\omega_n\| + \frac{2|t_{n+1}-t_n|}{t_{n+1}}\|\omega_n - q\|.$$

Let $M = \frac{1}{k}\sum_{i=1}^{k} 2M_i < \infty$, since

$$\|\omega_{n+1}-\omega_n\| \leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n+1}^{(i)} - u_n^{(i)}\| \leq \|x_{n+1}-x_n\| + M|r_{n+1}-r_n|,$$

hence

$$\begin{aligned}&\left\|\tfrac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)P_{C_2}\omega_{n+1}ds - \tfrac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\right\| \\ &\leq \|x_{n+1}-x_n\| + M|r_{n+1}-r_n| + \tfrac{2|t_{n+1}-t_n|}{t_{n+1}}\|\omega_n - q\|.\end{aligned} \tag{3.4}$$

Suppose $z_n = \frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n A)\Lambda_n}{1-\beta_n}$, where $\Lambda_n := \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds$. It follows from (3.3), (3.4)

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|\frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)\Lambda_{n+1}}{1-\beta_{n+1}} \\ &\quad -\frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I-\alpha_n A)\Lambda_n}{1-\beta_n}\| \\ &= \|\frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1})\Lambda_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}A\Lambda_{n+1}}{1-\beta_{n+1}} \\ &\quad -\frac{\alpha_n\gamma f(x_n)}{1-\beta_n} - \frac{(1-\beta_n)\Lambda_n}{1-\beta_n} + \frac{\alpha_n A\Lambda_n}{1-\beta_n}\| \\ &= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - A\Lambda_{n+1}) + \frac{\alpha_n}{1-\beta_n}(A\Lambda_n - \gamma f(x_n)) + (\Lambda_{n+1}-\Lambda_n)\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - A\Lambda_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|A\Lambda_n - \gamma f(x_n)\| + \|\Lambda_{n+1}-\Lambda_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - A\Lambda_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|A\Lambda_n - \gamma f(x_n)\| + \|x_{n+1}-x_n\| \\ &\quad +M|r_{n+1}-r_n| + \frac{2|t_{n+1}-t_n|}{t_{n+1}}\|\omega_n - q\|.\end{aligned}$$

(C1), (C3) and (C4) implies that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

By Lemma 2.4

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Consequently

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.5}$$

Moreover, for any $i \in \{1,2,\ldots,k\}$,

$$\|u_n^{(i)} - q\|^2 \leq \|(x_n - q) - r_n(\Psi_i x_n - \Psi_i q)\|^2$$

$$= \|x_n - q\|^2 - 2r_n\langle x_n - q, \Psi_i x_n - \Psi_i q\rangle + r_n^2\|\Psi_i x_n - \Psi_i q\|^2$$
$$\leq \|x_n - q\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i q\|^2,$$

and then

$$\begin{aligned}\|\omega_n - q\|^2 &= \|\sum_{i=1}^{k}\frac{1}{k}(u_n^{(i)} - q)\|^2 \leq \frac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)} - q\|^2 \\ &\leq \|x_n - q\|^2 - \tfrac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i q)\|^2.\end{aligned} \tag{3.6}$$

By (3.6), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha_n(\gamma f(x_n) - Aq) + \beta_n(x_n - q) + ((1-\beta_n)I - \alpha_n A)(\Lambda_n - q)\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \beta_n\|x_n - q\|^2 + (1-\beta_n - \alpha_n\lambda)\|\Lambda_n - q\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \beta_n\|x_n - q\|^2 + (1-\beta_n - \alpha_n\lambda)\|\omega_n - q\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \beta_n\|x_n - q\|^2 + (1-\beta_n - \alpha_n\lambda)\{\|x_n - q\|^2 \\ &\quad -\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i q)\|^2\} \\ &\leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 \\ &\quad -(1-\beta_n - \alpha_n\lambda)\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i q\|^2,\end{aligned}$$

and hence

$$\begin{aligned}&(1-\beta_n - \alpha_n\lambda)\tfrac{1}{k}\sum_{i=1}^{k} b(2\mu_i - a)\|\Psi_i x_n - \Psi_i q\|^2 \\ &\quad \leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\quad \leq \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_{n+1} - x_n\|(\|x_{n+1} - q\| - \|x_n - q\|).\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, it follows that

$$\lim_{n\to\infty}\|\Psi_i x_n - \Psi_i q\| = 0, \forall i = 1, 2, \ldots, k. \tag{3.7}$$

(iii) By Lemma 2.2, for any $i = 1, 2, \ldots, k$,

$$\begin{aligned}\|u_n^{(i)} - q\|^2 &\leq \langle (I - r_n\Psi_i)x_n - (I - r_n\Psi_i)q, u_n^{(i)} - q\rangle \\ &= \frac{1}{2}\{\|(I - r_n\Psi_i)x_n - (I - r_n\Psi_i)q\|^2 + \|u_{n,i} - q\|^2 \\ &\quad -\|(I - r_n\Psi_i)x_n - (I - r_n\Psi_i)q - (u_n^{(i)} - q)\|^2\} \\ &\leq \frac{1}{2}\{\|x_n - q\|^2 + \|u_n^{(i)} - q\|^2 - \|x_n - u_n^{(i)} - r_n(\Psi_i x_n - \Psi_i q)\|^2\} \\ &= \frac{1}{2}\{\|x_n - q\|^2 + \|u_n^{(i)} - q\|^2 - (\|x_n - u_n^{(i)}\|^2 \\ &\quad -2r_n\langle x_n - u_n^{(i)}, \Psi_i x_n - \Psi_i q\rangle + r_n^2\|\Psi_i x_n - \Psi_i q\|^2).\end{aligned}$$

