TAMKANG JOURNAL OF MATHEMATICS
Volume 46, Number 3, 229-243, September 2015
doi:10.5556/j.tkjm.46.2015.1694
This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

INVERSE PROBLEMS FOR THE DIFFERENTIAL OPERATOR ON THE GRAPH WITH A CYCLE WITH DIFFERENT ORDERS ON DIFFERENT EDGES

NATALIA BONDARENKO

Abstract. We consider a variable order differential operator on a graph with a cycle. We study inverse spectral problems for this operator by the system of spectra. Uniqueness theorems are proved, and constructive algorithms are obtained for the solution of the inverse problems.

1. Introduction

Differential operators on geometrical graphs (networks) play a fundamental role in many problems in science and engineering. Main results for second-order operators on graphs and their applications are described in [1, 2, 3, 4, 5, 6, 7]. In this paper, we focus our attention on inverse spectral problems that consist in recovering the coefficients of differential operators on graphs by their spectral characteristics. Thus we assume that the structure of the graph, boundary and matching conditions in the vertices are known a priori.

Although the inverse spectral theory for second-order differential operators has been developed fairly completely, there are only a few works for higher-order operators [8]. In paper [9], V.A. Yurko started to study inverse problems for various order differential operators, i.e. when the orders of differential equations are different on different edges of the graph. Papers [10, 11] describe some mechanical models with variable order differential operators.

In work [9], an inverse problem is solved on a star-type graph. Now we plan to investigate inverse problems for a variable order operator on a graph with a cycle. We use the system of spectra, corresponding to different boundary and matching conditions, for recovering the potential of the differential operator. Our problem statement is a natural generalization of the classical inverse Sturm-Liouville problem on a finite interval by two spectra (see monographs [13, 12]).

Received June 27, 2014, accepted January 20, 2015.
2010 *Mathematics Subject Classification.* 34A55, 34L05, 47E05, 34B45.
Key words and phrases. Geometrical graphs, differential operators, inverse spectral problems, Weyl-type matrices, method of spectral mappings.

Let us come to the formulation of the problem. Consider a compact graph G with the vertices $V = \{v_0, \dots, v_m\}$ and the edges $\mathscr{E} = \{e_0, \dots, e_m\}$, where $e_j = [v_j, v_0]$, $j = \overline{1,m}$, and e_0 is a cycle containing only the vertex v_0. Thus v_j, $j = \overline{1,m}$, are boundary vertices and v_0 is the only internal vertex (see Figure 1). For each edge $e_j \in \mathscr{E}$, we introduce the parameter $x_j \in [0,1]$ in such a way, that for $j = \overline{1,m}$, the end $x_j = 0$ corresponds to the vertex v_j, and the end $x_j = 1$ corresponds to v_0. For $j = 0$, both ends correspond to the vertex v_0.

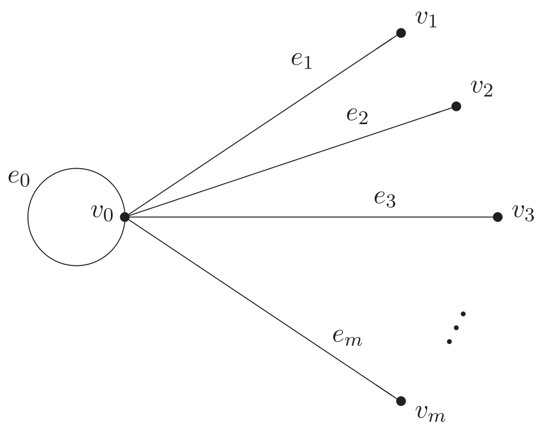

Figure 1:

Fix the integers $2 = n_0 \le n_1 \le \dots \le n_m$. Consider the following differential equations of variable orders:

$$y_j^{(n_j)} + \sum_{\mu=0}^{n_j-2} q_{\mu j}(x_j) y_j^{(\mu)}(x_j) = \lambda y_j(x_j), \quad j = \overline{0,m}, \tag{1.1}$$

where $q_{\mu j} \in L[0,1]$. We call the collection $q := \{q_{\mu j}\}_{j=\overline{0,m}, \mu=\overline{0,n_j-2}}$ the potential on the graph G.

Now we are going to introduce matching conditions in the internal vertex v_0, that generalize Kirchhoff's matching conditions for Sturm-Liouville operators on graphs [7] and matching conditions for higher-order differential operators [8]. Consider the linear forms

$$U_{j\nu}(y_j) = \sum_{\mu=0}^{\nu} \gamma_{j\nu\mu} y_j^{(\mu)}(1), \quad \gamma_{j\nu} := \gamma_{j\nu\nu} \ne 0, \quad j = \overline{1,m}, \nu = \overline{0, n_j - 1},$$

$$U_{0\nu}(y_0) = y_0^{(\nu)}(1), \quad \nu = 0, 1,$$

where $\gamma_{j\nu\mu}$ are some complex numbers. Define the continuity condition $\mathrm{Cont}(\nu)$ and the Kirchhoff's condition $\mathrm{Kirch}(\nu)$ of the ν-th order:

$$\mathrm{Cont}(\nu): \begin{cases} U_{m\nu}(y_m) = U_{j\nu}(y_j), \ j = \overline{0, m-1} \colon \nu < n_j - 1, \\ y_0(0) = U_{0\nu}(y_0), \qquad \text{if } \nu = 0; \end{cases}$$

$$\mathrm{Kirch}(\nu)\colon \sum_{j\colon \nu<n_j} U_{j\nu}(y_j) = \delta_{1\nu} y_0'(0).$$

Here and below δ_{jk} is the Kronecker delta.

Fix an edge number $s = \overline{1,m}$ and orders $k = \overline{1,n_s-1}$, $\mu = \overline{k,n_s}$. Let $\Lambda_{sk\mu} = \{\lambda_{lsk\mu}\}_{l\ge1}$ be the spectrum of the boundary value problem $L_{sk\mu}$ for the system (1.1) under the boundary conditions

$$\begin{cases} y_s^{(\nu-1)}(0) = 0, & \nu = \overline{1,k-1},\mu, \\ y_j^{(\nu-1)}(0) = 0, & \nu = \overline{1,n_j-k},\ j = \overline{1,m}\backslash s\colon n_j > k, \\ y_j(0) = 0, & j = \overline{1,m}\colon n_j \le k, \end{cases} \tag{1.2}$$

and the matching conditions $\mathrm{Cont}(\nu)$, $\nu = \overline{0,k-1}$, $\mathrm{Kirch}(\nu)$, $\nu = \overline{k,n_s-1}$, in the vertex v_0. Note that the total number of the boundary conditions and the matching conditions equals $\sum_{j=0}^{m} n_j$, i.e. the sum of the orders on the edges. In Section 3 we discuss the question of regularity for these conditions.

