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TWO-DIMENSIONAL GENERALIZED WEYL FRACTIONAL CALCULUS PERTAINING TO TWO-DIMENSIONAL $\overline{H}$-TRANSFORMS

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Abstract. The aim of this paper is to establish a relation between the two-dimensional $\overline{H}$-transform involving a general polynomials and the Weyl type two-dimensional Saigo operator of fractional integration.

1. Introduction

Our purpose of this paper is to establish a theorem on two-dimensional $\overline{H}$-transforms involving a general class of polynomials with Weyl type two-dimensional Saigo operators. The results established here are basic in nature and include the results given earlier by Saigo, Saxena and Ram [19], Saxena and Ram [22], Nishimoto and Saxena [12], Saxena and Kiryakova [21], etc.

2. Fractional Integrals and Derivatives

An interesting and useful generalization of both the Riemann-Liouville and Erdélyi-Kober fractional integration operators is introduced by Saigo [14], [15] in terms of Gauss's hypergeometric function as given below.

Assuming that a, b and c are complex numbers and let $x \in R_+ = (0, \infty)$. Following [14], [15] the fractional integral ($Re(a) > 0$) and derivative ($Re(a) < 0$) of the first kind of a function $f(x)$ on R_+ are defined respectively in the forms

$$I_{0,x}^{a,b,c} f = \frac{x^{-a-b}}{\Gamma(a)} \int_0^x (x-t)^{a-1}\ {}_2F_1(a+b, -c; a; 1-\frac{t}{x}) f(t)dt, \quad Re(a) > 0 \qquad (1)$$

$$= \frac{d^n}{dx^n} I_{0,x}^{a+n,b-n,c-n} f,\ 0 < Re(a) + n \leq 1\ (n = 1, 2, \ldots), \qquad (2)$$

where ${}_2F_1(\alpha, \beta; \gamma; .)$ is Gauss's hypergeometric function. The fractional integral ($Re(a) >$

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0) and derivative ($Re(a) < 0$) of the second kind are given by

$$J^{a,b,c}_{x,\infty} f = \frac{1}{\Gamma(a)} \int_x^\infty (t-x)^{a-1} t^{-a-b} \, {}_2F_1(a+b, -c; a; 1-\frac{x}{t}) f(t) dt, \quad Re(a) > 0 \quad (3)$$

$$= (-1)^n \frac{d^n}{dx^n} I^{a+n, b-n, c}_{x,\infty} f, \; 0 < Re(a) + n \leq 1 \; (n = 1, 2, \ldots). \quad (4)$$

The Riemann-Liouville, Weyl and Erdélyi-Kober fractional calculus operators follow as special cases of the operators I and J as detailed below.

$$R^a_{0,x} f = I^{a,-a,c}_{0,x} f = \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t) dt, \quad Re(a) > 0 \quad (5)$$

$$= \frac{d^n}{dx^n} R^{a+n}_{0,x} f, \; 0 < Re(a) + n \leq 1 \; (n = 1, 2, \ldots) \quad (6)$$

$$W^a_{x,\infty} f = J^{a,-a,c}_{x,\infty} f = \frac{1}{\Gamma(a)} \int_x^\infty (t-x)^{a-1} f(t) dt, \quad Re(a) > 0 \quad (7)$$

$$= (-1)^n \frac{d^n}{dx^n} W^{a+n}_{x,\infty} f, \; 0 < Re(a) + n \leq 1 \; (n = 1, 2, \ldots) \quad (8)$$

$$E^{a,c}_{0,x} f = I^{a,0,c}_{0,x} f = \frac{x^{-a-c}}{\Gamma(a)} \int_0^x (x-t)^{a-1} t^c f(t) dt, \quad Re(a) > 0 \quad (9)$$

$$K^{a,c}_{x,\infty} f = J^{a,0,c}_{x,\infty} f = \frac{x^c}{\Gamma(a)} \int_x^\infty (t-x)^{a-1} t^{-a-c} f(t) dt, \quad Re(a) > 0 \quad (10)$$

Following Miller [11, p.82], we denote the class of functions $f(x)$ on R_+, which are infinitely differentiable with partial derivatives of any other behaving as $0(|x|^{-\eta})$ when $x \to \infty$ for all η, by $\mathcal{U}_1$. Similarly we denote the class of functions $f(x, y)$ on $R_+ \times R_+$, which are infinitely differentiable with partial derivatives of any order behaving as $0(|x|^{-\eta_1}, |y|^{-\eta_2})$ when $x \to \infty$, $y \to \infty$ for all η_i $(i = 1, 2)$ by $\mathcal{U}_2$.