This implies

$$\|u_n^{(i)}-q\|^2 \le \|x_n-q\|^2-\|x_n-u_n^{(i)}\|^2+2r_n\|x_n-u_n^{(i)}\|\|\Psi_i x_n-\Psi_i q\|, \tag{3.8}$$

and hence

$$\begin{aligned}\|\omega_n-q\|^2 &= \|\sum_{i=1}^{k}\frac{1}{k}(u_n^{(i)}-q)\|^2\\ &\le \tfrac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)}-q\|^2\\ &\le \|x_n-q\|^2-\tfrac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)}-x_n\|^2+\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_n^{(i)}\|\|\Psi_i x_n-\Psi_i q\|.\end{aligned} \tag{3.9}$$

Observe that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \alpha_n\|\gamma f(x_n)-Aq\|^2+\beta_n\|x_n-q\|^2+(1-\beta_n-\alpha_n\lambda)\|\omega_n-q\|^2\\ &\le \alpha_n\|\gamma f(x_n)-Aq\|^2+\beta_n\|x_n-q\|^2+(1-\beta_n-\alpha_n\lambda)\{\|x_n-q\|^2\\ &\quad -\frac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)}-x_n\|^2+\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_n^{(i)}\|\|\Psi_i x_n-\Psi_i q\|\}.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\beta_n-\alpha_n\lambda)\tfrac{1}{k}\sum_{i=1}^{k}\|u_n^{(i)}-x_n\|^2 &\le \alpha_n\|\gamma f(x_n)-Aq\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad +(1-\beta_n-\alpha_n\lambda)\tfrac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_n^{(i)}\|\|\Psi_i x_n-\Psi_i q\|\\ &\le \alpha_n\|\gamma f(x_n)-Aq\|^2+\|x_{n+1}-x_n\|(\|x_n-q\|-\|x_{n+1}-q\|)\\ &\quad +(1-\beta_n-\alpha_n\lambda)\tfrac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_n^{(i)}\|\|\Psi_i x_n-\Psi_i q\|.\end{aligned}$$

Since $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$, we have

$$\lim_{n\to\infty}\|u_n^{(i)}-x_n\|=0. \tag{3.10}$$

It is easy to prove

$$\lim_{n\to\infty}\|\omega_n-x_n\|=0. \tag{3.11}$$

The definition of $\{x_n\}$ shows

$$\begin{aligned}\|\Lambda_n-x_n\| &\le \|x_{n+1}-x_n\|+\|x_{n+1}-\Lambda_n\|\\ &\le \|x_{n+1}-x_n\|+\|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)\Lambda_n-\Lambda_n\|\\ &\le \|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-A\Lambda_n\|+\beta_n\|x_n-\Lambda_n\|.\end{aligned}$$

That is

$$\|\Lambda_n - x_n\| \le \frac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - A\Lambda_n\|.$$

The condition (C1) together (3.5) implies that

$$\lim_{n\to\infty}\|\Lambda_n - x_n\| = 0. \tag{3.12}$$

Moreover, $\|\omega_n - \Lambda_n\| \le \|\omega_n - x_n\| + \|x_n - \Lambda_n\|$, we get

$$\lim_{n\to\infty}\|\Lambda_n - \omega_n\| = 0. \tag{3.13}$$

Then,

$$\lim_{n\to\infty}\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\right\| = 0,$$
$$\lim_{n\to\infty}\left\|\omega_n - \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\right\| = 0. \qquad \square$$

Theorem 3.2. *Suppose all assumptions of Theorem 3.1 are hold. Then the sequence $\{x_n\}$ is strongly convergent to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^k F(S) \cap GEPS(F_i, \Psi_i)$, which solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, \bar{x} - x\rangle \le 0.$$

Equivalently, $\bar{x} = P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(\bar{x})$.

Proof. For all $x, y \in H$, we have

$$\begin{aligned}
&\|P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(x) - P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(y)\| \\
&\le \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\
&\le \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\
&\le (1-\lambda)\|x - y\| + \gamma\rho\|x - y\| \\
&= (1 - (\lambda - \gamma\rho))\|x - y\|.
\end{aligned}$$

This implies that $P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)$ is a contraction of H into itself. Since H is complete, then there exists a unique element $\bar{x} \in H$ such that

$$\bar{x} = P_{\bigcap_{i=1}^k F(S)\cap GEPS(F_i,\Psi_i)}(I - A + \gamma f)(\bar{x}).$$

Next, we prove

$$\limsup_{n\to\infty}\langle (A - \gamma f)\bar{x}, \bar{x} - \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds\rangle \le 0.$$

Let $\tilde{x} = P_{\bigcap_{i=1}^{k} F(S)\cap GEPS(F_i,\Psi_i)} x_1$, set

$$\Sigma = \{\bar{y} \in C_2 : \|\bar{y} - \tilde{x}\| \leq \|x_1 - \tilde{x}\| + \frac{\|\gamma f(\tilde{x}) - A\tilde{x}\|}{\lambda - \gamma\rho}\}.$$

It is clear, Σ is nonempty closed bounded convex subset of C_2 and $S = \{T(s) : s \in [0,\infty)\}$ is a nonexpansive semigroup on Σ.

