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INEQUALITIES OF QUERMASSINTEGRALS ABOUT MIXED BLASCHKE MINKOWSKI HOMOMORPHISMS

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Abstract. In this article, we establish some inequalities of quermassintegrals associated with mixed Blaschke Minkowski homomorphisms. In particular, Minkowski and Brunn-Minkowski type inequalities for quermassintegrals differences of mixed Blaschke Minkowski homomorphisms are established. In addition, we also give an isolated form of Brunn-Minkowski type inequality of quermassintegrals established by Schuster.

1. Introduction and Main Results

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and $V(K)$ denotes the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we denote $\omega_n = V(B)$.

If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty,\infty)$, is defined by (see [4, 16])

$$h(K,x) = \max\{x\cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x\cdot y$ denotes the standard inner product of x and y.

The projection body ΠK of $K \in \mathscr{K}^n$ is the convex body whose support function is given for $u \in S^{n-1}$ by (see [4, 16])

$$h(\Pi K,u) = V_{n-1}(K|u^{\perp}),$$

where V_{n-1} denotes $(n-1)$-dimensional volume and $K|u^{\perp}$ is the image of orthogonal projection of K onto the subspace orthogonal to u.

Important volume inequalities for the polars of projection bodies are the Petty projection inequalities (see [14]). Among bodies of given volume the polar projection bodies have maximal volume precisely for ellipsoids and minimal volume precisely for simplices. The corresponding results for the volume of projection body itself are major open problems in convex

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geometry (see [9]). Projection bodies and their polars have received considerable attention over the last decades (see [2]−[8], [10, 11, 18]).

The notion of the classical projection body is extended to the mixed projection body by Lutwak (see [11, 12]). For each $K \in \mathscr{K}^n$ and $i = 0, \ldots, n-1$, the mixed projection body, $\Pi_i K$, of K is the origin-symmetric convex body whose support function is defined by

$$h(\Pi_i K, u) = w_i(K|u^{\perp}),$$

for all $u \in S^{n-1}$, where $w_i(K|u^{\perp})$ denotes the $(n-1)$-dimensional quermassintegral of $K|u^{\perp}$, and $\Pi_0 K = \Pi K$. Besides, Lutwak, Yang and Zhang also gave the definition of L_p-projection body in [8]. With regard to the study of L_p-projection body, see [8, 15], [19]−[22]. Recently, Wang and Leng in [23] gave the notion of the L_p-mixed projection body and got many important results.

Further, according to the well know properties of the projection operator $\Pi : \mathscr{K}^n \to \mathscr{K}^n$ established in [11] and [10], Schuster in [18] gave the definition of Blaschke Minkowski homomorphism as follows:

A map $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ is called Blaschke Minkowski homomorphism if it satisfies the following conditions

(a) Φ is continuous.

(b) Φ is Blaschke Minkowski additive, i.e., for all $K, L \in \mathscr{K}^n$

$$\Phi(K \dotplus L) = \Phi K + \Phi L.$$

(c) Φ intertwines rotation, i.e., for all $K \in \mathscr{K}^n$ and $\vartheta \in SO(n)$

$$\Phi(\vartheta K) = \vartheta \Phi K.$$

Here $\Phi K + \Phi L$ denotes the Minkowski sum (see (2.1)) of the Blaschke Minkowski homomorphisms ΦK and ΦL and $K \dotplus L$ is the Blaschke sum of the convex bodies K and L (see (2.4)). $SO(n)$ is the group of rotation in n dimensions.

By the above definition of Blaschke Minkowski homomorphism, Schuster in [18] obtained the following result which generalizes the notion of mixed projection bodies, and call it as mixed Blaschke Minkowski homomorphisms.