The two-dimensional Saige operator of Weyl type fractional integration of orders $Re(a) > 0$, $Re(\gamma) > 0$ is defined in the class $\mathcal{U}_2$ by

$$J^{a,b,c}_{x,\infty} J^{\gamma,\sigma,\rho}_{y,\infty} [f(x,y)] = \frac{x^b y^\sigma}{\Gamma(a)\Gamma(\gamma)} \int_x^\infty \int_y^\infty (s-x)^{a-1} (w-y)^{\gamma-1} s^{-a-b} w^{-\gamma-\sigma}$$
$$\times \, {}_2F_1(a+b, -c; a; 1-\frac{x}{s}) \, {}_2F_1(\gamma+\sigma, -\rho; \gamma; 1-\frac{y}{w}) f(s,w) ds dw, \quad (11)$$

where b, σ, c, ρ are real numbers. More generally, a Saigo operator of Weyl type ftactional calculus in two-variables is defined by the differ-integral expression

$$J^{a,b,c}_{x,\infty} J^{\gamma,\sigma,\rho}_{y,\infty} [f(x,y)] = \frac{(-1)^{m+n} x^b y^\sigma}{\Gamma(a+m)\Gamma(\gamma+n)} \frac{\partial^{m+n}}{\partial x^m \partial y^n} \Big\{ \int_x^\infty \int_y^\infty (s-x)^{a+m-1} (w-y)^{\gamma+n-1}$$
$$\times s^{-a-b} w^{-\gamma-\sigma} \, {}_2F_1(a+b, -c; a; 1-\frac{x}{s}) \, {}_2F_1(\gamma+\sigma, -\rho; \gamma; 1-\frac{y}{w}) f(s,w) ds dw \Big\}, \quad (12)$$

for arbitrary real (complex) a and γ, $m, n = 0, 1, \ldots$. For $f(x, y) \in \mathcal{U}_2$, this differ-integral exists and also belongs to $\mathcal{U}_2$ [11, p.82].

In particular, if $Re(a) < 0$, $Re(\gamma) < 0$ and m, n are positive integers such that $Re(a) + m > 0$, $Re(\gamma) + n > 0$, then (12) yields the partial fractional derivative of $f(x, y)$.

Letting $b = \sigma = 0$, (12) yields the Weyl type Erdélyi-Kober operators in two-dimensions:

$$K^{a,c}_{x,\infty} K^{\gamma,\rho}_{y,\infty}[f(x,y)] = J^{a,0,c}_{x,\infty} J^{\gamma,0,\rho}_{y,\infty}[f(x,y)]$$
$$= \frac{(-1)^{m+n} x^c y^\rho}{\Gamma(a+m)\Gamma(\gamma+n)} \frac{\partial^{m+n}}{\partial x^m \partial y^n}$$
$$\times \Big\{ \int_x^\infty \int_y^\infty (s-x)^{a+m-1} (w-y)^{\gamma+n-1} s^{-a-c} w^{-\gamma-\rho} f(s,w) ds dw \Big\}. \quad (13)$$

3. Two-dimensional Laplace Transform and $\overline{\text{H}}$-Transforms Involoving a General Class of Polynomials

The Laplace transform $\zeta(g, h)$ of a function $f(x, y) \in \mathcal{U}_2$ is defined as

$$\zeta(g,h) = \mathcal{L}[f(x,y); g,h] = \int_0^\infty \int_0^\infty e^{-gx-hy} f(x,y) dx dy, \quad (14)$$

where $Re(g) > 0$, $Re(h) > 0$.

Similarly, the Laplace transform of $f[p\sqrt{x^2-u^2}H(x-u), q\sqrt{y^2-v^2}H(y-v)]$ is defined by the Laplace transform of $F(x, y)$, where

$$F(x,y) = f\Big[p\sqrt{x^2-u^2}H(x-u), q\sqrt{y^2-v^2}H(y-v)\Big], \; x > u > 0; \; y > v > 0 \quad (15)$$

and $H(t)$ denotes Heaviside's unit step function.

Definition. By two-dimensional $\overline{H}$-transform $\phi(g, h)$ involving a general class of polynomials of a function $F(x, y)$, we mean the following repeated integral involving two different $\overline{H}$-functions with a general class of polynomials

$$\phi(g,h) = \phi^{M_1,N_1,M;M_2,N_2,M'}_{P_1,Q_1,N;P_2,Q_2,N'}[F(x,y); \alpha, \beta; g, h]$$
$$= \int_u^\infty \int_v^\infty (gx)^{\alpha-1} (hy)^{\beta-1} \overline{H}^{M_1,N_1}_{P_1,Q_1}\Big[(gx)^{r_1}\Big|^{(a_j,\alpha_j;A_j)_{1,N_1},(a_j,\alpha_j)_{N_1+1,P_1}}_{(b_j,\beta_j)_{1,M_1},(b_j,\beta_j;B_j)_{M_1+1,Q_1}}\Big]$$
$$\times S^M_N\{(gx)^{t_1}\} \overline{H}^{M_2,N_2}_{P_2,Q_2}\Big[(hy)^{r_2}\Big|^{(c_j,\kappa_j;C_j)_{1,N_2},(c_j,\kappa_j)_{N_2+1,P_2}}_{(d_j,\tau_j)_{1,M_2},(d_j,\tau_j;D_j)_{M_2+1,Q_2}}\Big] S^{M'}_{N'}\{(hy)^{t_2}\}$$
$$\times F(x,y) dx dy \quad (16)$$

Here we suppose that $u > 0, v > 0, r_1 > 0, r_2 > 0$; $\phi(g, h)$ exists and belongs to $\mathcal{U}_2$. Further suppose that

$$|\arg\ g^{r_1}| < \frac{1}{2} T_1 \pi, \quad |\arg h^{r_2}| < \frac{1}{2} T_2 \pi, \quad (17)$$

where

$$T_1 = \sum_{j=1}^{M_1} |\beta_j| + \sum_{j=1}^{N_1} A_j a_j - \sum_{j=M_1+1}^{Q_1} |B_j \beta_j| - \sum_{j=N_1+1}^{P_1} \alpha_j > 0,$$

$$T_2 = \sum_{j=1}^{M_2} |\tau_j| + \sum_{j=1}^{N_2} C_j c_j - \sum_{j=M_2+1}^{Q_2} |D_j \tau_j| - \sum_{j=N_2+1}^{P_2} \kappa_j > 0$$