Let $\Lambda_n = \frac{1}{t_n}\int_0^{t_n} T(s)P_{C_2}\omega_n ds$, since $\{\Lambda_n\} \subset \Sigma$ is bounded, there is a subsequence $\{\Lambda_{n_j}\}$ of $\{\Lambda_n\}$ such that

$$\limsup_{n\to\infty} \langle (A-\gamma f)\bar{x}, \bar{x} - \Lambda_n\rangle = \lim_{j\to\infty} \langle (A-\gamma f)\bar{x}, \bar{x} - \Lambda_{n_j}\rangle. \tag{3.14}$$

As $\{\omega_{n_j}\}$ is also bounded, there exists a subsequence $\{\omega_{n_{j_l}}\}$ of $\{\omega_{n_j}\}$ such that $\omega_{n_{j_l}} \rightharpoonup \xi$. Without loss of generality, let $\omega_{n_j} \rightharpoonup \xi$. From (iii) in Theorem 3.1, we have $\Lambda_{n_j} \to \xi$.
Since $\{\omega_n\} \subset C_1$ and $\{\Lambda_n\} \subset C_2$ and C_1, C_2 are two closed convex subsets in H, we obtain that $\xi \in C_1 \cap C_2$.
Now, we prove the following items:

(i) $\xi \in F(S) = \bigcap_{s\geq 1} F(T(s))$.
Since $\{\Lambda_n\} \subset C_2$, we have

$$\begin{aligned}\|\Lambda_n - P_{C_2}\omega_n\| &= \|P_{C_2}\Lambda_n - P_{C_2}\omega_n\| \\ &\leq \|\Lambda_n - \omega_n\|.\end{aligned}$$

By (iii) in Theorem 3.1, we have

$$\lim_{n\to\infty} \|\Lambda_n - P_{C_2}\omega_n\| = 0. \tag{3.15}$$

By using (iii) in Theorem 3.1 and (3.15), we obtain

$$\lim_{n\to\infty} \|\omega_n - P_{C_2}\omega_n\| = 0. \tag{3.16}$$

This shows that the sequence $P_{C_2}\omega_{n_j} \rightharpoonup \xi$ as $j \to \infty$.
For each $h > 0$, we have

$$\begin{aligned}\|T(h)P_{C_2}\omega_n - P_{C_2}\omega_n\| &\leq \|T(h)P_{C_2}\omega_n - T(h)\Lambda_n\| + \|T(h)\Lambda_n - \Lambda_n\| + \|\Lambda_n - P_{C_2}\omega_n\| \\ &\leq 2\|\Lambda_n - P_{C_2}\omega_n\| + \|T(h)\Lambda_n - \Lambda_n\|.\end{aligned}$$

The lemma 2.5 implies that

$$\lim_{n\to\infty} \|T(h)\Lambda_n - \Lambda_n\| = 0, \tag{3.17}$$

the equalities (3.15, 3.16) and (3.17)implies that

$$\lim_{n\to\infty}\|T(h)P_{C_2}\omega_n-\omega_n\|=0.$$

Note that $F(TP_C)=F(T)$ for any mapping $T:C\to C$. The Lemma 2.7 implies that $\xi\in F(T(h)P_{C_2})=F(T(h))$ for all $h>0$. This shows that $\xi\in F(S)$.

(ii) $\xi\in\bigcap_{i=1}^k GEPS(F_i,\Psi_i)$.

Since $\{\omega_n\}$ is bounded and as respects (3.13), there exists a subsequence $\{\omega_{n_j}\}$ of $\{\omega_n\}$ such that $\omega_{n_j}\rightharpoonup\xi$. By intuition from [8],

$$F_i(u_n^{(i)},y)+\langle\Psi_i x_n,y-u_n^{(i)}\rangle+\frac{1}{r_n}\langle y-u_n^{(i)},u_n^{(i)}-x_n\rangle\ge 0,\ \text{for all } y\in C_1.$$

By $(A2)$, we have

$$\langle\Psi_i x_n,y-u_n^{(i)}\rangle+\frac{1}{r_n}\langle y-u_n^{(i)},u_n^{(i)}-x_n\rangle\ge F_i(y,u_n^{(i)}).$$

Substitute n by n_j, we get

$$\langle\Psi_i x_{n_j},y-u_{n_j}^{(i)}\rangle+\langle y-u_{n_j}^{(i)},\frac{u_{n_j}^{(i)}-x_{n_j}}{r_{n_j}}\rangle\ge F_i(y,u_{n_j}^{(i)}).\tag{3.18}$$

For $0<l\le 1$ and $y\in C_1$, set $y_l=ly+(1-l)\xi$. We have $y_l\in C_1$ and

$$\begin{aligned}\langle y_l-u_{n_j}^{(i)},\Psi_i y_l\rangle &\ge \langle y_l-u_{n_j}^{(i)},\Psi_i y_l\rangle-\langle\Psi_i x_{n_j},y_l-u_{n_j}^{(i)}\rangle\\ &\quad-\langle y_l-u_{n_j}^{(i)},\frac{u_{n_j}^{(i)}-x_{n_j}}{r_{n_j}}\rangle+F_i(y_l,u_{n_j}^{(i)})\\ &=\langle y_l-u_{n_j}^{(i)},\Psi_i y_l-\Psi_i u_{n_j}^{(i)}\rangle+\langle y_l-u_{n_j}^{(i)},\Psi_i u_{n_j}^{(i)}-\Psi_i x_{n_j}\rangle\\ &\quad-\langle y_l-u_{n_j}^{(i)},\frac{u_{n_j}^{(i)}-x_{n_j}}{r_{n_j}}\rangle+F_i(y_l,u_{n_j}^{(i)}).\end{aligned}$$