We will use the spectra $\{\Lambda_{sk\mu}\}$ for recovering of the potential $\{q_{\mu j}\}$, but this information is insufficient, and we need additional data related to the cycle. Let $S_0(x_0,\lambda)$ and $C_0(x_0,\lambda)$ be the solutions of the differential equation (1.1) on the edge e_0 ($n_0 = 2$), satisfying the initial conditions

$$S_0(0,\lambda) = C_0'(0,\lambda) = 0, \quad S_0'(0,\lambda) = C_0(0,\lambda) = 1.$$

Denote

$$h(\lambda) := S_0(1,\lambda), \quad H(\lambda) := C_0(1,\lambda) - S_0'(1,\lambda), \quad d(\lambda) := C_0(1,\lambda) + S_0'(1,\lambda). \tag{1.3}$$

Note that the functions $h(\lambda)$, $H(\lambda)$ and $d(\lambda)$ are entire in λ of order $1/2$. Let $\{\nu_n\}_{n\ge1}$ be the zeros of $h(\lambda)$, and $\omega_n := \operatorname{sign} H(\nu_n)$. Here we assume for the sake of simplicity, that the potential $q_{00}(x_0)$ is real-valued. For the non-self-adjoint case, one can use the approach described in [14].

Inverse problem 1. *Given the spectra $\Lambda_{sk\mu}$, $s = \overline{1,m}$, $k = \overline{1,n_s-1}$, $\mu = \overline{k,n_s}$, and the signs $\Omega := \{\omega_n\}_{n\ge1}$, construct the potentials $q_{\mu j}$, $j = \overline{0,m}$, $\mu = \overline{0,n_j-2}$.*

We will prove the unique solvability of Inverse problem 1 and develop a constructive procedure for its solution. Our approach is based on the method of spectral mappings [13, 15] and some ideas of paper [16] concerning an inverse problem for Sturm-Liouville operator on a graph with a cycle. Our general strategy is to solve auxiliary inverse problems on the boundary edges. These problems are not local problems on intervals, since they use information from the whole graph, but they are close to local problems by their properties. Then the problem is reduced to the well-studied Sturm-Liouville periodic inverse problem for the cycle [18, 17, 16].

One can avoid the use of the additional data Ω by variation of a parameter in matching conditions. Introduce the following condition

$$\mathrm{Cont}(0,\alpha): \quad U_{m0}(y_m) = U_{j0}(y_j),\ j = \overline{0, m-1}, \quad \alpha y_0(0) = y_0(1),$$

depending on the complex parameter $\alpha \neq 0$. Let $m > 1$. The case $m = 1$ requires minor modifications. Fix an edge number $s = \overline{1, m}$ and orders $k = \overline{1, n_s - 1}$, $\mu = \overline{k, n_s}$. Let $\Lambda^{\alpha}_{sk\mu}$ be the spectrum of the boundary value problem $L^{\alpha}_{sk\mu}$ for the system (1.1) under the boundary conditions (1.2) and the matching conditions $\mathrm{Cont}(0, \alpha_s)$, $\mathrm{Cont}(\nu)$, $\nu = \overline{1, k-1}$, $\mathrm{Kirch}(\nu)$, $\nu = \overline{k, n_s - 1}$. Here α_s, $s = \overline{1, m}$, are some nonzero numbers, not all equal to each other. We assume that the conditions are regular.

Inverse problem 2. *Given the spectra* $\Lambda^{\alpha}_{sk\mu}$, $s = \overline{1, m}$, $k = \overline{1, n_s - 1}$, $\mu = \overline{k, n_s}$, *construct the potential* q *on the graph* G.

We show that Inverse Problem 2 is uniquely solvable, and can be solved by a method, analogous to the solution of Inverse Problem 1.

The paper is organized as follows. In Section 2, we introduce so-called Weyl-type matrices for each of the boundary edges and show how to construct them by the given spectra. In Section 3, we study asymptotics of special solutions of system (1.1). In Section 4, we discuss auxiliary inverse problems on the boundary edges and on the cycle. In Section 5, we arrive at the main results of our paper for Inverse Problem 1. Inverse Problem 2 is studied in Section 6. We also provide Appendix with an example.

2. Weyl-type solutions and Weyl-type matrices

In this section, we introduce some special solutions of system (1.1) and study their structural and analytical properties.

Fix $j = \overline{0, m}$. Let $\{C_{kj}(x_j, \lambda)\}_{k=\overline{1, n_j}}$ be a fundamental system of solutions of equation (1.1) on the edge e_j under initial conditions $C^{(\mu-1)}_{kj}(0, \lambda) = \delta_{k\mu}$, $k, \mu = \overline{1, n_j}$. For each fixed $x_j \in [0, 1]$, the functions $C^{(\mu-1)}_{kj}(x_j, \lambda)$ are entire in λ-plane of order $1/n_j$. We also have

$$\det \left[C^{(\mu-1)}_{kj}(x_j, \lambda) \right]_{k,\mu} \equiv 1, \quad j = \overline{0, m}. \tag{2.1}$$

Fix $s = \overline{1, m}$ and $k = \overline{1, n_s - 1}$. Let $\Psi_{sk} = \{\psi_{skj}\}_{j=\overline{1, m}}$ be the solutions of system (1.1) satisfying the conditions

$$\begin{cases} \psi_{sks}^{(\nu-1)}(0)=\delta_{k\nu}, & \nu=\overline{1,k}, \\ \psi_{skj}^{(\xi-1)}(0)=0, & \xi=\overline{1,n_j-k}, \quad j=\overline{1,m}\setminus s\colon\, k<n_j, \\ \psi_{skj}(0)=0, & j=\overline{1,m}\colon\, k\geq n_j, \end{cases} \tag{2.2}$$

$$\operatorname{Cont}(\nu), \quad \nu=\overline{0,k-1}, \quad \operatorname{Kirch}(\nu), \quad \nu=\overline{k,n_s-1}. \tag{2.3}$$

The vector-function Ψ_{sk} is called *the Weyl-type solution* of order k for the boundary vertex v_s. Additionally define $\psi_{sn_js}(x_s,\lambda)=C_{n_js}(x_s,\lambda)$, $s=\overline{1,m}$.