The $\overline{H}$-function appearing in (16), introduced by Inayat-Hussain ([6], see also [2]) in terms of Mellin-Barnes type contour integral, is defined by

$$\overline{H}^{M,N}_{P,Q}\left[z\Big|^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Psi(\xi) z^{\xi} d\xi, \tag{18}$$

where

$$\Psi(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{N} \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod_{j=M+1}^{Q} \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi)}, \tag{19}$$

which contains fractional powers of some of the Γ-functions. Here and throughout the paper a_j $(j = 1, \ldots, P)$ and b_j $(j = 1, \ldots, Q)$ are complex parameters, $\alpha_j \geq 0$ $(j = 1, \ldots, P)$, $\beta_j \geq 0$ $(j = 1, \ldots, Q)$ (not all zero simultaneously) and the exponents A_j $(j = 1, \ldots, N)$ and B_j $(j = M + 1, \ldots, Q)$ can take on non-integer values. The contour in (18) is imaginary axis $Re(\xi) = 0$. It is suitably indented in order to avoid the singularities of the Γ-functions and to keep these singularities on appropriate sides. Again, for A_j $(j = 1, \ldots, N)$ not an integer, the poles of the Γ-functions of the numerator in (19) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - \beta_j \xi)$ $(j = 1, \ldots, M)$ and $\Gamma(1 - a_j + \alpha_j \xi)$ $(j = 1, \ldots, N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

For the sake of brevity

$$T = \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} A_j \alpha_j - \sum_{j=M+1}^{Q} |B_j \beta_j| - \sum_{j=N+1}^{P} \alpha_j > 0. \tag{20}$$

Further, a general class of polynomials appearing in (16), introduced by Srivastava ([23], p.185, eqn.(7)), is defined by

$$S_N^M(x) = \sum_{s=0}^{[N/M]} \frac{(-N)_{Ms}}{s!} A[N, s] x^s, \tag{21}$$

where M is arbitrary positive integer and the coefficient $A[N, s]$ is arbitrary constant, real or complex.

4. Relationship Between Two-dimensional $\overline{\text{H}}$-Transform Involving a General Class of Polynomials in Terms of Two-dimensional Saigo Operator of Weyl Type

To prove the theorem in this section, we need the two-dimensional $\overline{H}$-transform $\phi_1(g,h)$ involving a general class of polynomials $S_N^M[x]$ of $F(x,y)$ defined by

$$\phi_1(g,h) = \phi^{M_1+2,N_1,M;M_2+2,N_2,M'}_{P_1+2,Q_1+2,N;P_2+2,Q_2+2,N'}[f(x,y);\alpha,\beta;g,h]$$
$$= \int_u^\infty \int_v^\infty (gx)^{\alpha-1}(hy)^{\beta-1}\overline{H}^{M_1+2,N_1}_{P_1+2,Q_1+2}$$
$$\times\Big[(gx)^{r_1}\Big|^{(a_j,\alpha_j;A_j)_{1,N_1},(a_j,\alpha_j)_{N_1+1,P},(1-\alpha-s_1t_1,r_1),(a+b+c-\alpha+1-s_1t_1,r_1)}_{(b_j,\beta_j)_{1,M_1},(b-\alpha+1-s_1t_1,r_1),(c-\alpha+1-s_1t_1,r_1),(b_j,\beta_j;B_j)_{M_1+1,Q_1}}\Big]S_N^M\{(gx)^{t_1}\}$$
$$\times\overline{H}^{M_2+2,N_2}_{P_2+2,Q_2+2}\Big[(hy)^{r_2}\Big|^{(c_j,\kappa_j;C_j)_{1,N_2},(c_j,\kappa_j)_{N_2+1,P_2},(1-\beta-s_2t_2,r_2),(\gamma+\sigma+\rho-\beta+1-s_2t_2,r_2)}_{(d_j,\tau_j)_{1,M_2},(\sigma-\beta+1-s_2t_2,r_2),(\rho-\beta+1-s_2t_2,r_2),(d_j,\tau_j;D_j)_{M_2+1,Q_2}}\Big]$$
$$\times S_{N'}^{M'}\{(hy)^{t_2}\}F(x,y)dxdy, \qquad (22)$$

where it is supposed that $\phi_1(g,h)$ exists and belongs to $\mathcal{U}_2$ as well as $r_1>0$, $r_2>0$ and other conditions on the parameters, in which additional parameters a,b,γ,σ,c,ρ included correspond to those in (11).

Theorem 1. *For $Re(a)>0$, $Re(\gamma)>0$, $u>0$, $v>0$, $r_1>0$ and $r_2>0$, also let $\phi(g,h)$ be given by* (16)*, then the following formula*

$$J^{a,b,c}_{g,\infty}J^{\gamma,\sigma,\rho}_{h,\infty}[\phi(g,h)] = \phi_1(g,h) \qquad (23)$$

holds, provided that $\phi_1(g,h)$ exists and belongs to class $\mathcal{U}_2$.

