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RECENT DEVELOPMENTS ON PSEUDO-DIFFERENTIAL OPERATORS (I)

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Abstract. In recent years the analysis of (pseudo-)differential operators on manifolds with second and higher order corners made considerable progress, and essential new structures have been developed. The main objective of this series of paper is to give a survey on the development of this theory in the past twenty years. We start with a brief background of the theory of pseudo-differential operators which including its symbolic calculus on $\mathbb{R}^n$. Next we introduce pseudo-differential calculus with operator-valued symbols. This allows us to discuss elliptic boundary value problems on smooth domains in $\mathbb{R}^n$ and elliptic problems on manifolds. This paper is based on the first part of lectures given by the authors while they visited the National Center for Theoretical Sciences in Hsinchu, Taiwan during May-July of 2014.

1. Introduction

One of powerful tools to study partial differential equations is Fourier transform. Let us start with a well-known example. Consider the equation

$$A(u) = \big(1-\Delta\big)u = \Big(1-\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}\Big)u = f \tag{1}$$

with Δ being the Laplacian in $\mathbb{R}^n$ where $f \in C_0^\infty(\mathbb{R}^n)$. It is easy to see that the Fourier transform of $\frac{\partial u}{\partial x_j}$ is

$$\mathscr{F}\Big(\frac{\partial u}{\partial x_j}\Big)(\xi) = \widehat{\Big(\frac{\partial u}{\partial x_j}\Big)}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\frac{\partial u}{\partial x_j}(x)\,dx = -i\xi_j\widehat{u}(\xi). \tag{2}$$

Therefore,

$$\Big(1+\sum_{k=1}^{n}\xi_k^2\Big)\widehat{u}(\xi) = \big(1+|\xi|^2\big)\widehat{u}(\xi) = \widehat{f}(\xi).$$

It follows that

$$u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{u}(\xi)\,d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\big(1+|\xi|^2\big)^{-1}\widehat{f}(\xi)\,d\xi.$$

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Now let us move to partial differential operators with variable coefficients. Consider the equation

$$A(x,D)u = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u = f. \tag{3}$$

It follows that

$$A(x,D)u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\,\widehat{u}(\xi)\,đ\xi$$

where $đ\xi = \frac{1}{(2\pi)^n} d\xi$ and $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$.

Freezing the coefficients at x_0, can we use $A(x_0,D)u(x)$ to approximate the solution of the equation (3)? In general, this is not true. Denote

$$\sigma_\psi(A)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \qquad x\in\mathbb{R}^n, \qquad \xi\in\mathbb{R}^n\setminus\{0\}$$

the *homogeneous principal symbol* of $a(x,\xi)$. We say that A is *elliptic* if $\sigma_\psi(A)$ does not vanish for all $(x,\xi)\in T^*\mathbb{R}^n\setminus 0$. As usual, T^*X denotes the cotangent bundle of a the manifold X and 0 indicates the zero section, locally represented by $\xi = 0$. In this case, an approximate solution of (3) should be obtained in the form

$$u(x) \approx \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\widehat{f}(\xi)đ\xi$$

for a suitable function $p(x,\xi)$ that is not necessarily a polynomial in ξ. In general, an operator that is able to express solvability of elliptic equations is expected to be of the form

$$\mathrm{Op}(a)(f)(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{f}(\xi)\,đ\xi, \qquad f\in C_0^\infty(\mathbb{R}^n),$$

for a so-called *symbol* $a(x,\xi)$. The first part of this paper is to study basic properties of symbols. We will impose the condition that $a(x,\xi)$ has polynomial growth while $|\xi|\to\infty$. When $\mathrm{Op}(a)$ is a differential operator, then $a(x,\xi)$ is indeed a polynomial, otherwise called a pseudo-differential operator. In fact, we wish to generalize the idea of differential operators to a much bigger class of operators $\mathscr{A}$ with the following properties:

(1) $\mathscr{A}$ at least contains differential operators and parametrices of elliptic operators, hypoelliptic operators and convolution operators with smooth kernels.
(2) $\mathscr{A}$ is closed under addition, composition, adjoint which is invariant under coordinates change.
(3) $\mathscr{A}$ has the pseudo-local property, *i.e.,*

$$\text{sing supp}(A(u)) \subseteq \text{sing supp}(u) \qquad \text{for all} \quad A\in\mathscr{A}.$$

Here sing supp(u) is the *singular support of* u which is the smallest closed set C such that u is smooth off C.