The condition $(A4)$, monotonicity of Ψ_i and (3.10) implies that $\langle y_l-u_{n_j}^{(i)},\Psi_i y_l-\Psi_i u_{n_j}^{(i)}\rangle\ge 0$ and $\|\Psi_i u_{n_j}^{(i)}-\Psi_i x_{n_j}\|\to 0$ as $j\to\infty$. Hence

$$\langle y_l-\xi,\Psi_i y_l\rangle\ge F_i(y_l,\xi).\tag{3.19}$$

Now, (A1) and (A4) together (3.19) show

$$\begin{aligned}0=F_i(y_l,y_l)&\le lF_i(y_l,y)+(1-l)F_i(y_l,\xi)\\ &\le lF_i(y_l,y)+(1-l)\langle y_l-\xi,\Psi_i y_l\rangle\end{aligned}$$

$$= lF_i(y_l, y) + (1-l)l\langle y-\xi, \Psi_i y_l\rangle,$$

which yields $F_i(y_l, y) + (1-l)\langle y-\xi, \Psi_i y_l\rangle \geq 0$.
By taking $l \to 0$, we have

$$F_i(\xi, y) + \langle y-\xi, \Psi_i\xi\rangle \geq 0.$$

This shows $\xi \in GEPS(F_i, \Psi_i)$, for all $i = 1, 2, \ldots, k$. Then, $\xi \in \bigcap_{i=1}^{k} GEPS(F_i, \Psi_i)$.

Now, in view of (3.14), we see

$$\limsup_{n\to\infty}\langle (A-\gamma f)\bar{x}, \bar{x}-\Lambda_n\rangle = \langle (A-\gamma f)\bar{x}, \bar{x}-\xi\rangle \leq 0. \tag{3.20}$$

Finally, we prove $\{x_n\}$ is strongly convergent to $\bar{x}$.

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\Lambda_n - \bar{x}\|^2\\
&= \|\alpha_n(\gamma f(x_n) - A\bar{x}) + \beta_n(x_n-\bar{x}) + ((1-\beta_n)I-\alpha_n A)(\Lambda_n-\bar{x})\|^2\\
&= \|\beta_n(x_n-\bar{x}) + ((1-\beta_n)I-\alpha_n A)(\Lambda_n-\bar{x})\|^2 + \alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2\\
&\quad +2\alpha_n\beta_n\langle x_n-\bar{x}, \gamma f(x_n)-A\bar{x}\rangle\\
&\quad +2\alpha_n\langle ((1-\beta_n)I-\alpha_n A)(\Lambda_n-\bar{x}), \gamma f(x_n)-A\bar{x}\rangle\\
&\leq \{(1-\beta_n-\alpha_n\lambda)\|\Lambda_n-\bar{x}\| + \beta_n\|x_n-\bar{x}\|\}^2 + \alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2\\
&\quad +2\alpha_n\beta_n\gamma\langle x_n-\bar{x}, f(x_n)-f(\bar{x})\rangle + 2\alpha_n\beta_n\langle x_n-\bar{x}, \gamma f(\bar{x})-A\bar{x}\rangle\\
&\quad +2(1-\beta_n)\gamma\alpha_n\langle \Lambda_n-\bar{x}, f(x_n)-f(\bar{x})\rangle\\
&\quad +2(1-\beta_n)\alpha_n\langle \Lambda_n-\bar{x}, \gamma f(\bar{x})-A\bar{x}\rangle - 2\alpha_n^2\langle A(\Lambda_n-\bar{x}), \gamma f(x_n)-A\bar{x}\rangle.
\end{aligned}$$

Consequently

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq \{(1-\alpha_n\lambda)^2 + 2\rho\alpha_n\beta_n\gamma + 2\rho(1-\beta_n)\gamma\alpha_n\}\|x_n-\bar{x}\|^2\\
&\quad +\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2 + 2\alpha_n\beta_n\langle x_n-\bar{x}, \gamma f(\bar{x})-A\bar{x}\rangle\\
&\quad +2\alpha_n(1-\beta_n)\langle \Lambda_n-\bar{x}, \gamma f(\bar{x})-A\bar{x}\rangle - 2\alpha_n^2\langle A(\Lambda_n-\bar{x}), \gamma f(x_n)-A\bar{x}\rangle\\
&\leq (1-2\alpha_n(\lambda-\rho\gamma))\|x_n-\bar{x}\|^2 + \lambda^2\alpha_n^2\|x_n-\bar{x}\|^2 + \alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2\\
&\quad +2\alpha_n\beta_n\langle x_n-\bar{x}, \gamma f(\bar{x})-A\bar{x}\rangle + 2\alpha_n(1-\beta_n)\langle \Lambda_n-\bar{x}, \gamma f(\bar{x})-A\bar{x}\rangle\\
&\quad +2\alpha_n^2\|A(\Lambda_n-\bar{x})\|\|\gamma f(x_n)-A\bar{x}\|\\
&= (1-2\alpha_n(\lambda-\rho\gamma))\|x_n-\bar{x}\|^2 + \alpha_n\{\alpha_n(\lambda^2\|x_n-\bar{x}\|^2\\
&\quad +\|\gamma f(x_n)-A\bar{x}\|^2 + 2\|A(\Lambda_n-\bar{x})\|\|\gamma f(x_n)-A\bar{x}\|\\
&\quad +2\beta_n\langle x_n-\bar{x}, \gamma f(\bar{x})-A\bar{x}\rangle + 2(1-\beta_n)\langle \Lambda_n-\bar{x}, \gamma f(\bar{x})-A\bar{x}\rangle\}.
\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{\Lambda_n\}$ are bounded, one can take a constant $\Gamma > 0$ such that

$$\Gamma \geq \lambda^2\|x_n-\bar{x}\|^2 + \|\gamma f(x_n)-A\bar{x}\|^2 + 2\|A(\Lambda_n-\bar{x})\|\|\gamma f(x_n)-A\bar{x}\|.$$

Let

$$\Xi_n = 2\beta_n\langle x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle + 2(1-\beta_n)\langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle + \Gamma\alpha_n.$$

Hence

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - 2\alpha_n(\lambda - \rho\gamma))\|x_n - \bar{x}\|^2 + \alpha_n \Xi_n. \tag{3.21}$$

With respect to (3.20), $\limsup_{n\to\infty} \Xi_n \leq 0$ and so all conditions of Lemma 2.6 are satisfied for (3.21). Consequently, the sequence $\{x_n\}$ is strongly convergent to $\bar{x}$. □

As a result, by intuition from [8], the following mean ergodic theorem for a nonexpansive mapping in Hilbert space is proved.