Let $M_{sk\mu}(\lambda):=\psi_{sks}^{(\mu-1)}(0,\lambda)$. For each fixed $s=\overline{1,m}$, the matrix $M_s(\lambda):=[M_{sk\mu}(\lambda)]_{k,\mu=1}^{n_s}$ is called *the Weyl-type matrix* with respect to the boundary vertex v_s. The notion of the Weyl-type matrices is a generalization of the notion of the Weyl function (m-function) for the classical Sturm-Liouville operator (see [19, 13]) and the notion of Weyl matrices for higher-order differential operators (see [8, 9, 15]).

It follows from (2.2), that $M_{sk\mu}(\lambda)=\delta_{k\mu}$ for $k\geq\mu$. Moreover,

$$\psi_{sks}(x_s,\lambda)=C_{ks}(x_s,\lambda)+\sum_{\mu=k+1}^{n_s}M_{sk\mu}(\lambda)C_{\mu s}(x_s,\lambda), \quad s=\overline{1,m}, \quad k=\overline{1,n_s}. \tag{2.4}$$

Now we plan to study the connection between the Weyl-type matrices, the spectra $\Lambda_{sk\mu}$ and the functions, defined in (1.3). For this purpose, one can easily expand the functions $\psi_{skj}(x_j,\lambda)$ by the fundamental systems $C_{\mu j}(x_j,\lambda)$ and substitute these expansions into the matching conditions (2.3). Solving the resulting linear system E_{sk}, one gets for $s=\overline{1,m}$, $1\leq k<\mu\leq n_s$:

$$M_{sk\mu}(\lambda)=-\frac{\Delta_{sk\mu}(\lambda)}{\Delta_{skk}(\lambda)}. \tag{2.5}$$

Here $\Delta_{sk\mu}(\lambda)$, $k\leq\mu$, is he characteristic function for the boundary value problem $L_{sk\mu}$, and its zeros coincide with the eigenvalues $\Lambda_{sk\mu}$. The functions $\Delta_{sk\mu}$ are entire in λ and, consequently, $M_{sk\mu}(\lambda)$ are meromorphic in λ. Similarly to [9], one can easily prove the following fact.

Lemma 1. *Each characteristic function $\Delta_{sk\mu}(\lambda)$ can be determined uniquely by its zeros $\Lambda_{sk\mu}=\{\lambda_{lsk\mu}\}_{l\geq1}$.*

Furthermore, analysing the structure of determinants in the systems E_{sk} (see the example in Appendix for clarity), we obtain the relations

$$\Delta_{sk\mu}(\lambda)=\big(d(\lambda)-2\big)F_{s\mu}(\lambda)+h(\lambda)G_{s\mu}(\lambda), \quad k=1, \tag{2.6}$$

$$\Delta_{sk\mu}(\lambda)=h(\lambda)G_{sk\mu}(\lambda), \quad k>1, \tag{2.7}$$

where $F_{s\mu}(\lambda)$, $G_{s\mu}(\lambda)$, $G_{sk\mu}(\lambda)$ are some combinations of $C_{lj}^{(\nu)}(1,\lambda)$. We will use formulas (2.6), (2.7) to find the data, associated with the cycle, from the characteristic determinants $\Delta_{sk\mu}(\lambda)$.

3. Asymptotic behavior of the Weyl-type solution

Fix $j = \overline{0,m}$. Let $\lambda = \rho_j^{n_j}$. The ρ-plane can be partitioned into sectors of angle $\dfrac{\pi}{n_j}$:

$$S_{\nu j} = \left\{ \arg \rho_j \in \left(\frac{\nu\pi}{n_j}, \frac{(\nu+1)\pi}{n_j} \right) \right\}, \quad \nu = \overline{0, 2n_j - 1}.$$

Let us fix one of them and call it simply S_j. Then the roots $R_{1j}, R_{2j}, \dots, R_{n_j j}$ of the equation $R^{n_j} - 1 = 0$ can be numbered in such a way that

$$\operatorname{Re}(\rho_j R_{1j}) < \operatorname{Re}(\rho_j R_{2j}) < \dots < \operatorname{Re}(\rho_j R_{n_j j}), \quad \rho_j \in S_j. \tag{3.1}$$

Denote

$$\Omega_{0j} := 1, \quad \Omega_{kj} := \det \left[R_{\xi j}^{\nu-1} \right]_{\xi,\nu=1}^{k}, \quad \omega_{kj} := \frac{\Omega_{k-1,j}}{\Omega_{kj}}, \quad j = \overline{0,m},\ k = \overline{1,n_j},$$

$$[1]_j := 1 + O(\rho_j^{-1}), \quad |\rho_j| \to \infty.$$

The following Lemma has been proved in [9]:

Lemma 2. *Fix $j = \overline{0,m}$ and a sector S_j with property (3.1). Let $k = \overline{1, n_j - 1}$ and let $y_j(x_j,\lambda)$ and $z_j(x_j,\lambda)$ be solutions of equation (1.1) on the edge e_j under the initial conditions*

$$y_j(0) = y_j'(0) = \dots = y_j^{(k-1)}(0) = 0,$$
$$z_j(0) = z_j'(0) = \dots = z_j^{(k-2)}(0) = 0, \quad z_j^{(k-1)}(0) = 1.$$

