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FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS SATISFYING AN IMPLICIT RELATION

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Abstract. In this paper, we give a common fixed point theorem for multivalued mappings with Hausdorff metric.

1. Introduction

Throughout this paper, X stands for a metric space with the metric d whereas $CB(X)$ denotes the family of all nonempty closed bounded subsets of X. Let

$$H(A,B)=\max\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(y,A)\},$$

where $A,B\in CB(X)$ and $d(x,A)=\inf\{d(x,y):y\in A\}$. The function H is a metric on $CB(X)$ and is called Hausdorff metric. It is well known that if X is a complete metric space, then so is the metric space $(CB(X),H)$. Let $A,B\in CB(X)$ and $k>1$. In the sequel the following well known fact will be used [4]: for each $a\in A$, there is $b\in B$ such that $d(a,b)\le kH(A,B)$.

Kaneko and Sessa extend the definition of compatibility to include multivalued mappings in the following way.

Definition 1. ([3]) The mappings $f:X\to X$ and $S:X\to CB(X)$ are compatible if $fSx\in CB(X)$ for all $x\in X$ and

$$\lim_{n\to\infty}H(Sfx_n,fSx_n)=0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Sx_n=A\in CB(X)$ and $\lim_{n\to\infty}fx_n=t\in A$.

Now, we consider the following conditions.

Condition 1. The mappings $f:X\to X$ and $S:X\to CB(X)$ are said to be satisfy Condition 1 iff $fSx\in CB(X)$ for all $x\in X$ and

$$\lim_{n\to\infty}H(Sfx_n,fSx_n)\le\lim_{n\to\infty}H(Sfx_n,Sx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Sx_n=A\in CB(X)$ and $\lim_{n\to\infty}fx_n=t\in A$.

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Condition 2. The mappings $f : X \to X$ and $S : X \to CB(X)$ are said to be satisfy Condition 2 iff for all $x \in X$ and

$$\lim_{n\to\infty} d(ffx_n, fx_n) \le \lim_{n\to\infty} H(Sfx_n, Sx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = A \in CB(X)$ and $\lim_{n\to\infty} fx_n = t \in A$.

Remark 1. Note that, two compatible maps f and S are satisfy Condition 1.

The following example shows that f and S are satisfy Condition 1 and Condition 2 but they are not compatible.

Example 1. Let $X = [0,\infty)$ be endowed with the Euclidean metric d. Let $fx = \dfrac{x^2+2}{2}$ and $Sx = [x^2+1, x^2+2]$ for each $x \ge 0$. It is clear that f and S are continuous. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} fx_n = t, \lim_{n\to\infty} Sx_n = A \in CB(X),$$

then $t \in A$ if and only if $t = 1$. Indeed, if $fx_n \to t$, then $t \ge 1$ since $fx = \dfrac{x^2+2}{2} \ge 1$. Again, if $fx_n \to t$, then $x_n^2 \to 2t-2$ and so $Sx_n \to A = [2t-1, 2t]$.

Now, if $t \in A$, then $t \in [2t-1, 2t]$, that is, $2t-1 \le t$ and so $t \le 1$. Now, since $t \ge 1$ and $t \le 1$, then $t = 1$. On the contrary, if $t = 1$, then $x_n \to 0$ since $fx_n \to t = 1$. Thus $Sx_n \to [1,2] = A$ and so $t \in A$. Therefore, we have

$$0 \ne \lim_{n\to\infty} H(Sfx_n, fSx_n) = \frac{1}{2} \le 1 = \lim_{n\to\infty} H(Sfx_n, Sx_n).$$

Thus f and S are satisfy Condition 1 but they are not compatible. On the other hand

$$\lim_{n\to\infty} d(ffx_n, fx_n) = \frac{1}{2} \le 1 = \lim_{n\to\infty} H(Sfx_n, Sx_n).$$

Thus f and S are satisfy Condition 2.

2. Implicit relation

Implicit relations on metric spaces have been used in many articles (see [2] [5] [6] [8]).

Let $\mathscr{F}$ be the set of all continuous functions $F : R_+^6 \to R_+$ satisfying the following conditions:

$F_1 : F(t_1, \ldots, t_6)$ is decreasing in $t_2, \ldots, t_6$.