Corollary 3.3. *Suppose all assumptions of Theorem 3.1 are holds. Let $\{T^i\}$ be a family of non-expansive mappings on C_1 for all $i = 1, 2, \ldots, k$ such that $\bigcap_{i=1}^k F(T^i) \cap GEPS(F_i, \Psi_i) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n^{(i)}\} \subset C_1$ be sequences generated in the following manner:*

$$\begin{cases} x_1 \in H \ \ \text{choosen} \ \ \text{arbitrary}, \\ F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)}\rangle + \frac{1}{r_n}\langle y - u_n^{(1)}, u_n^{(1)} - x_n\rangle \geq 0, \ \text{for all } y \in C_1, \\ F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)}\rangle + \frac{1}{r_n}\langle y - u_n^{(2)}, u_n^{(2)} - x_n\rangle \geq 0, \ \text{for all } y \in C_1, \\ \vdots \\ F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)}\rangle + \frac{1}{r_n}\langle y - u_n^{(k)}, u_n^{(k)} - x_n\rangle \geq 0, \ \text{for all } y \in C_1, \\ \omega_n = \frac{1}{k}\sum_{i=1}^k u_n^{(i)}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{n+1}\sum_{i=0}^n P_{C_2} T^i \omega_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$ and $\{r_n\} \subset (0,\infty)$ is a real sequence. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $0 < b < r_n < a < 2\mu_i$ *for* $i \in \{1, 2, \ldots, k\}$.

Then the sequence $\{x_n\}$ is strongly convergent to a point $\bar{x}$, where

$$\bar{x} = P_{\bigcap_{i=1}^k F(T^i) \cap GEPS(F_i, \Psi_i)}(I - A + \gamma f)(\bar{x}),$$

is the unique solution of the variational inequality

$$\langle (A - \gamma f)\bar{x}, \bar{x} - x\rangle \leq 0, \forall x \in \bigcap_{i=1}^k F(T^i) \cap GEPS(F_i, \Psi_i).$$

4. Application

If $T(s)=T$ for all $s>0$ and $C_1=C_2=C$, then we have the following corollary.

Corollary 4.1. *Let H be a real Hilbert space, $F_1,F_2,\dots,F_k$ be bifunctions from $C\times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $\Psi_1,\Psi_2,\dots,\Psi_k$ be μ_i-inverse strongly monotone mapping on H, A be a strongly positive linear bounded operator on H with coefficient λ and $0<\gamma<\frac{\lambda}{\rho}$, $f:H\to H$ be a $\rho-$contraction. Suppose that T be a nonexpansive mapping on C such that $\bigcap_{i=1}^k F(T)\cap GEPS(F_i,\Psi_i)\neq\emptyset$. Define the sequence $\{x_n\}$ as follows.*

$$\begin{cases} x_1\in H, \text{ and } u_n^{(i)}\in C,\\ F_1(u_n^{(1)},y)+\langle\Psi_1x_n,y-u_n^{(1)}\rangle+\frac{1}{r_n}\langle y-u_n^{(1)},u_n^{(1)}-x_n\rangle\geq 0, \text{ for all } y\in C,\\ F_2(u_n^{(2)},y)+\langle\Psi_2x_n,y-u_n^{(2)}\rangle+\frac{1}{r_n}\langle y-u_n^{(2)},u_n^{(2)}-x_n\rangle\geq 0, \text{ for all } y\in C,\\ \vdots\\ F_k(u_n^{(k)},y)+\langle\Psi_kx_n,y-u_n^{(k)}\rangle+\frac{1}{r_n}\langle y-u_n^{(k)},u_n^{(k)}-x_n\rangle\geq 0, \text{ for all } y\in C,\\ \omega_n=\frac{1}{k}\sum_{i=1}^k u_n^{(i)},\\ x_{n+1}=\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)TP_C\omega_n ds, \end{cases}$$

where $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$ are the sequences satisfying the conditions (C1)−(C3) *in Theorem 3.2. Then the sequence $\{x_n\}$ converges strongly to a point $\bar{x}$, where $\bar{x}\in\bigcap_{i=1}^k F(T)\cap GEPS(F_i,\Psi_i)$ solves the variational inequality*

$$\langle(A-\gamma f)\bar{x},\bar{x}-x\rangle\leq 0.$$

We apply Theorem 3.2 for finding a common fixed point of a nonexpansive semigroup mappings and strictly pseudo-contractive mapping and inverse strongly monotone mapping. Recall that, a mapping $T:C\to C$ is called strictly pseudo-contractive if there exists k with $0\leq k\leq 1$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2, \text{ for all } x,y\in C.$$