Then for $x_j \in (0, T_j]$, $\nu = \overline{0, n_j - 1}$, $\rho_j \in S_j$, $|\rho_j| \to \infty$,

$$y_j^{(\nu)}(x_j,\lambda) = \sum_{\mu=k+1}^{n_j} A_{\mu j}(\rho_j)(\rho_j R_{\mu j})^{\nu} \exp(\rho_j R_{\mu j} x_j)[1]_j,$$

$$z_j^{(\nu)}(x_j,\lambda) = \frac{\omega_{kj}}{\rho_j^{k-1}}(\rho_j R_{kj})^{\nu} \exp(\rho_j R_{kj} x_j)[1]_j + \sum_{\mu=k+1}^{n_j} B_{\mu j}(\rho_j)(\rho_j R_{\mu j})^{\nu} \exp(\rho_j R_{\mu j} x_j)[1]_j,$$

where the coefficients $A_{\mu j}(\rho_j)$, $B_{\mu j}(\rho_j)$ do not depend on x_j. Here and below we assume that $\arg \rho_j = const$*, as $|\rho_j| \to \infty$.*

Now we are going to apply Lemma 2 to the Weyl-type solution, in order to study its asymptotic behavior. Fix an edge $s = \overline{1,m}$ and an order $k = \overline{1, n_s - 1}$. For brevity, further we omit the indices s, k it they are fixed, $\psi_j(x_j,\lambda) := \psi_{skj}(x_j,\lambda)$. Fix a ray $\{\lambda \colon \arg\lambda = \theta\}$, $\theta \neq 0, \pi$, which belongs to some sectors S_j with property (3.1) for each $j = \overline{0,m}$. It follows from (2.2) and Lemma 2 that

$$\psi_s^{(\nu)}(x_s,\lambda) = \frac{\omega_{ks}}{\rho_s^{k-1}}(\rho_s R_{ks})^{\nu} \exp(\rho_s R_{ks} x_s)[1]_s$$

$$+\sum_{\mu=k+1}^{n_s} A_{\mu s}(\rho_s)(\rho_s R_{\mu s})^\nu \exp(\rho_s R_{\mu s} x_s)[1]_s,\ \nu=\overline{0,n_s-1},$$

$$\psi_j^{(\nu)}(x_j,\lambda) = \sum_{\mu=\max(n_j-k,1)+1}^{n_j} A_{\mu j}(\rho_j)(\rho_j R_{\mu j})^\nu \exp(\rho_j R_{\mu j} x_j)[1]_j, \quad j=\overline{1,m}\setminus s, \quad \nu=\overline{0,n_s-1},$$

$$\psi_0^{(\nu)}(x_0,\lambda) = \sum_{\mu=1}^{2} A_{\mu 0}(\rho_0)(\rho_0 R_{\mu 0})^\nu \exp(\rho_0 R_{\mu 0} x_0)[1]_0, \quad \nu=0,1.$$

Substitution of these representations into the matching conditions (2.3) gives a linear system D_{sk} with respect to the coefficients $A_{\mu j}(\rho_j)$. Since each $A_{\mu j}(\rho_j)$ in this system is multiplied by the corresponding exponent $\exp(\rho_j R_{\mu j})$ and $[1]_j = 1+o(\lambda)$, $|\lambda|\to\infty$, we obtain the following asymptotics for the determinant of D_{sk}:

$$d_{sk}(\lambda) = d_{sk}^0 \lambda^{\nu_{sk}} \exp(P_{sk}(\lambda))(1+o(\lambda)), \quad |\lambda|\to\infty, \tag{3.2}$$

where

$$P_{sk}(\lambda) = \rho_s\left(\sum_{\mu=k+1}^{n_s} R_{\mu s}\right) + \sum_{\substack{j=\overline{1,m}\setminus s\\ k<n_j}} \rho_j \left(\sum_{\mu=\max(n_j-k,1)+1}^{n_j} R_{\mu j}\right),$$

and ν_{sk} is a rational power of λ. In order to have the main term of the asymptotics (3.2) distinct from zero, we impose the requirement

$$d_{sk}^0 \neq 0, \quad s=\overline{1,m}, \quad k=\overline{1,n_s-1}. \tag{3.3}$$

The matching conditions (2.3), satisfying (3.3), are called *regular*.

One can easily show that the determinant $\Delta_{sk\mu}(\lambda)$ asymptotically equals

$$d_{sk}^0(\rho_s R_{ks})^\mu \lambda^{\nu_{sk}} \exp(P_{sk}(\lambda))$$

up to a nonzero constant (under the current assumptions on λ). Consequently, if the matching conditions are regular, then $\Delta_{sk\mu}(\lambda) \not\equiv 0$. Hence the boundary value problems $L_{sk\mu}$ have only discrete spectra.

Solving the system D_{sk} by the Cramer's rule, we obtain, in particular

$$A_{\mu s}(\rho_s) = O(\rho_s^{1-k} \exp(\rho_s(R_{ks}-R_{\mu s})), \quad \mu=\overline{k+1,n_s},$$

and finally arrive at the following assertion.

Lemma 3. *Fix $s=\overline{1,m}$ and a sector S_s with property (3.1). For $x_s\in(0,T_s)$, $\nu=\overline{0,n_s-1}$, $k=\overline{1,n_s}$, the following asymptotic formula holds*

$$\psi_{sks}^{(\nu)}(x_s,\lambda) = \frac{\omega_{ks}}{\rho_s^{k-1}}(\rho_s R_{ks})^\nu \exp(\rho_s R_{ks} x_s)[1]_s, \quad \rho_s\in S_s, \quad |\rho_s|\to\infty.$$

4. Auxiliary inverse problems

In this section we consider auxiliary inverse problems of recovering the differential operator on each fixed edge. We start from the boundary edges. Fix $s = \overline{1,m}$ and consider the following inverse problems of the edge e_s.

IP(s). *Given the Weyl-type matrix M_s, construct the potential $q_s := \{q_{\mu s}\}_{\mu=0}^{n_s-2}$ on the edge e_s.*

In IP(s) we construct the potential on the single edge e_s, but the Weyl-type matrix M_s brings global information from the whole graph. In other words, IP(s) is not a local inverse problem related only to the edge e_s.

Let us prove the uniqueness theorem for the solution of IP(s). For this purpose together with q we consider a potential $\tilde{q}$. Everywhere below if a symbol α denotes an object related to q then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{q}$.