Proof. Let $Re(a)>0$, $Re(\gamma)>0$, then in view of (11) and (16) we have

$$J^{a,b,c}_{g,\infty}J^{\gamma,\sigma,\rho}_{h,\infty}[\phi(g,h)]$$
$$= \frac{g^b h^\sigma}{\Gamma(a)\Gamma(\gamma)}\int_g^\infty\int_h^\infty (s-g)^{a-1}(w-h)^{\gamma-1}s^{-a-b}w^{-\gamma-\sigma}$$
$$\times {}_2F_1(a+b,-c;a;1-\frac{g}{s})\,{}_2F_1(\gamma+\sigma,-\rho;\gamma;1-\frac{h}{w})\phi(s,w)dsdw$$
$$= \frac{g^b h^\sigma}{\Gamma(a)\Gamma(\gamma)}\int_g^\infty\int_h^\infty s^{-a-b}w^{-\gamma-\sigma}(s-g)^{a-1}(w-h)^{\gamma-1}$$
$$\times {}_2F_1(a+b,-c;a;1-\frac{g}{s})\,{}_2F_1(\gamma+\sigma,-\rho;\gamma;1-\frac{h}{w})$$
$$\times\Big\{\int_u^\infty\int_v^\infty (sx)^{\alpha-1}(wy)^{\beta-1}\overline{H}^{M_1,N_1}_{P_1,Q_1}\Big[(sx)^{r_1}\Big|^{(a_j,\alpha_j;A_j)_{1,N_1},(a_j,\alpha_j)_{N_1+1,P_1}}_{(b_j,\beta_j)_{1,M_1},(b_j,\beta_j;B_j)_{M_1+1,Q_1}}\Big]S_N^M\{(sx)^{t_1}\}$$
$$\times\overline{H}^{M_2,N_2}_{P_2,Q_2}\Big[(ty)^{r_2}\Big|^{(c_j,\kappa_j;C_j)_{1,N_2},(c_j,\kappa_j)_{N_2+1,P_2}}_{(d_j,\tau_j)_{1,M_2},(d_j,\tau_j;D_j)_{M_2+1,Q_2}}\Big]S_{N'}^{M'}\{(wy)^{t_2}\}F(x,y)dxdy\Big\}dsdw, \qquad (24)$$

On interchanging the order of integration which is permissible, and on evaluating the s- and w-integrals through the integral formula

$$\int_x^\infty s^{-\mu-\nu}(s-x)^{\nu-1}\ {}_2F_1(r,\omega;\nu,1-\frac{x}{s})\overline{H}_{P,Q}^{M,N}\left[zs^k\Big|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}\right]S_{n'}^{M'}\{zs^{k'}\}ds$$
$$=\frac{\Gamma(\nu)}{x^\mu}\overline{H}_{P+2,Q+2}^{M+2,N}\left[zx^k\Big|_{(b_j,\beta_j)_{1,M},(\mu-s'k',k),(\mu+\nu-r-\omega-s'k',k),(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},(\mu+\nu-r-k's',k),(\mu+\nu-\omega-s'k',k)}\right], \tag{25}$$

where $Re(\nu)>0$, $Re(\mu+\nu+\frac{k(1-a_j)}{\alpha_j})>0$, $Re(\mu+\nu-r-\omega+\frac{k(1-a_j)}{\alpha_j})>0$, $|\arg z|<\frac{1}{2}T\pi$ (T is given in (20)). (25) can be established by means of the formula [4, p.399]

$$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1}\ {}_2F_1(\alpha,\beta;\gamma;x)dx=\frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)},$$

for $Re(\gamma)>0$, $Re(\rho)>0$, $Re(\gamma+\rho-\alpha-\beta)>0$. The left hand side of (24) becomes

$$=\int_u^\infty\int_v^\infty (gx)^{\alpha-1}(hy)^{\beta-1}$$
$$\times\overline{H}_{P_1+2,Q_1+2}^{M_1+2,N_1}\left[(gx)^{r_1}\Big|_{(b_j,\beta_j)_{1,M_1},(b-\alpha+1-s_1t_1,r_1),(c-\alpha+1-s_1t_1,r_1),(b_j,\beta_j;B_j)_{M_1+1,Q_1}}^{(a_j,\alpha_j;A_j)_{1,N_1},(a_j,\alpha_j)_{N_1+1,P_1},(1-\alpha-s_1t_1,r_1),(a+b+c-\alpha+1-s_1t_1,r_1)}\right]$$
$$\times S_N^M\{(gx)^{t_1}\}$$
$$\times\overline{H}_{P_2+2,Q_2+2}^{M_2+2,N_2}\left[(hy)^{r_2}\Big|_{(d_j,\tau_j)_{1,M_2},(\sigma-\beta+1-s_2t_2,r_2),(1-\beta+\rho-s_2t_2,r_2),(d_j,\tau_j;D_j)_{M_2+1,Q_2}}^{(c_j,\kappa_j;C_j)_{1,N_2},(c_j,\kappa_j)_{N_2+1,P_2},(1-\beta-s_2t_2,r_2),(\gamma+\sigma+\rho-\beta+1-s_2t_2,r_2)}\right]$$
$$\times S_{N'}^{M'}\{(hy)^{t_2}\}F(x,y)dxdy$$
$$=\phi_1{}_{P_1+2,Q_1+2,N;P_2+2,Q_2+2,N'}^{M_1+2,N_1,M;M_2+2,N_2,M'}[F(x,y);\alpha,\beta;g,h]$$
$$=\phi_1(g,h)=\text{R.H.S. of (23)}.$$

Since the two-dimensional Weyl type Saigo operators $J_{x,\infty}^{a,b,c}J_{y,\infty}^{\gamma,\sigma,\rho}$ preserves the class $\mathcal{U}_2$, it follows that $\phi_1(g,h)$ also belongs to $\mathcal{U}_2$.