F_2 : there exist an increasing function $f : R_+ \to R_+, f(0) = 0$ and $k > 1$ with $f(s) < \frac{s}{k}$ such that the inequality

$(F_a) : u \le kt$ and $F(t, v, v, u, u+v, 0) \le 0$ or

$(F_b) : u \le kt$ and $F(t, v, u, v, 0, u+v) \le 0$ implies $t \le f(v)$.

$F_3 : F(u, u, u, 0, u, u) > 0$ and $F(u, u, 0, u, u, u) > 0, \forall u > 0$.

Remark 2. Note that, if we choose $f(s) = hs$ with $hk < 1$ in F_2, we obtain implicit relation of Popa [6]

Example 2. $F(t_1,\dots,t_6) = t_1 - m\max\{t_2, t_3, t_4, \frac{1}{2}(t_5+t_6)\}$, where $m \in (0,1)$.

F_1 : Obviously.

F_2 : Let $u > 0$, $u \le kt$ and $F(t, v, v, u, u+v, 0) = t - m\max\{u, v\} \le 0$, where $1 < k < \frac{1}{m}$. If $u \ge v$ then $u \le kt \le kmu < u$, a contradiction. Thus $u > v$ and $t \le mv$. Similarly, $u > 0$, $u \le kt$ and $F(t, v, u, v, 0, u+v) \le 0$ imply $t \le mv$. If $u = 0$, then $u \le v$ and $t \le mv$. Thus F_2 is satisfying with $f(s) = ms$.

$F_3 : F(u, u, u, 0, u, u) = F(u, u, 0, u, u, u) = u - mu > 0$, $\forall u > 0$.

Example 3. $F(t_1,\dots,t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5+t_6)\})$, where $\phi : R_+ \to R_+$ increasing, $\phi(0) = 0$ and $k > 1$ with $\phi(s) < \frac{s}{k}$ for $s > 0$.

F_1 : Obviously.

F_2 : Let $u > 0, u \le kt$ and $F(t, v, v, u, u+v, 0) = t - \phi(\max\{u, v\}) \le 0$. If $u \ge v$, then $u \le kt \le k\phi(u) < u$, a contradiction. Thus $u < v$ and $t \le \phi(v)$. Similarly $u > 0$, $u \le kt$ and $F(t, v, u, v, 0, u+v) \le 0$ imply $t \le \phi(v)$. If $u = 0$, then $u \le \phi(v)$. Thus F_2 is satisfying with $f = \phi$.

$F_3 : F(u, u, u, 0, u, u) = F(u, u, 0, u, u, u) = u - \phi(u) > 0$, $\forall u > 0$.

3. Main Result

We need the following lemma for the proof of our main theorem.

Lemma 1. [(7)] *For any $t > 0, \gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the composition of γ $n-$times with itself.*

Now we give our main theorem.

Theorem 1. *Let (X,d) be a complete metric space. Let $f, g : X \to X$ and $S, T : X \to CB(X)$ be a continuous mapping such that f and S as well as g and T satisfy Condition* 1 *and Condition* 2*. Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and for all $x, y \in X$*

$$F(H(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) \le 0, \tag{3.1}$$

where $F \in \mathscr{F}$. Then f, g, S and T have a common fixed point.

Proof. Let x_0 be arbitrary point in X. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ of elements in X and sequence $\{A_n\}$ of elements in $CB(X)$. Since $S(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $y_1 = gx_1 \in Sx_0$. Then there exists an element $y_2 = fx_2 \in Tx_1 = A_1$, because $T(X) \subseteq f(X)$, such that

$$d(y_1, y_2) = d(gx_1, fx_2) \le kH(Sx_0, Tx_1).$$

Since $S(X) \subseteq g(X)$, we may choose $x_3 \in X$ such that $y_3 = gx_3 \in Sx_2 = A_2$ and

$$d(y_2, y_3) \le kH(Tx_1, Sx_2).$$

By induction we produce the sequences $\{x_n\}$, $\{y_n\}$ and $\{A_n\}$ such that

$$y_{2n+1} = gx_{2n+1} \in Sx_{2n} = A_{2n}, \tag{3.2}$$

$$y_{2n+2} = fx_{2n+2} \in Tx_{2n+1} = A_{2n+1}, \tag{3.3}$$

$$d(y_{2n+1}, y_{2n}) \leq kH(Sx_{2n}, Tx_{2n-1}), \tag{3.4}$$

$$d(y_{2n+1}, y_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1}) \tag{3.5}$$