Theorem 1.A. *There is a continuous operator*

$$\Phi : \underbrace{\mathscr{K}^n \times \cdots \times \mathscr{K}^n}_{n-1} \to \mathscr{K}^n,$$

symmetric in its arguments such that, for $K_1,\dots,K_m \in \mathcal{K}^n$ *and* $\lambda_1,\dots,\lambda_m \ge 0$*,*

$$\Phi(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_{n-1}} \lambda_{i_1}\cdots\lambda_{i_{n-1}} \Phi(K_{i_1},\dots,K_{i_{n-1}}), \tag{1.1}$$

where the sum is with respect to Minkowski addition. If $K_1 = \cdots = K_{n-i-1} = K$*,* $K_{n-i} = \cdots = K_{n-1} = B$*, we write* $\Phi_i K$ *for* $\Phi(K,\dots,K,B,\dots,B)$*. For* $0 \le i < n$*, we write* $\Phi_i(K,L)$ *for* $\Phi(\underbrace{K,\dots,K}_{n-i-1},\underbrace{L,\dots,L}_{i})$ *and write* $\Phi_0 K$ *as* ΦK*.*

Schuster in [18] also gave the following Minkowski and Brunn-Minkowski type inequalities of quermassintegrals of mixed Blaschke Minkowski homomorphisms.

Theorem 1.B. *If* $K, L \in \mathcal{K}^n$ *and* $0 \le i \le n-1$*,* $1 \le j < n-1$*, then*

$$W_i(\Phi_j(K,L))^{n-1} \ge W_i(\Phi K)^{n-j-1} W_i(\Phi L)^j, \tag{1.2}$$

with equality if and only if K *and* L *are homothetic. For* $M \in \mathcal{K}^n$*,* $W_i(M)$ *denotes the quermassintegrals of* M*.*

Theorem 1.C. *If* $K, L \in \mathcal{K}^n$ *and* $0 \le i \le n-1$*,* $0 \le j < n-2$*, then*

$$W_i(\Phi_j(K+L))^{\frac{1}{(n-i)(n-j-1)}} \ge W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}}, \tag{1.3}$$

with equality if and only if K *and* L *are homothetic.*

In this article, we shall continuously study the mixed Blaschke Minkowski homomorphisms. Our results were stated as follows: Firstly, corresponding to Theorem 1.B, we give a Minkowski type inequality of quermassintegrals differences with regard to the mixed Blaschke Minkowski homomorphisms.

Theorem 1.1. *Let* K*,* L*,* D *and* D' *be convex bodies in* $\mathbb{R}^n$*,* $D \subseteq K$*,* $D' \subseteq L$*,* D' *is a homothetic of* D*, then for* $0 \le i \le n-1$*,* $1 \le j < n-1$*,*

$$\begin{aligned}&\left[W_i(\Phi_j(K,L)) - W_i(\Phi_j(D,D'))\right]^{n-1}\\ &\qquad \ge \left[W_i(\Phi K) - W_i(\Phi D)\right]^{n-j-1}\left[W_i(\Phi L) - W_i(\Phi D')\right]^j,\end{aligned}$$

with equality if and only if K *and* L *are homothetic and* $W_i(\Phi K)/W_i(\Phi D) = W_i(\Phi L)/W_i(\Phi D')$*.*

Secondly, associated with Theorem 1.C, we obtain a Brunn-Minkowski type inequality for quermassintegrals differences of mixed Blaschke Minkowski homomorphisms.

Theorem 1.2. *Let* K*,* L*,* D *and* D' *be convex bodies in* $\mathbb{R}^n$*,* $D \subseteq K$*,* $D' \subseteq L$*, and let* D' *be a homothetic of* D*, then for* $0 \le i < n$*,* $0 \le j < n-2$*,*

$$\left[W_i(\Phi_j(K+L)) - W_i(\Phi_j(D+D'))\right]^{\frac{1}{(n-i)(n-j-1)}}$$

$$\geq \left[W_i(\Phi_j K) - W_i(\Phi_j D)\right]^{\frac{1}{(n-i)(n-j-1)}} + \left[W_i(\Phi_j L) - W_i(\Phi_j D')\right]^{\frac{1}{(n-i)(n-j-1)}},$$

with equality if and only if K and L are homothetic and $(W_i(\Phi_j K), W_i(\Phi_j L)) = v(W_i(\Phi_j D), W_i(\Phi_j D'))$, where v is a constant.