It is interesting to note that the statement of Theorem 1 can be easily extended for arbitrary real a,γ by using the definition (12) for the generalized Weyl type fractional calculus operators and differentiating under the signs of the integrals.

5. Interesting Special Cases

Taking $c=\rho=0$ in Theorem 1, we have the following Theorem 1(a).

Theorem 1.(a). *For $Re(a)>0$, $Re(\gamma)>0$, $u>0$, $v>0$, $r_1>0$, $r_2>0$ and also let $\phi(g,h)$ be given by* (16)*, then the following formula*

$$J_{g,\infty}^{a,b,0}J_{h,\infty}^{\gamma,\sigma,0}[\phi(g,h)]=\phi_2(g,h), \tag{26}$$

holds, provided that $\phi_2(g,h)$ exists and belongs to class $\mathcal{U}_2$, where ϕ_2 is represented by the repeated integral

$$\phi_2(g,h) = \int_u^\infty \int_v^\infty (gx)^{\alpha-1}(hy)^{\beta-1} \times \overline{H}^{M_1+1,N_1}_{P_1+1,Q_1+1}\left[(gx)^{r_1}\Big|^{(a_j,\alpha_j;A_j)_{1,N_1},(a_j,\alpha_j)_{N_1+1,P_1},(a+b-\alpha+1-s_1t_1,r_1)}_{(b_j,\beta_j)_{1,M_1},(b-\alpha+1-s_1t_1,r_1),(b_j,\beta_j;B_j)_{M_1+1,Q_1}}\right] S_N^M\{(gx)^{t_1}\} \times \overline{H}^{M_2+1,N_2}_{P_2+1,Q_2+1}\left[(hy)^{r_2}\Big|^{(c_j,\kappa_j;C_j)_{1,N_2},(c_j,\kappa_j)_{N_2+1,P_2},(\gamma+\sigma-\beta+1-s_2t_2,r_2)}_{(d_j,\tau_j)_{1,M_2},(\sigma-\beta+1-s_2t_2,r_2),(d_j,\tau_j;D_j)_{M_2+1,Q_2}}\right] \times S_{N'}^{M'}\{(hy)^{t_2}\}F(x,y)dxdy, \tag{27}$$

For $A_j = B_j = 1$, the $\overline{H}$-function in (18) *reduces to Fox's H-function* [5], [9] *and then Theorem* 1(*a*) *reduces to*

$$J^{a,b,0}_{g,\infty} J^{\gamma,\sigma,0}_{h,\infty}[\phi(g,h)] = \phi_3(g,h) \tag{28}$$

provided that $\phi_3(g,h)$ exists and belongs to $\mathcal{U}_2$, where ϕ_3 is represented by the repeated integral

$$\phi_3(g,h) = \int_u^\infty \int_v^\infty (gx)^{\alpha-1}(hy)^{\beta-1} H^{M_1+1,N_1}_{P_1+1,Q_1+1}\left[(gx)^{r_1}\Big|^{(a_{P_1},\alpha_{P_1}),(a+b-\alpha+1-s_1t_1,r_1)}_{(b-\alpha+1-s_1t_1,r_1),(b_{Q_1},\beta_{Q_1})}\right] \times S_N^M\{(gx)^{t_1}\} H^{M_2+1,N_2}_{P_2+1,Q_2+1}\left[(hy)^{r_1}\Big|^{(c_{P_2},\kappa_{P_2}),(\gamma+\sigma-\beta+1-s_2t_2,r_2)}_{(\sigma-\beta+1-s_2t_2,r_2),(d_{Q_2},\tau_{Q_2})}\right] \times S_{N'}^{M'}\{(hy)^{t_2}\}F(x,y)dxdy, \tag{29}$$

On employing the identity

$$H^{M,N}_{P,Q}\left[x\Big|^{(a_P,1)}_{(b_Q,1)}\right] = G^{M,N}_{P,Q}\left[x\Big|^{a_1,\dots,a_P}_{b_1,\dots,b_Q}\right], \tag{30}$$

we see that the two-dimensional H-transform reduces to the corresponding two-dimensional G-transform $\Psi(g,h)$ defined by

$$\Psi(g,h) = G^{M_1,N_1,M;M_2,N_2,M'}_{P_1,Q_1,N;P_2,Q_2,N'}[F(x,y);\alpha,\beta;g,h] = \int_u^\infty \int_v^\infty (gx)^{\alpha-1}(hy)^{\beta-1} G^{M_1,N_1}_{P_1,Q_1}\left[(gx)^{r_1}\Big|^{a_1,\dots,a_{P_1}}_{b_1,\dots,b_{Q_2}}\right] S_N^M\{(gx)^{t_1}\} \times G^{M_2,N_2}_{P_2,Q_2}\left[(hy)^{r_2}\Big|^{c_1,\dots,c_{P_2}}_{d_1,\dots,d_{Q_2}}\right] S_{N'}^{M'}\{(hy)^{t_2}\}F(x,y)dxdy \tag{31}$$

provided that $\Psi(g,h)$ exists and belongs to class $\mathcal{U}_2$, where r_1 and r_2 are positive integers, $u > 0$, $v > 0$, $P_1 \le Q_1$, $P_2 \le Q_2$, $|\arg g^{r_1}| < \frac{T_1^*\pi}{2}$ and $|\arg h^{r_2}| < \frac{T_2^*\pi}{2}$ with $T_1^* = 2N_1+2M_1-P_1-Q_1$ and $T_2^* = 2N_2+2M_2-P_2-Q_2 \cdot G^{M,N}_{P,Q}[.]$ appearing in (30) *and* (31) *represents Meijer's G-function whose detailed account is available from the monograph of Mathai and Saxena* [8].