Theorem 1. *Fix $s = \overline{1,m}$. If $M_s = \tilde{M}_s$, then $q_s = \tilde{q}_s$. Thus, the specification of the Weyl-type matrix M_s uniquely determines the potential q_s on the edge e_s.*

Proof. Denote $\psi_s(x_s,\lambda) := [\psi_{sks}^{(\nu-1)}(x_s,\lambda)]_{k,\nu=1}^{n_s}$, $C_s(x_s,\lambda) := [C_{ks}^{(\nu-1)}(x_s,\lambda)]_{k,\nu=1}^{n_s}$. Then by (2.4) we get

$$\psi_s(x_s,\lambda) = C_s(x_s,\lambda) M_s^T(\lambda), \tag{4.1}$$

where T is the sign for the trasposition. Define the matrix $\mathscr{P}_s(x_s,\lambda) = [\mathscr{P}_{sjk}(x_s,\lambda)]_{j,k=1}^{n_s}$ by the formula

$$\mathscr{P}_s(x_s,\lambda) = \psi_s(x_s,\lambda)(\tilde{\psi}_s(x_s,\lambda))^{-1}.$$

Applying Lemma 3, we get

$$\mathscr{P}_{s1k}(x_s,\lambda) - \delta_{1k} = O(\rho_s^{-1}), \quad k = \overline{1,n_s}, \quad x_s \in (0,1), \quad \arg\lambda \neq 0,\pi, \quad |\lambda| \to \infty. \tag{4.2}$$

Transform the matrix $\mathscr{P}_s(x_s,\lambda)$, using (4.1) and $M_s = \tilde{M}_s$:

$$\mathscr{P}_s(x_s,\lambda) = C_s(x_s,\lambda)(\tilde{C}_s(x_s,\lambda))^{-1}.$$

Taking (2.1) into account, we conclude that for each fixed x_s, the matrix-valued function $\mathscr{P}_s(x_s,\lambda)$ is an entire analytic function in λ of order $1/n_s$. Together with (4.2), this yields $\mathscr{P}_{s11}(x_s,\lambda) \equiv 1$, $\mathscr{P}_{s1k}(x_s,\lambda) \equiv 0$, $k = \overline{2,n_s}$. Consequently, $\psi_{sks}(x_s,\lambda) \equiv \tilde{\psi}_{sks}(x_s,\lambda)$ and $q_s = \tilde{q}_s$. □

Using the method of spectral mappings, one can get a constructive procedure for the solution of IP(s). It can be obtained by the same arguments as for n-th order differential operators on a finite interval (see [15, Ch. 2] for detais).

For the Sturm-Liouville operator on the cycle e_0, we consider the following auxiliary inverse problem.

IP(0). *Given* $d(\lambda)$, $h(\lambda)$ *and* Ω, *construct* $q_{00}(x_0)$.

This inverse problem was studied in [18, 17] and other papers. In fact, one can easily construct Dirichlet spectral data $\{\nu_n, \alpha_n\}_{n\ge1}$ by the data $\{d(\lambda), h(\lambda), \Omega\}$, and reduce IP(0) to the classical Sturm-Liouville problem [12, 19, 20, 13]. Thus, IP(0) has a unique solution which can be found by the following algorithm.

Algorithm 1. ([16]) Given $d(\lambda)$, $h(\lambda)$ and Ω.

1. Find the zeros of $h(\lambda)$, $\{\nu_n\}_{n\ge1}$.
2. Calculate $H(\nu_n) := \omega_n\sqrt{d^2(\nu_n)-4}$.
3. Find $S_0'(1,\nu_n) := (d(\nu_n) - H(\nu_n))/2$.
4. Calculate $\alpha_n := \dot{h}(\nu_n)S_0'(T_0,\nu_n)$, $\dot{h}(\lambda) := \dfrac{dh(\lambda)}{d\lambda}$.
5. Construct q_{00} from the given spectral data $\{\nu_n, \alpha_n\}_{n\ge1}$ by solving the classical Sturm-Liouville problem.

5. Solution of Inverse Problem 1

Now we are ready to present a constructive procedure for the solution of Inverse Problem 1.

Algorithm 2. Given the spectra $\Lambda_{sk\mu}$, $s=\overline{1,m}$, $k=\overline{1,n_s-1}$, $\mu=\overline{k,n_s}$, and the signs Ω.

1. Construct the characteristic functions $\Delta_{sk\mu}(\lambda)$ by their zeros $\Lambda_{sk\mu}$.
2. Find the Weyl-type matrices $M_s(\lambda)$, $s=\overline{1,m}$, via (2.5).
3. For each $s=\overline{1,m}$, solve the inverse problem IP(s) and find the potential q_s on the edge e_s.
4. Construct the solutions $C_{ks}(x_s,\lambda)$, $s=\overline{1,m}$, $k=\overline{1,n_s}$.
5. Find $h(\lambda)$, $d(\lambda)$ from (2.6), (2.7).
6. Solve IP(0) by $d(\lambda)$, $h(\lambda)$ and Ω, using Algorithm 1, and and construct the potential on the cycle e_0.

On step 5, we assume that there exist at least one edge with the order $n_s > 2$. The case of all $n_s = 2$ was considered in [16]. Then $h(\lambda)$ can be easily determined from (2.7), and then $d(\lambda)$ from (2.6).

Remark. Note that there are often considered inverse problems by the Weyl functions and their generalizations. But in the present case, the functions $d(\lambda)-2$ and $h(\lambda)$ can not be uniquely recovered from the Weyl matrices M_s, if these functions have common zeros.

Theorem 1 together with the uniqueness of the solution for IP(0) yields the following result.

Theorem 2. *The spectra $\Lambda_{sk\mu}$, $k=\overline{1,n_s-1}$, $\mu=\overline{k,n_s}$, and the signs Ω determine the potential q on the graph G uniquely.*