Further, the following result is an isolated form of Theorem 1.C.

Theorem 1.3. *If $K, L \in \mathcal{K}^n$, $0 \leq i \leq n-1$ and $0 \leq j < n-1$, then for $0 \leq \alpha \leq 1$*

$$\begin{aligned} &W_i(\Phi_j(K+L))^{\frac{1}{(n-i)(n-j-1)}} \\ &\quad \geq W_i(\Phi_j((1-\alpha)K+\alpha L))^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j(\alpha K+(1-\alpha)L))^{\frac{1}{(n-i)(n-j-1)}} \\ &\quad \geq W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}}, \end{aligned}$$

there is equality in the left inequality if and only if $(1-\alpha)K+\alpha L$ and $\alpha K+(1-\alpha)L$ are homothetic and equality in the right inequality if and only if K and L are homothetic.

Finally, together with the definition of mixed Blaschke Minkowski homomorphisms, we extend an inequality of Lutwak.

Theorem 1.4. *For $K \in \mathcal{K}^n$, and $0 \leq i < j < n-1$. If $0 \leq m < n$, then*

$$W_m(\Phi_j K)^{n-i-1} \geq r_{\Phi}^{(n-m)(j-i)} \omega_n^{j-i} W_m(\Phi_i K)^{n-j-1},$$

with equality if and only if $\Phi_i K$ and $\Phi_j K$ are both balls and they are homothetic. Here r_Φ denotes the radius of the ball ΦB.

2. Preliminaries

In this section, we collect some basic notion and notation that are needed in the proofs of the main theorems, and they can be found in the books [4] and [16].

For $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1, \lambda_2 \geq 0$ (not both zero), the support function of the Minkowski linear combination $\lambda_1 K_1 + \lambda_2 K_2$ is

$$h(\lambda_1 K_1 + \lambda_2 K_2, \cdot) = \lambda_1 h(K_1, \cdot) + \lambda_2 h(K_2, \cdot). \tag{2.1}$$

The volume of Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$ of convex bodies $K_1, \ldots, K_m$ was given by

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \ldots, i_n} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}. \tag{2.2}$$

The coefficients $V(K_{i_1},\dots,K_{i_n})$ are called mixed volumes of $K_{i_1},\dots,K_{i_n}$. These functions are nonnegative, symmetric and translation invariant. Moreover, they are monotone (with respect to set inclusion), multilinear with respect to Minkowski addition and their diagonal form is ordinary volume, i.e., $V(K,\dots,K)=V(K)$.

Denote by $V_i(K,L)$ the mixed volume $V(K,\dots,K,L,\dots,L)$, where K appears $n-i$ times and L appears i times. For $0\le i\le n-1$, write $W_i(K,L)$ for the mixed volume $V(K,\dots,K,B,\dots,B,L)$, where K appears $n-i-1$ times and the unit ball B appears i times. The mixed volume $W_i(K,K)$ will be written as $W_i(K)$ and is called the quermassintegrals of K.

For $K_1,\dots,K_{n-1}\in\mathscr{K}^n$, a Borel measure on S^{n-1}, $S(K_1,\dots,K_{n-1},\cdot)$, is called the mixed surface area measure of $K_1,\dots,K_{n-1}$. It is symmetric and has the property that, for each $K\in\mathscr{K}^n$,

$$V(K,K_1,\dots,K_{n-1})=\frac{1}{n}\int_{S^{n-1}}h(K,u)\,dS(K_1,\dots,K_{n-1},u). \tag{2.3}$$

The measures $S_j(K,\cdot)=S(K,\dots,K,B,\dots,B,\cdot)$, where K appears j times and B appears $n-j-1$ times, are called the surface area measures of order j of K. If $j=n-1$, then write $S_{n-1}(K,\cdot)$ for $S(K,\cdot)$, and is called the surface area measure of K.