If $k=0$, then T is nonexpansive. Put $J=I-T$, where $T:C\to C$ is a strictly pseudo-contractive mapping. J is $\frac{1-k}{2}$-inverse strongly monotone and $J^{-1}(0)=F(T)$. Indeed, for all $x,y\in C$ we have

$$\|(I-J)x-(I-J)y\|^2\leq\|x-y\|^2+k\|Jx-Jy\|^2.$$

Also

$$\|(I-J)x-(I-J)y\|^2\leq\|x-y\|^2+\|Jx-Jy\|^2-2\langle x-y,Jx-Jy\rangle.$$

So, we have

$$\langle x-y,Jx-Jy\rangle\geq\frac{1-k}{2}\|Jx-Jy\|^2.$$

Corollary 4.2. *Let H be a real Hilbert space, $F_1, F_2, \dots, F_k$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $\Psi_1, \Psi_2, \dots, \Psi_k$ be μ_i-inverse strongly monotone mapping on H, A be a strongly positive linear bounded operator on H with coefficient λ and $0 < \gamma < \frac{\lambda}{\rho}$, $f : H \to H$ be a $\rho-$contraction. Suppose that $T : C \to H$ be a k-strictly pseudo-contractive mapping for some $0 \le k < 1$ such that $\bigcap_{i=1}^{k} F(T) \cap GEPS(F_i, \Psi_i) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows.*

$$\begin{cases} x_1 \in H, \text{ and } u_n^{(i)} \in C, \\ F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)} \rangle + \frac{1}{r_n} \langle y - u_n^{(1)}, u_n^{(1)} - x_n \rangle \ge 0, \text{ for all } y \in C, \\ F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)} \rangle + \frac{1}{r_n} \langle y - u_n^{(2)}, u_n^{(2)} - x_n \rangle \ge 0, \text{ for all } y \in C, \\ \vdots \\ F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)} \rangle + \frac{1}{r_n} \langle y - u_n^{(k)}, u_n^{(k)} - x_n \rangle \ge 0, \text{ for all } y \in C, \\ \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_n^{(i)}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) P_C J \omega_n ds, \end{cases}$$

where $J : C \to H$ is a mapping defined by $Jx = kx + (1-k)Tx$ and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$ are the sequences satisfying the conditions (C1)−(C3) *in Theorem 3.1. Then the sequence $\{x_n\}$ is strongly convergent to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^{k} F(T) \cap GEPS(F_i, \Psi_i)$ solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, \bar{x} - x \rangle \le 0.$$

Proof. Note that $S : C \to H$ is a nonexpansinve mapping and $F(T) = F(S)$. By Lemma 2.3 in [24] and Lemma 2.2 in [21], we have $P_C S : C_2 \to C$ is a nonexpansive mapping and $F(P_C S) = F(S) = F(S)$. Therefore, the result follows from Corollary 4.1. ☐

5. Numerical Examples

In this section, we show numerical examples which grantee the main theorem. The programming has been provided with Matlab according to the following algorithm.

Example 5.1. Suppose that $H = \mathbb{R}, C_1 = [-1,1], C_2 = [0,1]$ and

$$F_1(x, y) = -3x^2 + xy + 2y^2, F_2(x, y) = -4x^2 + xy + 3y^2, F_3(x, y) = -5x^2 + xy + 4y^2.$$

Also, we consider $\Psi_1(x) = x, \Psi_2 = 2x$ and $\Psi_3(x) = \frac{x}{10}$. Suppose that $A = \frac{x}{10}, f(x) = \frac{x}{10}$ with coefficient $\gamma = 1$ and $T(s) = e^{-s}$ is a nonexpansive semigroup on C_2. It is easy to check that $\Psi_1, \Psi_2, \Psi_3, A, f$ and $T(s)$ satisfy all conditions in Theorem 3.1. For each $y \in C_1$ there exists $z \in C_1$ such that

$$F_1(z, y) + \langle \Psi_1 x, y - z \rangle + \frac{1}{r} \langle y - z, z - x \rangle \ge 0$$

$$\Leftrightarrow -3z^2 + zy + 2y^2 + x(y-z) + \frac{1}{r}(y-z)(z-x) \geq 0$$
$$\Leftrightarrow 2ry^2 + ((r+1)z - (r-1)x)y - 3rz^2 - xzr - z^2 + zx \geq 0.$$

Set $G(y) = 2ry^2 + ((r+1)z-(r-1)x)y - 3rz^2 - xzr - z^2 + zx$. Then $G(y)$ is a quadratic function of y with coefficients $a = 2r, b = (r+1)z-(r-1)x$ and $c = -3rz^2 - xzr - z^2 + zx$. So

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= [(r+1)z-(r-1)x]^2 - 8r(-3rz^2 - xzr - z^2 + zx) \\ &= x^2(r-1)^2 + 2zx(r-1)(5r+1) + z^2(5r+1)^2 \\ &= [(x(r-1)+z(5r+1))]^2.\end{aligned}$$

Since $G(y) \geq 0$ for all $y \in C_1$, if and only if $\Delta = [(x(r-1)+z(5r+1))]^2 \leq 0$. Therefore, $z = \frac{1-r}{5r+1}x$, which yields $T_{r_n^{(1)}} = u_n^{(1)} = \frac{1-r_n}{5r_n+1}x_n$. By the same argument, for F_2 and F_3, one can conclude

$$\begin{aligned}T_{r_n^{(2)}} &= u_n^{(2)} = \frac{1-2r_n}{7r_n+1}x_n, \\ T_{r_n^{(3)}} &= u_n^{(3)} = \frac{10-r_n}{90r_n+10}x_n.\end{aligned}$$

Then

$$\begin{aligned}\omega_n &= \frac{u_n^{(1)} + u_n^{(2)} + u_n^{(3)}}{3} \\ &= \frac{1}{3}[\frac{1-r_n}{5r_n+1} + \frac{1-2r_n}{7r_n+1} + \frac{10-r_n}{90r_n+10}]x_n.\end{aligned}$$

By choosing $r_n = \frac{n+8}{n}$, $t_n = n$, and $\alpha_n = \frac{9}{10n}, \beta_n = \frac{2n-1}{10n-9}$, we have the following algorithm for the sequence $\{x_n\}$