6. Solution of Inverse Problem 2

Consider Inverse Problem 2 by the system of spectra $\{\Lambda^\alpha_{sk\mu}\}$. Introduce the Weyl-type solutions $\Psi^\alpha_{sk}=\{\psi^\alpha_{skj}\}_{j=\overline{1,m}}$, $s=\overline{1,m}$, $k=\overline{1,n_s-1}$, satisfying the boundary conditions (1.2) and the matching conditions $\mathrm{Cont}(0,\alpha_s)$, $\mathrm{Cont}(v)$, $v=\overline{1,k-1}$, $\mathrm{Kirch}(v)$, $v=\overline{k,n_s-1}$. Define the Weyl-type matrices $M^\alpha_s(\lambda)=[M^\alpha_{sk\mu}(\lambda)]_{k,m=1}^{n_s}$, $s=\overline{1,m}$, similarly to $M_s(\lambda)$. One can investigate the asymptotic behavoir of the Weyl-type solutions and the regularity of the boundary conditions analogously to Inverse Problem 1. In can be shown, that the characteristic functions of the boundary value problems $L^\alpha_{sk\mu}$ have the form

$$\begin{cases} \Delta^\alpha_{sk\mu}(\lambda)=d_{\alpha_s}(\lambda)F_{s\mu}(\lambda)+h(\lambda)G_{s\mu}(\lambda), & k=1, \\ \Delta^\alpha_{sk\mu}(\lambda)=h(\lambda)G_{sk\mu}(\lambda), & k>1, \end{cases} \tag{6.1}$$

where $h(\lambda)=S_0(1,\lambda)$, $d_{\alpha_s}(\lambda)=C_0(1,\lambda)+\alpha_s S_0'(1,\lambda)-\alpha_s-1$, and the functions $F_{s\mu}(\lambda)$, $G_{s\mu}(\lambda)$, $G_{sk\mu}(\lambda)$ depends only on the potentials $q_{\mu s}$ on the boundary edges e_s, $s=\overline{1,m}$ (see the example in Appendix). Having the functions d_{α_s} for at least two different values α_s, one can easily find $S_0'(1,\lambda)$. It remains to solve the classical inverse problem by two spectra (zeros of the characteristic functions $S_0(1,\lambda)$ and $S_0'(1,\lambda)$), and find the potential q_{00} on the cycle e_0. Finally, we arrive at the following algorithm for the solution of Inverse Problem 2.

Algorithm 3. Given the spectra $\Lambda^\alpha_{sk\mu}$, $s=\overline{1,m}$, $k=\overline{1,n_s-1}$, $\mu=\overline{k,n_s}$.

1. Construct the characteristic functions $\Delta^\alpha_{sk\mu}(\lambda)$ by their zeros $\Lambda^\alpha_{sk\mu}$.
2. Find the Weyl-type matrices: $M^\alpha_{sk\mu}(\lambda)=-\dfrac{\Delta^\alpha_{sk\mu}(\lambda)}{\Delta^\alpha_{skk}(\lambda)}$.
3. For each $s=\overline{1,m}$, solve the inverse problem IP(s) by M^α_s and find the potential q_s on the edge e_s.
4. Construct the solutions $C_{ks}(x_s,\lambda)$, $s=\overline{1,m}$, $k=\overline{1,n_s}$.
5. Find $h(\lambda)$, $d_{\alpha_s}(\lambda)$ from (6.1).

6. Find $S_0'(1,\lambda)$ from two different $d_{\alpha_s}(\lambda)$, find the sequences Λ_0 and Λ_1 of zeros of the functions $S_0(1,\lambda)$ and $S_0'(1,\lambda)$, respectively.
7. Solve the inverse problem by two spectra Λ_0 and Λ_1 (see [12, 13]) on the cycle e_0.

Remark 1. If $m = 1$ and $n_1 > 2$, the similar results can be obtained for the boundary value problems for the system (1.1) under conditions (1.2), (2.3) with $\mathrm{Cont}(0,\alpha_k)$ instead $\mathrm{Cont}(0)$, where the collection $\{\alpha_k\}_{k=1}^{n_1}$ contains at least two distinct values.

Acknowledgement

This research was supported by Grants 13-01-00134 and 15-01-04864 of Russian Foundation for Basic Research and by the Ministry of Education and Science of the Russian Federation (Grant 1.1436.2014K).

Appendix. Example

In this section, we consider an example, that illustrates how the entries of the Weyl-type matrices can be found from the linear system E_{sk} (see Section 2) and shows the structure of determinants. We also check the regularity of the matching conditions for the example.

Let $s = \overline{1,m}$ and $k = \overline{1,n_s-1}$ be fixed. For brevity, in this section we omit the indices s and k, when they are fixed. So we write $\psi_j(x_j,\lambda)$ instead of $\psi_{skj}(x_j,\lambda)$. We substitute the following expansions

$$\psi_j(x_j,\lambda) = \sum_{\mu=1}^{n_j} M_j^{\mu}(\lambda) C_{\mu j}(x_j,\lambda), \quad j = \overline{1,m}.$$

into the mathcing conditions (2.3) and obtain the coefficients $M_k^{\mu}(\lambda)$ from a linear system by the Cramer's rule: $M_k^{\mu}(\lambda) = -\dfrac{\Delta_\mu(\lambda)}{\Delta_0(\lambda)}$.

Let $m = 2$, $n_1 = 3$, $n_2 = 4$.

Fix $s = 1$, $k = 1$. Then the boundary conditions (2.2) take the form:

$$\psi_1(0,\lambda) = 1, \quad \psi_2(0,\lambda) = \psi_2'(0,\lambda) = \psi_2''(0,\lambda) = 0.$$

Consequently,

$$\psi_0(x_0,\lambda) = M_0^1(\lambda) C_0(x_0,\lambda) + M_0^2(\lambda) S_0(x_0,\lambda),$$
$$\psi_1(x_1,\lambda) = C_{11}(x_1,\lambda) + M_1^2(\lambda) C_{21}(x_1,\lambda) + M_1^3(\lambda) C_{31}(x_1,\lambda),$$
$$\psi_2(x,\lambda) = M_2^4(\lambda) C_{42}(x_2,\lambda).$$

Let $U_{j\nu}(\psi_j) = \psi_j^{(\nu)}(1,\lambda)$. Then we have the following matching conditions:

$$\begin{aligned} \mathrm{Cont}(0):&\ \psi_0(0,\lambda) = \psi_0(1,\lambda) = \psi_1(1,\lambda) = \psi_2(1,\lambda) = 0,\\ \mathrm{Kirch}(1):&\ \psi_0'(1,\lambda) + \psi_1'(1,\lambda) + \psi_2'(1,\lambda) = \psi_0'(0,\lambda),\\ \mathrm{Kirch}(2):&\ \psi_1''(1,\lambda) + \psi_2''(1,\lambda) = 0, \end{aligned}$$

which give the following system (we omit the arguments $(1,\lambda)$ of $C_{\mu j}$):

$$\begin{bmatrix} -1+C_0 & S_0 & 0 & 0 & 0\\ -1 & 0 & C_{21} & C_{31} & 0\\ C_0' & -1+S_0' & C_{21}' & C_{31}' & C_{42}'\\ 0 & 0 & C_{21}'' & C_{31}'' & C_{42}''\\ -1 & 0 & 0 & 0 & C_{42} \end{bmatrix} \begin{bmatrix} M_0^1\\ M_0^2\\ M_1^2\\ M_1^3\\ M_2^4 \end{bmatrix} + \begin{bmatrix} 0\\ C_{11}\\ C_{11}'\\ C_{11}''\\ 0 \end{bmatrix} = 0.$$