If $K_1,K_2\in\mathscr{K}^n$ and $\lambda_1,\lambda_2\ge 0$ (not both zero), then there exists a convex body $\lambda_1\cdot K_1\dot{+}\lambda_2\cdot K_2$, such that

$$S(\lambda_1\cdot K_1\dot{+}\lambda_2\cdot K_2,\cdot)=\lambda_1 S(K_1,\cdot)+\lambda_2 S(K_2,\cdot). \tag{2.4}$$

This addition and scalar multiplication are called Blaschke addition and scalar multiplication. For $K\in\mathscr{K}^n$ and $\lambda\ge 0$, we have $\lambda\cdot K=\lambda^{\frac{1}{n-1}}K$.

One of the most general and fundamental inequality for mixed volumes is Aleksandrov-Fenchel inequality: If $K_1,\dots,K_n\in\mathscr{K}^n$ and $1\le m\le n$, then

$$V(K_1,\dots,K_n)^m\ge\prod_{j=1}^{m}V(\underbrace{K_j,\dots,K_j}_{m},K_{m+1},\dots,K_n). \tag{2.5}$$

An important special case of inequality (2.5) is the following Minkowski type inequality: If $K,L\in\mathscr{K}^n$, $0\le i\le n-2$, then

$$W_i(K,L)^{n-i}\ge W_i(K)^{n-i-1}W_i(L), \tag{2.6}$$

with equality if and only if K and L are homothetic.

3. The Proofs of Theorems

In this section, we complete the proofs of Theorems 1.1–1.4. For the proof of Theorem 1.1, we require a Lemma as follows:

Lemma 3.1 ([24]). *If $a, b, c, d > 0$, $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta = 1$. Let $a > b$ and $c > d$, then*

$$a^\alpha c^\beta - b^\alpha d^\beta \geq (a-b)^\alpha (c-d)^\beta, \tag{3.1}$$

with equality if and only if $a/b = c/d$.

Proof of Theorem 1.1. Using inequality (1.2), we have for $0 \leq i \leq n-1$ and $1 \leq j < n-1$

$$W_i(\Phi_j(K,L)) \geq W_i(\Phi K)^{\frac{n-j-1}{n-1}} W_i(\Phi L)^{\frac{j}{n-1}}, \tag{3.2}$$

with equality if and only if K and L are homothetic. Since D' is a homothetic of D,

$$W_i(\Phi_j(D,D')) = W_i(\Phi D)^{\frac{n-j-1}{n-1}} W_i\left(\Phi D'\right)^{\frac{j}{n-1}}. \tag{3.3}$$

Combine with (3.2) and (3.3), apply inequality (3.1) to get

$$\begin{aligned} &W_i(\Phi_j(K,L)) - W_i(\Phi_j(D,D')) \\ &\quad \geq W_i(\Phi K)^{\frac{n-j-1}{n-1}} W_i(\Phi L)^{\frac{j}{n-1}} - W_i(\Phi D)^{\frac{n-j-1}{n-1}} W_i\left(\Phi D'\right)^{\frac{j}{n-1}} \\ &\quad \geq [W_i(\Phi K) - W_i(\Phi D)]^{\frac{n-j-1}{n-1}} [W_i(\Phi L) - W_i(\Phi D')]^{\frac{j}{n-1}}, \end{aligned}$$

i.e.,

$$\begin{aligned} &[W_i(\Phi_j(K,L)) - W_i(\Phi_j(D,D'))]^{n-1} \\ &\quad \geq [W_i(\Phi K) - W_i(\Phi D)]^{n-j-1} [W_i(\Phi L) - W_i(\Phi D')]^j. \end{aligned} \tag{3.4}$$

According to the equality conditions of inequalities (3.1) and (3.2), we see that equality holds in (3.4) if and only if K and L are homothetic and $W_i(\Phi K)/W_i(\Phi D) = W_i(\Phi L)/W_i(\Phi D')$.

Taking $i = 0$, $j = 1$ in inequality (3.4), inequality (3.4) changes to the following result.