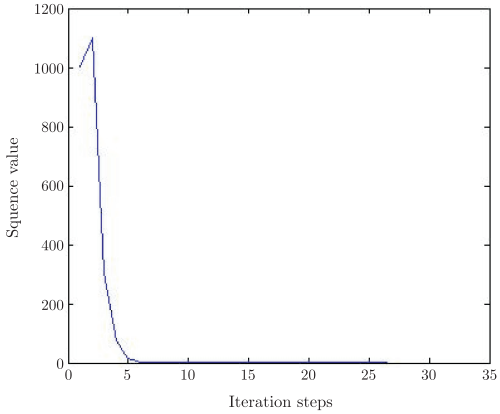
$$x_{n+1} = \frac{200n^2 - 10n - 81}{100n^2 - 90n}x_n + \frac{800n^2 - 890n + 81}{1000n^2 - 900n}(\frac{1-e^{-n}}{n})\omega_n.$$

Choose $x_1 = 1000$. By using MATLAB software, we obtain the following table and figure of the result.

Example 5.2. Theorem 3.2 can be illustrated by the following numerical example where the parameters are given as follows:

$$\begin{aligned}H = [-10,10]\ ,\ C_1 = [-1,1], C_2 = [0,1], A = I, f(x) = \frac{x}{5} \\ \Psi_1(x) = x\ ,\ \Psi_2 = 2x, \Psi_3(x) = \frac{x}{10}, \Psi_4(x) = 3x, \Psi_5(x) = 4x \\ \alpha_n = \frac{1}{2n}\ ,\ \beta_n = \frac{n}{2n+1}, r_n = \frac{n+1}{n}\end{aligned}$$

n	x_n	n	x_n	n	x_n
1	1000	11	0.001780587335	21	0.0000000003224455074
2	1098.409511	12	0.0003806958425	22	0.00000000006766467135
3	305.4917461	13	0.00008116570943	23	0.00000000001418205647
4	73.2742215	14	0.00001726216226	24	0.000000000002969041259
5	16.71274356	15	0.000003663196562	25	0.0000000000006208956961
6	3.722532274	16	0.0000007758152752	26	0.00000000000007698339387
7	0.8179356809	17	0.0000001640073634	27	0.0000000000001297089212
8	0.1781302867	18	0.00000003461280635	28	0.00000000000002707018951
9	0.03854703081	19	0.000000007293416021	29	0.000000000000005644205174
10	0.008300951942	20	0.000000001534585156	30	0.000000000000001175770771



$$T(s) = e^{-s}\ , \gamma = 1, t_n = n.$$

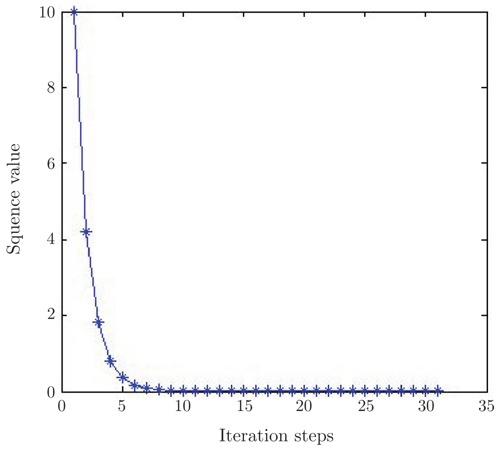
Moreover,

$$F_1(x,y) = -3x^2 + xy + 2y^2\ , F_4(x,y) = -6x^2 + xy + 5y^2$$
$$F_2(x,y) = -4x^2 + xy + 3y^2\ , F_5(x,y) = -8x^2 + xy + 7y^2$$
$$F_3(x,y) = -5x^2 + xy + 4y^2.$$

By the same argument in Example 5.1, we compute $u_n^{(i)}$ for $i = 1,2,3,4,5$ as follows:

$$T_{r_n^{(1)}} = u_n^{(1)} = \frac{1-r_n}{5r_n+1}x_n,$$
$$T_{r_n^{(2)}} = u_n^{(2)} = \frac{1-2r_n}{7r_n+1}x_n,$$
$$T_{r_n^{(3)}} = u_n^{(3)} = \frac{10-r_n}{90r_n+10}x_n,$$
$$T_{r_n^{(4)}} = u_n^{(4)} = \frac{1-3r_n}{11r_n+1}x_n,$$
$$T_{r_n^{(5)}} = u_n^{(3)} = \frac{1-4r_n}{15r_n+1}x_n.$$

n	x_n	n	x_n	n	x_n
1	10	11	0.004288752976	21	0.000003269796382
2	4.187523857	12	0.002073897432	22	0.000001606140362
3	1.810011289	13	0.001005587742	23	0.0000007895721035
4	0.8111798915	14	0.0004887121222	24	0.0000003884328747
5	0.37213169	15	0.0002379837774	25	0.000000191218571
6	0.173392393	16	0.0001160888527	26	0.00000009419138076
7	0.08167872103	17	0.00005671409128	27	0.00000004642361668
8	0.03878661445	18	0.00002774424594	28	0.00000002289261628
9	0.0185321898	19	0.00001358854452	29	0.0000000112944317
10	0.008897703796	20	0.000006662503243	30	0.00000000557482625



Then

$$\begin{aligned}\omega_n &= \frac{u_n^{(1)} + u_n^{(2)} + \ldots + u_n^{(5)}}{5} \\ &= \frac{1}{5}\left[\frac{1-r_n}{5r_n+1} + \frac{1-2r_n}{7r_n+1} + \frac{10-r_n}{90r_n+10} + \frac{1-3r_n}{11r_n+1} + \frac{1-4r_n}{15r_n+1}\right]x_n.\end{aligned}$$

Choose $x_1 = 10$. The detailed results of proposed iterative in Theorem 3.2 are presented in the following table and figure.