(4) Every element $A \in \mathscr{A}$ induces maps $A: C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ and $A: \mathscr{E}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$.

As we mentioned before, when (3) is elliptic, an approximate solution is given by

$$u(x) \approx \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \widehat{f}(\xi)\, đ\xi \tag{4}$$

where $p(x,\xi)$ is the inverse of $a(x,\xi)$ under Leibniz multiplication, discussed below. More precisely, to apply the argument leading to (4), we need the operators to satisfy

$$A_1(x,D) \circ A_2(x,D) \approx (A_1 \circ A_2)(x,D)$$

in some well-defined sense so that $A_1(x,D) \circ A_1^{-1}(x,D) \approx I +$ acceptable error terms. In the first part of this article, we will derive an asymptotic expansion for composition of pseudo-differential operators so that these formulas become precise.

Before we go further, we observe that the operator (1) in $\mathbb{R}^n$ is elliptic not only with respect to $\sigma_\psi(A)$ but also with respect to other symbols, referred to as exit symbols when we interpret $\mathbb{R}^n$ as a manifold with conical exit $|x| \to \infty$, see also Remark 4 below.

Given a pseudo-differential operator A of order m in $\mathbb{R}^n$, where the coefficients $a_\alpha(x)$ are classical symbols in x of order zero, cf. Definition 3 (i) below, denote

$$\sigma_{\mathrm{e}}(A)(x,\xi) = \sum_{|\alpha|\le m} a_{\alpha,(0)}(x)\xi^\alpha \qquad x \neq 0, \qquad \xi \in \mathbb{R}^n$$

and

$$\sigma_{\psi,\mathrm{e}}(A)(x,\xi) = \sum_{|\alpha|= m} a_{\alpha,(0)}(x)\xi^\alpha \qquad x \neq 0, \qquad \xi \neq 0.$$

Here $a_{\alpha,(0)}(x)$ is the homogeneous principal part of a_α in x of order 0.

An essential aspect of ellipticity with respect to the symbols $(\sigma_\psi, \sigma_e, \sigma_{\psi,e})$ is the following result.

Let $H^s(\mathbb{R}^n)$ be the standard Sobolev space on $\mathbb{R}^n$ of order $s \in \mathbb{R}$, *i.e.*, the space of all $u \in \mathscr{S}(\mathbb{R}^n)$ such that $\langle\xi\rangle^s(\mathscr{F}u)(\xi) \in L^2(\mathbb{R}^n)$. It can be shown that

$$A\colon H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n),$$

is continuous for all $s \in \mathbb{R}$. It is a Fredholm operator if and only if $\sigma_\psi(A)(x,\xi) \neq 0$ for $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, $\sigma_{\mathrm{e}}(A)(x,\xi) \neq 0$ for $(x,\xi) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$, and $\sigma_{\psi,\mathrm{e}}(A)(x,\xi) \neq 0$ for $(x,\xi) \in (\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$.

In particular, for the operator $A = 1 - \Delta$ considered at the very beginning we have

$$\sigma_\psi(A)(\xi) = |\xi|^2, \quad \sigma_{\mathrm{e}}(A) = 1 + |\xi|^2, \quad \sigma_{\psi,\mathrm{e}}(A) = |\xi|^2. \tag{5}$$

The second part of this article is devoted to elements of the pseudo-differential calculus to boundary value problems from the point of view of expressing parametrices of boundary value problem:

$$\begin{aligned} A(x,D)u &= f \qquad \text{in} \qquad \Omega, \\ \nu(u) &= g \qquad \text{on} \qquad \partial\Omega \end{aligned} \tag{6}$$

where ν represents a boundary (or trace) operator, and $\Omega \subset \mathbb{R}^n$ is a subdomain with smooth boundary.