Corollary 3.1. *Let K, L, D and D' be convex bodies in $\mathbb{R}^n$, $D \subseteq K$, $D' \subseteq L$, D' is a homothetic of D, then*

$$\begin{aligned} &\left[V(\Phi_1(K,L)) - V(\Phi_1(D,D'))\right]^{n-1} \\ &\geq [V(\Phi K) - V(\Phi D)]^{n-2} \left[V(\Phi L) - V(\Phi D')\right], \end{aligned}$$

with equality if and only if K and L are homothetic and $V(\Phi K)/V(\Phi D) = V(\Phi L)/V(\Phi D')$.

Lemma 3.2 ([1]). *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two series of positive real numbers and let $p > 1$. If $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \leq \left((a_1+b_1)^p - \sum_{i=2}^n (a_i+b_i)^p\right)^{\frac{1}{p}}, \tag{3.5}$$

with equality if and only if $a = vb$, where v is a constant.

Proof of Theorem 1.2. From Theorem 1.C, we have for $0 \le i < n$ and $0 \le j < n-2$

$$W_i(\Phi_j(K+L)) \ge [W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}}]^{(n-i)(n-j-1)}, \tag{3.6}$$

with equality if and only if K and L are homothetic. In view of D' is a homothetic of D, then

$$W_i(\Phi_j(D+D')) = [W_i(\Phi_j D)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j D')^{\frac{1}{(n-i)(n-j-1)}}]^{(n-i)(n-j-1)}, \tag{3.7}$$

From (3.6) and (3.7), it follows from inequality (3.5) that

$$\begin{aligned}
&\left[W_i(\Phi_j(K+L)) - W_i(\Phi_j(D+D'))\right]^{\frac{1}{(n-i)(n-j-1)}} \\
&\quad \ge \{[W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}}]^{(n-i)(n-j-1)} \\
&\qquad -[W_i(\Phi_j D)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j D')^{\frac{1}{(n-i)(n-j-1)}}]^{(n-i)(n-j-1)}\}^{\frac{1}{(n-i)(n-j-1)}} \\
&\quad \ge \left[W_i(\Phi_j K) - W_i(\Phi_j D)\right]^{\frac{1}{(n-i)(n-j-1)}} + \left[W_i(\Phi_j L) - W_i(\Phi_j D')\right]^{\frac{1}{(n-i)(n-j-1)}}.
\end{aligned}$$

That is

$$\begin{aligned}
&\left[W_i(\Phi_j(K+L)) - W_i(\Phi_j(D+D'))\right]^{\frac{1}{(n-i)(n-j-1)}} \\
&\quad \ge \left[W_i(\Phi_j K) - W_i(\Phi_j D)\right]^{\frac{1}{(n-i)(n-j-1)}} + \left[W_i(\Phi_j L) - W_i(\Phi_j D')\right]^{\frac{1}{(n-i)(n-j-1)}}.
\end{aligned} \tag{3.8}$$

By the equality conditions of inequalities (3.5) and (3.6), equality holds in (3.8) if and only if K and L are homothetic and $(W_i(\Phi_j K), W_i(\Phi_j L)) = v(W_i(\Phi_j D), W_i(\Phi_j D'))$, where v is a constant.

Let $i = 0,\ j = 0$ in inequality (3.8), we obtain the following result.

Corollary 3.2. *Let K, L, D and D' be convex bodies in $\mathbb{R}^n$, $D \subseteq K$, $D' \subseteq L$, and let D' be a homothetic of D, then*

$$\begin{aligned}
&\left[V(\Phi(K+L)) - V(\Phi(D+D'))\right]^{\frac{1}{n(n-1)}} \\
&\quad \ge [V(\Phi K) - V(\Phi D)]^{\frac{1}{n(n-1)}} + \left[V(\Phi L) - V(\Phi D')\right]^{\frac{1}{n(n-1)}},
\end{aligned}$$

with equality if and only if K and L are homothetic and $(V(\Phi K), V(\Phi L)) = v(V(\Phi D), V(\Phi D'))$, where v is a constant.