Thus we obtain the following Theorem 1(b).

Theorem 1.(b). *For $Re(a) > 0$, $Re(\gamma) > 0$, $u > 0$, $v > 0$, r_1 and r_2 being positive integers and also let $\Psi(g,h)$ be given by* (31), *then the following formula*

$$J_{g,\infty}^{a,b,c} J_{h,\infty}^{\gamma,\sigma,\rho}[\Psi(g,h)] = \Psi_1(g,h), \tag{32}$$

holds, provided that $\Psi_1(g,h)$ exists and belongs to class $\mathcal{U}_2$ for other conditions on the parameters, in which additional parameters $a, b, \gamma, c, \sigma, \rho$ included correspond to those in (31)*. Here*

$$\begin{aligned}\Psi_1(g,h) = r_1^{-a} r_2^{-\gamma} &\int_u^\infty \int_v^\infty (gx)^{\alpha-1}(hy)^{\beta-1} \\ &\times G_{P_1+2,Q_1+2}^{M_1+2,N_1}\Big[(gx)^{r_1}\Big|_{\Delta(r_1,b-\alpha+1-s_1t_1),\Delta(r_1,c-\alpha+1-s_1t_1),b_1,\ldots,b_{Q_1}}^{a_1,\ldots,a_{P_1},\Delta(r_1,-\alpha-s_1t_1),\Delta(r_1,a+b+c-\alpha+1-s_1t_1)}\Big] S_N^M\{(gx)^{t_1}\} \\ &\times G_{P_2+2,Q_2+2}^{M_2+2,N_2}\Big[(hy)^{r_2}\Big|_{\Delta(r_2,\sigma-\beta+1-s_2t_2),\Delta(r_2,\rho-\beta+1-s_2t_2),d_1,\ldots,d_{Q_2}}^{c_1,\ldots,c_{P_2},\Delta(r_2,1-\beta-s_2t_2),\Delta(r_2,\gamma+\sigma+\rho-\beta+1-s_2t_2)}\Big] \\ &\times S_{N'}^{M'}\{(hy)^{t_2}\}F(x,y)dxdy,\end{aligned} \tag{33}$$

and the symbol $\Delta(n,\alpha)$ represents the sequence of parameters

$$\frac{\alpha}{n}, \frac{\alpha+1}{n}, \ldots, \frac{\alpha+n-1}{n}.$$

On taking $c = \rho = 0$, (32) *becomes*

$$J_{g,\infty}^{a,b,0} J_{h,\infty}^{\gamma,\sigma,0}[\Psi(g,h)] = \Psi_2(g,h) \tag{34}$$

provided $\Psi_2(g,h)$ exists and belongs to class $\mathcal{U}_2$, where Ψ_2 is represented by the integral

$$\begin{aligned}\Psi_2(g,h) = r_1^{-a} r_2^{-\gamma} &\int_u^\infty \int_v^\infty (gx)^{\alpha-1}(hy)^{\beta-1} \\ &\times G_{P_1+1,Q_1+1}^{M_1+1,N_1}\Big[(gx)^{r_1}\Big|_{\Delta(r_1,b-\alpha+1-s_1t_1),b_1,\ldots,b_{Q_1}}^{a_1,\ldots,a_{P_1},\Delta(r_1,a+b-\alpha+1-s_1t_1)}\Big] S_N^M\{(gx)^{t_1}\} \\ &\times G_{P_2+1,Q_2+1}^{M_2+1,N_2}\Big[(hy)^{r_2}\Big|_{\Delta(r_2,\sigma-\beta+1-s_2t_2),d_1,\ldots,d_{Q_2}}^{c_1,\ldots,c_{P_2},\Delta(r_2,\gamma+\sigma-\beta+1-s_2t_2)}\Big] \\ &\times S_{N'}^{M'}\{(hy)^{t_2}\}F(x,y)dxdy.\end{aligned} \tag{35}$$

On using the representation of the Whittaker function [9]

$$G_{12}^{20}\left(x\Big|_{\frac{1}{2}+\beta,\frac{1}{2}-\beta}^{1-\alpha}\right) = e^{-x/2}W_{\alpha,\beta}(x), \tag{36}$$

we find that the two-dimensional H-transform involving a general class of polynomials reduces to the two-dimensional Whittaker transform

$$\begin{aligned}\Psi_3(g,h) &= W_{\lambda',\mu'}^{\lambda,\mu}[f(x,y);\alpha,\beta;g,h] \\ &= \int_u^\infty \int_v^\infty (gx)^{\alpha-1}(hy)^{\beta-1}\exp\Big[-\frac{1}{2}(gx+hy)\Big]W_{\lambda,\mu}(gx) \\ &\quad\times S_N^M\{(gx)^{t_1}\}W_{\lambda',\mu'}(hy)S_{N'}^{M'}\{(hy)^{t_2}\}F(x,y)dxdy\end{aligned} \tag{37}$$