Note that only the first two columns of the matrix depends on C_0 and S_0, i.e. on the potential $q_{00}(x)$ on the cycle e_0. The first two columns contain only two nonzero minors:

$$\begin{vmatrix} -1+C_0 & S_0\\ -1 & 0 \end{vmatrix} = S_0, \quad \begin{vmatrix} -1+C_0 & S_0\\ C_0' & -1+S_0' \end{vmatrix} = 2 - C_0 - S_0' = 2 - d(\lambda),$$

since $C_0S_0' - C_0'S_0 = 1$. So the determinant of the system equals

$$\Delta_0(\lambda) = -\big(d(\lambda)-2\big)\cdot \begin{vmatrix} C_{21} & C_{31}\\ C_{21}'' & C_{31}'' \end{vmatrix} \cdot C_{42} + h(\lambda)\cdot \left(\begin{vmatrix} C_{21}' & C_{31}'\\ C_{21}'' & C_{31}'' \end{vmatrix} C_{42} - \begin{vmatrix} C_{21} & C_{31} & 0\\ C_{21}' & C_{31}' & C_{42}'\\ C_{21}'' & C_{31}'' & C_{42}'' \end{vmatrix} \right)$$

and $M_1^k(\lambda) = -\dfrac{\Delta_k(\lambda)}{\Delta_0(\lambda)}$, $k = 2,3$, where $\Delta_k(\lambda)$ can be obtained from $\Delta_0(\lambda)$ by change of C_{k1} to C_{11}.

For $s = 1$, $k = 2$ we have

$$\begin{aligned} &\psi_1(0,\lambda) = 0, \quad \psi_1'(0,\lambda) = 1, \quad \psi_2(0,\lambda) = \psi_2'(0,\lambda) = 0,\\ &\psi_1(x_1,\lambda) = C_{21}(x_1,\lambda) + M_1^3(\lambda)C_{31}(x_1,\lambda),\\ &\psi_2(x_2,\lambda) = M_2^3(\lambda)C_{32}(x_2,\lambda) + M_2^4(\lambda)C_{42}(x_2,\lambda). \end{aligned}$$

$$\begin{aligned} \mathrm{Cont}(0):&\ \psi_0(0,\lambda) = \psi_0(1,\lambda) = \psi_1(1,\lambda) = \psi_2(1,\lambda) = 0,\\ \mathrm{Cont}(1):&\ \psi_1'(1,\lambda) = \psi_2'(1,\lambda) = 0,\\ \mathrm{Kirch}(2):&\ \psi_1''(1,\lambda) + \psi_2''(1,\lambda) = 0. \end{aligned}$$

The determinant of the system is

$$\begin{vmatrix} -1+C_0 & S_0 & 0 & 0 & 0 \\ -1 & 0 & C_{31} & 0 & 0 \\ 0 & 0 & -C'_{31} & C'_{32} & C'_{42} \\ 0 & 0 & C''_{31} & C''_{32} & C''_{42} \\ -1 & 0 & 0 & C_{32} & C_{42} \end{vmatrix} = h(\lambda)\left(\begin{vmatrix} -C'_{31} & C'_{32} & C'_{42} \\ C''_{31} & C''_{32} & C''_{42} \\ 0 & C_{32} & C_{42} \end{vmatrix} - C_{31}\begin{vmatrix} C'_{32} & C'_{42} \\ C''_{32} & C''_{42} \end{vmatrix}\right).$$

Thus, formulas (2.6), (2.7) are valid in this case. For the general graph, the determinants have the similar structure.

Now let us take the condition $\mathrm{Cont}(0,\alpha)$ instead of $\mathrm{Cont}(0)$. Then we have $-\alpha$ instead of -1 in the first columns of the determinants. The dependence on the potential q_{00} is contained in the following functions

$$\begin{vmatrix} -\alpha+C_0 & S_0 \\ -\alpha & 0 \end{vmatrix} = \alpha S_0, \quad \begin{vmatrix} -\alpha+C_0 & S_0 \\ C'_0 & -1+S'_0 \end{vmatrix} = 1+\alpha-C_0-\alpha S'_0 = -d_\alpha(\lambda),$$

so the determinants have the form (6.1).

Furthermore, let us show, how to check the regularity of the matching conditions for our example (with $\alpha = 1$). Let $s = 1$, $k = 1$. Denote $\rho = \sqrt{\lambda}$, $\sigma = \sqrt[3]{\lambda}$, $\theta = \sqrt[4]{\lambda}$. Let R_j, $j = \overline{1,3}$ and r_j, $j = \overline{1,4}$ be the cube roots and the fourth roots from 1, respectively. Suppose they are numbered according to (3.1) in some sector. Lemma 2 gives

$$\begin{aligned} \psi_0(x_0,\lambda) &= A_{10}(\rho)\exp(-\rho x_0)[1] + A_{20}(\rho)\exp(\rho x_0)[1], \\ \psi_1(x_0,\lambda) &= \frac{1}{R_1}\exp(\sigma R_1 x_1)[1] + A_{21}(\sigma)\exp(\sigma R_2 x_1)[1] + A_{31}(\sigma)\exp(\sigma R_3 x_1)[1], \\ \psi_2(x_0,\lambda) &= A_{42}(\theta)\exp(\theta r_4 x_2)[1], \end{aligned}$$

where $[1] = 1 + o(\lambda)$, $|\lambda| \to \infty$ on some fixed ray.