Example 5.3. Let

$$H = [-10,10]\ ,\ C_1 = [0,1], C_2 = [-1,1], A = \frac{x}{10}, f(x) = \frac{x}{10}$$

$$\Psi_1(x) = \Psi_2(x) = 0\ ,\ \Psi_3(x) = x, \Psi_4(x) = 2x, \Psi_5(x) = \frac{x}{10}$$

$$\Psi_6(x) = 3x\ ,\ \Psi_7(x) = 4x$$

$$\alpha_n = \frac{1}{n}\ ,\ \beta_n = \frac{n}{3n+1}, r_n = \frac{n+1}{n}$$

$$T(s) = e^{-s}\ ,\ \gamma = 1, t_n = n.$$

Moreover,

$$\begin{aligned} F_1(x,y) &= (1-x^2)(x-y)\ ,\ F_5(x,y) = -5x^2 + xy + 4y^2;\\ F_2(x,y) &= -x^2(x-y)^2\ ,\ F_6(x,y) = -6x^2 + xy + 5y^2;\\ F_3(x,y) &= -3x^2 + xy + 2y^2\ ,\ F_7(x,y) = -8x^2 + xy + 7y^2;\\ F_4(x,y) &= -4x^2 + xy + 3y^2. \end{aligned}$$

By the same argument in Example 5.1, we compute $u_n^{(i)}$ for $i = 1,2$ as follows: For any $y \in C_1$ and $r > 0$, we have

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0 \Leftrightarrow (y-z)(rz^2 + z - r - x) \geq 0.$$

This implies that $rz^2+z-r-x = 0$. Therefore, $z = \frac{-1+\sqrt{1+4r(r+x)}}{2r}$ which yields $T_{r_n^{(1)}} = \frac{-1+\sqrt{1+4r_n(r_n+x_n)}}{2r_n}$. Also, we have

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0 \Leftrightarrow -rz^2y^2 + (2rz^3 + z - x)y - rz^4 - z^2 + zx \geq 0.$$

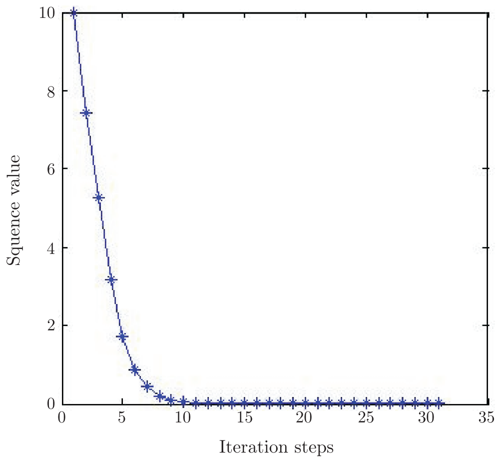
Set $J(y) = -rz^2y^2 + (2rz^3 + z - x)y - rz^4 - z^2 + zx$. Then $J(y)$ is a quadratic function of y with coefficients $a = -rz^2, b = 2rz^3 + z - x$ and $c = -rz^4 - z^2 + zx$. So

$$\begin{aligned} \Delta &= [2rz^3 + z - x]^2 + 4rz^2(-rz^4 - z^2 + zx)\\ &= (z-x)^2. \end{aligned}$$

Since $J(y) \geq 0$ for all $y \in H$, if and only if $\Delta = (z-x)^2 = 0$. Therefore, $z = T_{r_n^{(2)}} = x$. Then

$$\begin{aligned} T_{r_n^{(1)}} = u_n^{(1)} &= \frac{-1+\sqrt{1+4r_n(r_n+x_n)}}{2r_n},\\ T_{r_n^{(2)}} = u_n^{(2)} &= x_n,\\ T_{r_n^{(3)}} = u_n^{(3)} &= \frac{1-r_n}{5r_n+1}x_n,\\ T_{r_n^{(4)}} = u_n^{(4)} &= \frac{1-2r_n}{7r_n+1}x_n,\\ T_{r_n^{(5)}} = u_n^{(5)} &= \frac{10-r_n}{90r_n+10}x_n,\\ T_{r_n^{(6)}} = u_n^{(6)} &= \frac{1-3r_n}{11r_n+1}x_n,\\ T_{r_n^{(7)}} = u_n^{(7)} &= \frac{1-4r_n}{15r_n+1}x_n. \end{aligned}$$

n	x_n	n	x_n	n	x_n
1	10	11	0.0239156544	21	0.00001930391707
2	7.414282455	12	0.01166047726	22	0.000009519419651
3	5.25738677	13	0.005695329069	23	0.000004697209017
4	3.188403646	14	0.002786256225	24	0.000002319056949
5	1.720431319	15	0.001365047089	25	0.000001145527389
6	0.8702902283	16	0.000669619601	26	0.0000005661151488
7	0.4279164132	17	0.0003288520367	27	0.0000002798942327
8	0.2082805371	18	0.0001616632095	28	0.000000138439269
9	0.1011375703	19	0.00007954524563	29	0.00000006849968096
10	0.04914122712	20	0.00003917150569	30	0.00000003390554052



Then

$$\omega_n = \frac{u_n^{(1)} + u_n^{(2)} + \ldots + u_n^{(7)}}{7}.$$

We have

$$x_{n+1} = \frac{10n^2+3n+1}{30n^2-10n} x_n + \frac{20n^2+7n-1}{30n^2+10n} (\frac{1-e^{-n}}{n}) \omega_n.$$

Choose $x_1 = 10$. The detailed results of proposed iterative in Theorem 3.2 are presented in the following table and figure.

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