for every $n \in N$. Letting $x = x_{2n}, y = x_{2n+1}$ in (3.1), we have successively

$$F(H(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1}), d(fx_{2n}, Sx_{2n}), \\ d(gx_{2n+1}, Tx_{2n+1}), d(fx_{2n}, Tx_{2n+1}), d(gx_{2n+1}, Sx_{2n})) \leq 0,$$

and so

$$F(H(Sx_{2n}, Tx_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), d(y_{2n+1}, y_{2n+1})) \leq 0.$$

Thus

$$F(H(Sx_{2n}, Tx_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), 0) \leq 0. \tag{3.6}$$

From (3.5), (3.6) and (F_a), there exist an increasing function $f : R_+ \to R_+$, $f(0) = 0$ and $k > 1$ with $f(s) < \frac{s}{k}$, we have

$$H(Sx_{2n}, Tx_{2n+1}) = H(A_{2n}, A_{2n+1}) \leq f(d(y_{2n}, y_{2n+1})) \tag{3.7}$$

and so

$$d(y_{2n+1}, y_{2n+2}) \leq kf(d(y_{2n}, y_{2n+1})). \tag{3.8}$$

Similarly we obtain

$$H(Sx_{2n}, Tx_{2n-1}) = H(A_{2n}, A_{2n-1}) \leq f(d(y_{2n-1}, y_{2n})) \tag{3.9}$$

and so

$$d(y_{2n}, y_{2n+1}) \leq kf(d(y_{2n-1}, y_{2n})). \tag{3.10}$$

Since $kf(s) < s$ it follows from (3.8), (3.10) and Lemma 1 that $\{y_n\}$ is a Cauchy sequence. Hence there exists $z \in X$ such that $y_n \to z$. Therefore, $gx_{2n+1} \to z$ and $fx_{2n} \to z$. Also from (3.7) and (3.9) and the fact that $\{y_n\}$ is Cauchy sequence it follows that $\{A_k\}$ is Cauchy sequence in the complete metric space $(CB(X), H)$. Thus $A_k \to A \in CB(X)$. This implies $Tx_{2n+1} \to A$ and $Sx_{2n} \to A$ and therefore $z \in A$, because

$$d(z, A) = \lim_{n\to\infty} d(y_n, A) \leq \lim_{n\to\infty} H(A_{n-1}, A_n) = 0.$$

Since A is closed, $t \in A$ and f and S are satisfying Condition 1 and Condition 2 implies that

$$\lim_{n\to\infty} H(fSx_{2n}, Sfx_{2n}) \leq \lim_{n\to\infty} H(Sx_{2n}, Sfx_{2n})$$

and

$$\lim_{n\to\infty} d(ffx_{2n}, fx_{2n}) \le \lim_{n\to\infty} H(Sx_{2n}, Sfx_{2n}).$$

This along with the continuity of f and S imply that

$$H(fA, Sz) \le H(A, Sz) \tag{3.11}$$

and

$$d(fz, z) \le H(A, Sz). \tag{3.12}$$

Now

$$\begin{aligned} d(fz, Sz) &\le d(fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz) \\ &\le d(fz, fgx_{2n+1}) + H(fSx_{2n}, Sz) \\ &\le d(fz, fgx_{2n+1}) + H(fSx_{2n}, Sfx_{2n}) + H(Sfx_{2n}, Sz) \end{aligned}$$

and letting $n \to \infty$ we have

$$d(fz, Sz) \le \lim_{n\to\infty} H(fSx_{2n}, Sfx_{2n}) \le H(A, Sz).$$

Now using (3.1) we have

$$\begin{aligned} &F(H(Sz, Tx_{2n+1}), d(fz, gx_{2n+1}), d(fz, Sz), \\ &d(gx_{2n+1}, Tx_{2n+1}), d(fz, Tx_{2n+1}), d(gx_{2n+1}, Sz)) \le 0, \end{aligned}$$

and

$$\begin{aligned} &F(H(Sz, Tx_{2n+1}), d(fz, gx_{2n+1}), d(fz, Sz), \\ &d(gx_{2n+1}, Tx_{2n+1}), d(fz, fx_{2n+2}) + d(fx_{2n+2}, Tx_{2n+1}), d(gx_{2n+1}, Sz)) \le 0. \end{aligned}$$