provided that $Re(g) > 0$, $Re(h) > 0$ *and* $\Psi_3(g,h)$ *exists and belongs to* $\mathcal{U}_2$.
The Whittaker confluent hypergeometric function appearing in equations (36) *and* (37) *is defined by the integral equation* [24, p.340]

$$W_{\lambda,\mu}(x) = \frac{x^{\lambda} e^{-x/2}}{\Gamma(\frac{1}{2}-\lambda+\mu)} \int_0^{\infty} w^{-\frac{1}{2}-\lambda+\mu} \Big(1+\frac{w}{x}\Big)^{\lambda+\mu-\frac{1}{2}} e^{-w} dw, \tag{38}$$

where $Re(\frac{1}{2}-\lambda+\mu) > 0$.

Theorem 1.(c). *There holds the formula*

$$J^{a,b,c}_{g,\infty} J^{\gamma,\sigma,\rho}_{h,\infty}[\Psi_3(g,h)] = \Psi_4(g,h) \tag{39}$$

provided that

$$\begin{aligned}\Psi_4(g,h) = &\int_u^{\infty}\int_v^{\infty} (gx)^{\alpha-1}(hy)^{\beta-1} G^{4,0}_{3,4}\Big[(gx)\Big|^{1-\lambda,1-\alpha-s_1t_1,a+b+c-\alpha+1-s_1t_1}_{b-\alpha+1-s_1t_1,c-\alpha+1-s_1t_1,\frac{1}{2}+\mu,\frac{1}{2}-\mu}\Big] \\ &\times S^M_N\{(gx)^{t_1}\} G^{4,0}_{3,4}\Big[(hy)\Big|^{1-\lambda',1-\beta-s_2t_2,\gamma+\sigma+\rho-\beta+1-s_2t_2}_{\sigma-\beta+1-s_2t_2,\rho-\beta+1-s_2t_2,\frac{1}{2}+\mu',\frac{1}{2}-\mu'}\Big] \\ &\times S^{M'}_{N'}\{(hy)^{t_2}\} F(x,y)dxdy,\end{aligned} \tag{40}$$

exists and belongs to class $\mathcal{U}_2$.
On taking $b=\sigma=0$, (39) *becomes*

$$J^{a,0,c}_{g,\infty} J^{\gamma,0,\rho}_{h,\infty}[\Psi_3(g,h)] = \Psi_5(g,h) \tag{41}$$

provided that

$$\begin{aligned}\Psi_5(g,h) = &\int_u^{\infty}\int_v^{\infty} (gx)^{\alpha-1}(hy)^{\beta-1} G^{3,0}_{2,3}\Big[(gx)\Big|^{1-\lambda,a+c-\alpha+1-s_1t_1}_{c-\alpha+1-s_1t_1,\frac{1}{2}+\mu,\frac{1}{2}-\mu}\Big] S^M_N\{(gx)^{t_1}\} \\ &\times G^{3,0}_{2,3}\Big[(hy)\Big|^{1-\lambda',\gamma+\rho-\beta+1-s_2t_2}_{\rho-\beta+1-s_2t_2,\frac{1}{2}+\mu',\frac{1}{2}-\mu'}\Big] S^{M'}_{N'}\{(hy)^{t_2}\} F(x,y)dxdy,\end{aligned} \tag{42}$$

exists and belongs to class $\mathcal{U}_2$.

6. Some Interesting Known Deductions

(i) On taking $A_j = B_j = 1$, $N = N' = 0$ in Theorem 1, we arrive at the result obtained by Saigo, Saxena and Ram [19, p.67].
(ii) For $A_j = B_j = 1$ and $N = N' = 0 = c = \rho$ in Theorem 1, we have a result earlier proved by Saxena and Kiryakova [21, p.136].
(iii) Letting $N = N' = 0 = c = \rho$ in Theorem 1(b), we get a result earlier given by Nishimoto and Saxena [12, p.25].
(iv) When $N = N' = 0 = b = \sigma$ in Theorem 1(c), we find the reuslt earlier given by Saxena and Ram [22, p.28].

7. One-Dimensional Analogue of Theorem 1

The following one dimensional analogue can be established on the similar lines as given in Theorem 1.

Theorem 2. *Let $\phi(g)$ be the one-dimensional $\overline{H}$-transform involving a general class of polynomials of $F(x)$ defined by*

$$\phi(g) = \phi_{P,Q,N'}^{M,N,M'}[F(x);\alpha,g]$$
$$= \int_u^\infty (gx)^{\alpha-1}\overline{H}_{P,Q}^{M,N}\left[(gx)^r\Big|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}\right]S_{N'}^{M'}\{(gx)^t\}F(x)dx, \quad (43)$$

provided that $\phi(g)$ exists and belongs to class $\mathcal{U}_1$, where $r > 0$, $|\arg g^r| < \frac{1}{2}T\pi$;

$$F(x) = f\left(a\sqrt{x^2-u^2}\right)H(x-u). \quad (44)$$

For $Re(a) > 0$, $u > 0$, $r > 0$ and let $\phi_1(g)$ be defined as

$$\phi_1(g)=\int_u^\infty (gx)^{\alpha-1}\overline{H}_{P+2,Q+2}^{M+2,N}\left[(gx)^r\Big|_{(b_j,\beta_j)_{1,M},(b-\alpha+1-st,r),(c-\alpha+1-st,r),(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},(1-\alpha-st,r),(a+b+c-\alpha+1-st,r)}\right]$$
$$\times S_{N'}^{M'}\{(gx)^t\}F(x)dx, \quad (45)$$

then the following formula

$$\overline{J}_{g,\infty}^{a,b,c}[\phi(g)] = \phi_1(g), \quad (46)$$

holds, provided that $\phi_1(g)$ exists and belongs to class $\mathcal{U}_1$.