Lemma 3.3 ([18])**.** *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke Minkowski homomorphism, then there is a function $g \in \mathscr{C}(S^{n-1}, \hat{e})$ such that*

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * g,$$

where $\mathscr{C}(S^{n-1}, \hat{e})$ denotes the set of continuous zonal functions on S^{n-1} and the notation "$*$" denotes the convolution.

As a consequence of Lemma 3.3, we have for mixed Blaschke Minkowski homomorphism induced by Φ

$$h(\Phi_i K, \cdot) = S_{n-i-1}(K, \cdot) * g. \tag{3.9}$$

Proof of Theorem 1.3. For $0 \le \alpha \le 1$, from (2.1), we have for any $u \in S^{n-1}$

$$\begin{aligned} h(K+L,u) &= h(K,u)+h(L,u) = (1-\alpha)h(K,u)+\alpha h(K,u)+(1-\alpha)h(L,u)+\alpha h(L,u) \\ &= h((1-\alpha)K+\alpha L+\alpha K+(1-\alpha)L,u), \end{aligned}$$

i.e.,

$$K+L = (1-\alpha)K+\alpha L+\alpha K+(1-\alpha)L. \tag{3.10}$$

Thus it follows from inequality (1.3) that

$$\begin{aligned} &W_i(\Phi_j(K+L))^{\frac{1}{(n-i)(n-j-1)}} \\ &\quad = W_i(\Phi_j((1-\alpha)K+\alpha L+\alpha K+(1-\alpha)L))^{\frac{1}{(n-i)(n-j-1)}} \\ &\quad \ge W_i(\Phi_j((1-\alpha)K+\alpha L))^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j(\alpha K+(1-\alpha)L))^{\frac{1}{(n-i)(n-j-1)}}. \end{aligned} \tag{3.11}$$

From the equality condition in inequality (1.3), we know that equality holds in (3.11) if and only if $(1-\alpha)K+\alpha L$ and $\alpha K+(1-\alpha)L$ are homothetic.

Further, we prove the right inequality in Theorem 1.3. From formula (3.9), it follows that

$$\begin{aligned} &W_i(\Phi_j((1-\alpha)K+\alpha L))^{\frac{1}{(n-i)(n-j-1)}} \\ &\quad \ge W_i(\Phi_j((1-\alpha)K))^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j(\alpha L))^{\frac{1}{(n-i)(n-j-1)}} \\ &\quad = (1-\alpha)W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + \alpha W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}}. \end{aligned} \tag{3.12}$$

By the equality condition of (1.3), we know that with equality in (3.12) if and only if $(1-\alpha)K$ and αL are homothetic, that is K and L are homothetic. Similarly,

$$\begin{aligned} &W_i(\Phi_j(\alpha K+(1-\alpha)L))^{\frac{1}{(n-i)(n-j-1)}} \\ &\quad \ge \alpha W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + (1-\alpha)W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}}, \end{aligned} \tag{3.13}$$

with equality if and only if K and L are homothetic. Combining with (3.12) and (3.13), this gets the desired inequality.

According to the equality conditions of (3.12) and (3.13), we see that equality holds in the right inequality of Theorem 1.3 if and only if K and L are homothetic.

If we take the mixed projection body operator Π_i as the mixed Blaschke Minkowski homomorphism in Theorem 1.3, then we have the following

Corollary 3.3. *If $K, L \in \mathcal{K}^n$, $0 \le i \le n-1$ and $0 \le j < n-1$, then for $0 \le \alpha \le 1$*

$$\begin{aligned} & W_i(\Pi_j(K+L))^{\frac{1}{(n-i)(n-j-1)}} \\ & \quad \ge W_i(\Pi_j((1-\alpha)K+\alpha L))^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Pi_j(\alpha K+(1-\alpha)L))^{\frac{1}{(n-i)(n-j-1)}} \\ & \quad \ge W_i(\Pi_j K)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Pi_j L)^{\frac{1}{(n-i)(n-j-1)}}, \end{aligned}$$

there is equality in the left inequality if and only if $(1-\alpha)K+\alpha L$ and $\alpha K+(1-\alpha)L$ are homothetic and equality in the right inequality if and only if K and L are homothetic.