Letting $n \to \infty$ we obtain

$$F(H(Sz, A), d(fz, z), d(fz, Sz), d(z, A), d(fz, z) + d(z, A), d(z, Sz)) \le 0$$

and so

$$F(H(Sz, A), H(Sz, A), H(Sz, A), 0, H(Sz, A), H(Sz, A)) \le 0$$

which is a contradiction to F_3. Thus $H(Sz, A) = 0$ and so, from (3.11) and (3.12) we have $z = fz \in Sz$. Similarly we have $z = gz \in Tz$. Thus z is a common fixed point of this four mappings.

Corollary 1. *Let (X, d) be a complete metric space. Let $f, g : X \to X$ and $S, T : X \to CB(X)$ be a continuous mapping such that f and S as well as g and T satisfy Condition 1 and Condition 2. Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and that $H(Sx, Ty) \le r d(fx, gy)$ for all $x, y \in X$ where $0 < r < 1$. Then f, g, S and T have a common fixed point.*

Example 4. Let $X = [0,1]$ be endowed with the Euclidean metric d. Let $fx = \frac{x}{2}$, $gx = \frac{x^2}{3}$, $Sx = [0, \frac{x}{4}]$ and $Tx = [0, \frac{x^2}{6}]$. It is clear that these mappings are continuous and $S(X) = [0, \frac{1}{4}] \subset$

$[0, \frac{1}{2}] = f(X)$ and $T(X) = [0, \frac{1}{6}] \subset [0, \frac{1}{3}] = g(X)$. Also f and S as well as g and T are satisfy Condition 1 and Condition 2. Indeed, let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} fx_n = t, \lim_{n\to\infty} Sx_n = A \in CB(X),$$

then $t \in A$ if and only if $t = 0$. Therefore,

$$\lim_{n\to\infty} H(Sfx_n, fSx_n) = \lim_{n\to\infty} d(ffx_n, fx_n) = 0 = \lim_{n\to\infty} H(Sfx_n, Sx_n).$$

Thus f and S are satisfy Condition 1 and Condition 2. Again, for all $x, y \in X$, we have

$$\begin{aligned} H(Sx, Ty) &= H([0, \frac{x}{4}], [0, \frac{y^2}{6}]) \\ &= \left|\frac{x}{4} - \frac{y^2}{6}\right| \\ &= \frac{1}{2}\left|\frac{x}{2} - \frac{y^2}{3}\right| \\ &= \frac{1}{2} d(fx, gy). \end{aligned}$$

Consequently, these mappings are satisfy all conditions of Corollary 1, then they have a common fixed point in X.

Theorem 2. [(1)] *Let (X, d) be a complete metric space. Let $f, g : X \to X$ and $S, T : X \to CB(X)$ be a continuous mapping such that f is compatible with S and g is compatible with T. Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and that $H(Sx, Ty) \le r d(fx, gy)$for all $x, y \in X$ where $0 < r < 1$. Then there is a coincidence point for f and S, as g and T.*

Remark 3. In Theorem 2, Azam and Beg [1] find a coincidence point of mappings assumed compatibility. In Corollary 1, we find a common fixed point of mappings with Condition 1 and Condition 2 replaced by compatibility.

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References

[1] A. Azam and I. Beg, *Coincidence points of compatible multivalued mappings*, Demonstratio Math. **29**(1996), 17–22.

[2] M. Imdad, S. Kumar and M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Rad. Math. **11**(2002), 135–143.

[3] H. Kaneko and S. Sessa, *Fixed point theorems for compatible multivalued and single-valued mappings*, Internat. J. Math. and Math. Sci. **12**(1989), 257–262.

[4] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math. **20**(1969), 457–488.

[5] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonsratio Math. **32**(1999), 157–163.

[6] V. Popa, *A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation*, Demonstratio Math. **33**(2000), 159–164.

[7] S. P. Singh and B. A. Meade, *A common fixed point theorem*, Bull. Austral. Math. Soc. **16**(1977), 49–53.

[8] S. Sharma and B. Desphande, *On compatible mappings satisfying an implicit relation in common fixed point consideration*, Tamkang J. Math. **33**(2002), 245–252.

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