$$\text{Here } \overline{J}_{g,\infty}^{a,b,c}f = \frac{g^b}{\Gamma(a)}\int_g^\infty (t-g)^{a-1}t^{a-b}\ {}_2F_1(a+b,-c;a;1-\frac{g}{t})f(t)dt = g^b J_{g,\infty}^{a,b,c}f.$$

Special Cases

(i) For $A_j = B_j = 1$, the $\overline{H}$-function in (18) reduces to Fox's H-function and then (46) becomes

$$\overline{J}_{g,\infty}^{a,b,c}[\phi(g)] = \phi_2(g), \quad (47)$$

provided that $\phi_2(g)$ exists and belongs to class $\mathcal{U}_1$, where

$$\phi_2(g) = \int_u^\infty (gx)^{\alpha-1}\overline{H}_{P+2,Q+2}^{M+2,N}\left[(gx)^r\Big|_{(b-\alpha+1-st,r),(c-\alpha+1-st,r),(b_Q,B_Q)}^{(a_P,\alpha_P),(1-\alpha-st,r),(a+b+c-\alpha+1-st,r)}\right]$$
$$\times S_{N'}^{M'}\{(gx)^t\}F(x)dx. \quad (48)$$

Further, for $\alpha_j = \beta_j = 1$, the Fox's H-function reduces to Meijer's G-function and then (47) yields the following Theorem 2(a).

Theorem 2.(a). *For Re(a) > 0, u > 0 and Let*

$$\phi_3(g) = G^{M,N,M'}_{P,Q,N'}[F(x);\alpha,g]$$
$$= \int_u^\infty (gx)^{\alpha-1} G^{M,N}_{P,Q}\left[(gx)^r \Big|^{a_1,\ldots,a_P}_{b_1,\ldots,b_Q}\right] S^{M'}_{N'}\{(gx)^t\} F(x)dx, \quad (49)$$

where $M+N > \frac{P+Q}{2}$, $|\arg g^r| < (M+N-\frac{P+Q}{2})\pi$ *and* $P \le Q$, *be the one-dimensional G-transform involving a general class of polynomials of* $F(x)$ *and* $\phi_3(g)$ *belongs to class* $\mathcal{U}_1$*, then the following formula*

$$\overline{J}^{a,b,c}_{g,\infty}[\phi_3(g)] = \phi_4(g), \quad (50)$$

holds, provided that $\phi_4(g)$ *exists and belongs to class* $\mathcal{U}_2$. *Here*

$$\phi_4(g) = r^{-a}\int_u^\infty (gx)^{\alpha-1} G^{M+2,N}_{P+2,Q+2}\left[(gx)^r \Big|^{a_1,\ldots,a_P,\Delta(r,1-\alpha-st),\Delta(r,a+b+c-\alpha+1-st)}_{\Delta(r,b-\alpha+1-st),\Delta(r,c-\alpha+1-st),b_1,\ldots,b_Q}\right]$$
$$\times S^{M'}_{N'}\{(gx)^t\}F(x)dx, \quad (51)$$

(ii) For $b = 0$, Theorem 2 reduces to the following Theorem 2(b).

Theorem 2.(b). *Let* $\phi(g)$ *be given by* (43) *and let*

$$K^{a,c}_{g,\infty} f = \overline{J}^{a,0,c}_{g,\infty} f, \quad (52)$$

be the one-dimensional Erdélyi-Kober operator of fractional integration defined by (10)*, then the following formula*

$$K^{a,c}_{g,\infty}[\phi(g)] = \phi_5(g), \quad (53)$$

holds, provided that $\phi_5(g)$ *exists and belongs to class* $\mathcal{U}_1$*, where* $r > 0$, $u > 0$ *and*

$$\phi_5(g) = \int_u^\infty (gx)^{\alpha-1} \overline{H}^{M+1,N}_{P+1,Q+1}\left[(gx)^r \Big|^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},(a+c-\alpha+1-st,r)}_{(b_j,\beta_j)_{1,M},(c-\alpha+1-st,r),(b_j,\beta_j;B_j)_{M+1,Q}}\right]$$
$$\times S^{M'}_{N'}\{(gx)^t\}F(x)dx, \quad (54)$$

Deductions

(i) Taking $N = N' = 0$, $A_j = B_j = 1$, (46) reduces to the result obtained by Saigo, Saxena and Ram [19, p.70].
(ii) If we take $N = N' = 0$ in (49), we arrive at the result obtained by Saigo, Saxena and Ram [19, p.71].
(iii) On taking $A_j = B_j = 1$, $N = N' = 0$ in (53), we get the result earlier proved by Saigo, Saxena and Ram [19, p.71].

On account of the most general character of the $\overline{H}$-function and a general class of polynomials, a large number of interesting particular cases of the results established in this paper can be given by suitably specializing the parameters of the $\overline{H}$-function and $S_N^M[x]$.

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