Substitute these asymptotic formulas into the matching conditions. $\mathrm{Cont}(0)$ gives

$$A_{10}(\rho)[1] + A_{20}(\rho)[1] = A_{10}(\rho)\exp(-\rho) + A_{20}(\rho)\exp(\rho).$$

Suppose that on the considered ray $\mathrm{Re}(\rho) > 0$. Then $A_{10}(\rho) = A_{20}(\rho)\exp(\rho)[1]$, so we have already excluded one variable. Further,

$$\begin{aligned} \mathrm{Cont}(0): \quad & A_{10}(\rho)[1] = \frac{1}{R_1}\exp(\sigma R_1)[1] + A_{21}(\sigma)\exp(\sigma R_2)[1] + A_{31}(\sigma)\exp(\sigma R_3)[1], \\ & A_{10}(\rho)[1] = A_{42}(\theta)\exp(\theta r_4)[1], \\ \mathrm{Kirch}(1): \quad & 2\rho A_{10}(\rho)[1] + \frac{1}{R_1}\sigma R_1\exp(\sigma R_1)[1] + A_{21}(\sigma)\sigma R_2\exp(\sigma R_2)[1] \end{aligned}$$

$$+ A_{31}(\sigma)\sigma R_3 \exp(\sigma R_3)[1] + A_{42}(\theta)\theta r_4 \exp(\theta r_4) = 0,$$

$$\text{Kirch}(2): \quad \frac{1}{R_1}(\sigma R_1)^2 \exp(\sigma R_1)[1] + A_{21}(\sigma)(\sigma R_2)^2 \exp(\sigma R_2)[1]$$

$$+ A_{31}(\sigma)(\sigma R_3)^2 \exp(\sigma R_3)[1] + A_{42}(\theta)(\theta r_4)^2 \exp(\theta r_4)[1] = 0.$$

We have got a linear system with respect to the variables $A_{10}(\rho)$, $A_{21}(\sigma)$, $A_{31}(\sigma)$, $A_{42}(\theta)$, with the determinant

$$d_{sk}(\lambda) = \begin{vmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 2\rho & \sigma R_2 & \sigma R_3 & \theta r_4 \\ 0 & (\sigma R_2)^2 & (\sigma R_3)^2 & (\theta r_4)^2 \end{vmatrix} \exp(\sigma(R_2 + R_3) + \theta r_4)[1] = 2\rho\sigma^2(R_2^2 - R_3^2)[1] \neq 0.$$

We have used that $\theta = o(\sigma)$, $\sigma = o(\rho)$, as $|\lambda| \to \infty$. Thus, $d_{sk}(\lambda) \neq 0$ for sufficiently large $|\lambda|$. So the relation (3.3) is proved for $s = 1$, $k = 1$.

References

[1] J. Langese, G. Leugering and J. Schmidt, Modelling, analysis and control of dynamic elastic multi-link structures, Birkhäuser, Boston (1994).

[2] T. Kottos and U. Smilansky, *Quantum chaos on graphs*, Phys. Rev. Lett., **79** (1997), 4794–4797.

[3] P. Kuchment, *Quantum graphs. Some basic structures*, Waves Random Media, **14** (2004), S107–S128.

[4] M. I. Belishev, *Boundary spectral inverse problem on a class of graphs (trees) by the BC method*, Inverse Problems, **20** (2004), 647–672.

[5] Y. Pokornyi and A. Borovskikh, *Differential equations on networks* (*geometric graphs*), J. Math. Sci. (N.Y.) **119**(2004), 691–718.

[6] Analysis on Graphs and Its Applications, edited by P. Exner, J.P. Keating, P. Kuchment, T. Sunada and Teplyaev, A. Proceedings of Symposia in Pure Mathematics, AMS, 77 (2008).

[7] V. A. Yurko, *Inverse spectral problems for differential operators on arbitrary compact graphs*, J. Inverse and Ill-Posed Probl., **18**(2010), 245–261.

[8] V. A. Yurko, *Inverse problems for differential of any order on trees*, Matemat. Zametki, **83**(2008), 139–152; English transl. in Math. Notes **83**(2008), 125–137.

[9] V. A. Yurko, *Inverse problems on star-type graphs: differential operators of different orders on different edges*, Central European J. Math., **12**(2014), 483–499.

[10] Y. V. Pokornyi, T. V. Beloglazova and K. P. Lazarev, *On a Class of Ordinary Differential Equations of Various Orders on a Graph*, Mat. Zametki, **73**(2003), 469–470.

[11] Y. V. Pokornyi, T. V. Beloglazova, E. V. Dikareva and T. V. Perlovskaya, *Green function for a locally interacting system of ordinary equations of different orders*, Matem. Zametki, **74**(2003), 146–149; English transl. in Mathem. Notes **74**(2003), 141–143.

[12] B. M. Levitan, Inverse Sturm-Liouville Problems, Nauka, Moscow (1984) (Russian); English transl., VNU Sci. Press, Utrecht (1987).

[13] G. Freiling and V. Yurko, Inverse Sturm-Liouville problems and their applications, Huntington, NY: Nova Science Publishers, 305 p. (2001).

[14] V. A. Yurko, *Inverse problems for non-selfadjoint quasi-periodic differential pencils*, Analysis and Math. Physics **2**(2012), 215–230.

[15] V. A. Yurko, Method of Spectral Mappings in the Inverse Problem Theory. Inverse and Ill-Posed Problems Series, Utrecht: VSP (2002).

[16] V. A. Yurko, *Inverse problems for Sturm-Liouville operators on graphs with a cycle*, Operators and Matrices, **2**(2008), 543–553.

[17] I. V. Stankevich, *An inverse problem of spectral analysis for Hill's equations*, Doklady Akad. Nauk SSSR, **192**(1970), 34–37 (Russian).

[18] V. A. Marchenko and I. V. Ostrovskii, *A characterization of the spectrum of the Hill operator*, Mat. Sbornik, **97** (1975), 540–606 (Russian); English transl. in Math. USSR Sbornik **26**(1975), 493–554.

[19] V. A. Marchenko, Sturm-Liouville Operators and their Applications, Naukova Dumka, Kiev (1977) (Russian); English transl., Birkhauser (1986).

[20] J. Pöschel and E. Trubowitz, Inverse Spectral Theory, New York, Academic Press, 1987.

Department of Mechanics and Mathematics, Saratov State University, Astrakhanskaya 83, Saratov 410012, Russia.

E-mail: bondarenkonp@info.sgu.ru