Lemma 3.4 ([18])**.** *If $K, L \in \mathcal{K}^n$, and $0 \le i, j \le n-2$, then*

$$W_i(K, \Phi_j L) = W_j(L, \Phi_i K). \tag{3.14}$$

Lemma 3.5 ([18])**.** *If $K \in \mathcal{K}^n$, and $0 \le i \le n-2$, then*

$$W_{n-1}(\Phi_i K) = r_\Phi W_{i+1}(K). \tag{3.15}$$

Lemma 3.6 ([11])**.** *If $K \in \mathcal{K}^n$, and $0 \le i < j < n$, then*

$$\omega_n^{j-i} W_i(K)^{n-j} \le W_j(K)^{n-i}, \tag{3.16}$$

with equality if and only if K is a ball.

Proof of Theorem 1.4. For the case $m = n-1$ in inequality of Theorem 1.4, it follows from Lemma 3.5 that it can reduce to (3.16). Let therefore $m < n-1$ and $Q \in \mathcal{K}^n$. From (3.14), we have

$$W_m(Q, \Phi_j K) = W_j(K, \Phi_m Q). \tag{3.17}$$

Thus from (2.5), it follows that

$$\begin{aligned} W_j(K, \Phi_m Q)^{n-i-1} &= V(\underbrace{K, \dots, K}_{n-j-1}, \underbrace{B, \dots, B}_{j-i}, \underbrace{B, \dots, B}_{i}, \Phi_m Q)^{n-i-1} \\ &\ge W_{n-1}(\Phi_m Q)^{j-i} W_i(K, \Phi_m Q)^{n-j-1}. \end{aligned} \tag{3.18}$$

It follows from Lemma 3.5 and Lemma 3.6 that

$$W_{n-1}(\Phi_m Q)^{n-m} = r_\Phi^{n-m} W_{m+1}(Q)^{n-m} \ge r_\Phi^{n-m} \omega_n W_m(Q)^{n-m-1}, \tag{3.19}$$

with equality if and only if Q is a ball. And from Lemma 3.4 and inequality (2.6), we obtain

$$W_i(K, \Phi_m Q)^{n-m} = W_m(Q, \Phi_i K)^{n-m} \ge W_m(Q)^{n-m-1} W_m(\Phi_i K), \tag{3.20}$$

with equality if and only if Q and $\Phi_i K$ are homothetic. Therefore, from identity (3.17) and inequalities (3.18), (3.19) and (3.20), this yields

$$W_m(Q,\Phi_j K)^{n-i-1} \geq r_\Phi^{j-i}\omega_n^{\frac{j-i}{n-m}} W_m(Q)^{\frac{(n-i-1)(n-m-1)}{n-m}} W_m(\Phi_i K)^{\frac{n-j-1}{n-m}}. \tag{3.21}$$

Now take $\Phi_j K$ for Q in (3.21), we get

$$W_m(\Phi_j K)^{n-i-1} \geq r_\Phi^{(n-m)(j-i)}\omega_n^{j-i} W_m(\Phi_i K)^{n-j-1}. \tag{3.22}$$

From the equality conditions of inequalities (3.19) and (3.20), we see that equality holds in (3.22) if and only if $\Phi_i K$ and $\Phi_j K$ are both balls and they are homothetic.

Since the mixed projection body operator Π_i is a mixed Blaschke Minkowski homomorphism, we get the following result, and it was established by Lutwak in the reference [11].

Corollary 3.4. *For $K \in \mathscr{K}^n$, and $0 \leq i < j < n-1$. If $0 \leq m < n$, then*

$$W_m(\Pi_j K)^{n-i-1} \geq \omega_{n-1}^{(n-m)(j-i)}\omega_n^{j-i} W_m(\Pi_i K)^{n-j-1},$$

with equality if and only if K is a ball.

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