Pseudo-differential operators in the standard sense, outlined in this paper, are motivated by the task to reflect the solvability properties of elliptic differential equations and of elliptic boundary value problems. In order to outline new ideas from the calculus of pseudo-differential operators and boundary value problems and to keep the consideration self- contained, we formulate some structures from the well-known pseudo-differential calculus, which are for scalar operators standard and may be found in papers or textbooks, cf. [12], [10], [24], but then have analogues in other contexts. For example, various modifications refer to conical exits of the configuration to infinity. In a continuation of this material this information is the background of the calculus of operators on manifolds with singularities, also motivated to solving equations, but now for configurations with conical singularities, edge or higher corner singularities. For instance, Kondratyev's paper [13] studies elliptic boundary value problems for differential operators in domains with conical singularities. The task to express parametrices gives rise to the pseudo-differential calculus for conical singularities, together with that for manifolds with smooth boundary, see Boutet de Monvel [1], or Schulze [18]. Singularities of higher order appear in connection with mixed and transmission or crack problems, cf. the monographs [11], [8], or the paper [2]. It turns out that the details require new structures, not only new classes of weighted Sobolev spaces and edge spaces, as established in [17], but also new techniques of proving continuity of operators in those spaces, see the paper [23] of Seiler. Elements of the calculus for higher singularities are developed in [15], [16], [19], [3], [4], [20], [21], [22], and in many other papers. The corner theories, beginning with conical and edge singularities, give rise to new Mellin operators and Mellin quantisations, see Eskin's book [6] or the monographs [5], [18]. Important applications for asymptotics in many-particle systems have been started in [7]. Let us finally point out that pseudo-differential theories on manifolds with singularities seem to be at a new beginning where most of the basic problems are to be solved in future. This concerns index theories extending [1], [14], [15], edge conditions under non-vanishing of an analogue of the Atiyah-Bott obstruction, iterative structures with symbolic hierarchies in the sense of [20], and asymptotic properties, especially variable and boundary edge asymptotics.

2. Pseudo-differential operators

Let

$$A(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha = \sum_{\alpha_1+\cdots+\alpha_n\le m} a_{\alpha_1\dots\alpha_n}(x)\Big(\frac{1}{i}\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\cdots\Big(\frac{1}{i}\frac{\partial}{\partial x_n}\Big)^{\alpha_n}$$

be a differential operator in a domain $\Omega\subseteq\mathbb{R}^n$ with coefficients $a_\alpha(x)\in C^\infty(\Omega)$, regarded as an operator

$$A(x,D) : C_0^\infty(\Omega) \to C_0^\infty(\Omega). \tag{7}$$

Then $A(x,D)$ can be expressed by the Fourier transform $\mathscr{F}$ as

$$A(x,D) = \mathscr{F}^{-1} a(x,\xi)\mathscr{F} \qquad \text{with} \qquad a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha,$$

using the elementary identity $D_x^\alpha = \mathscr{F}^{-1}\xi^\alpha\mathscr{F}$. Thus

$$\begin{aligned} A(x,D)u(x) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\Big\{\int_{\mathbb{R}^n} e^{-iy\cdot\xi}u(y)\,dy\Big\}\,đ\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi)u(y)\,dy\,đ\xi. \end{aligned} \tag{8}$$

The expression on the right hand side is interpreted as an oscillatory integral.

Remark 1. Note that A through its action as a continuous operator (7), its symbol $a(x,\xi)$ is a unique way. In fact, using the Fourier inversion formula

$$u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)\,đ\xi,$$

we obtain by applying A on both sides

$$A(x,D)u(x) = \int_{\mathbb{R}^n}\Big(Ae^{ix\cdot\xi}\Big)\hat{u}(\xi)\,đ\xi = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\Big(e^{-ix\cdot\xi}Ae^{ix\cdot\xi}\Big)\hat{u}(\xi)\,đ\xi, \tag{9}$$

i.e.,

$$a(x,\xi) = e^{-ix\cdot\xi}Ae^{ix\cdot\xi}. \tag{10}$$

We begin by studying the appropriate classes of amplitude functions; for simplicity we also refer to those as symbols.

Definition 2.

(i) Let $\Omega\subseteq\mathbb{R}^m$ be open and $\mu\in\mathbb{R}$. Then $S^\mu(\Omega\times\mathbb{R}^n)$ denotes the space of all $a(x,\xi)\in C^\infty(\Omega\times\mathbb{R}^n)$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le c\big(1+|\xi|\big)^{\mu-|\beta|} \tag{11}$$

for all $\alpha\in(\mathbb{Z}_+)^m$, $\beta\in(\mathbb{Z}_+)^n$, $x\in K$, with arbitrary $K\Subset\Omega$, $\xi\in\mathbb{R}^n$, with constant $c = c(\alpha,\beta,K)>0$. The number μ is called the order of the symbol a, written $\mu = \mathrm{ord}(a)$.

(ii) $S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n)$ for $\mu, \nu \in \mathbb{R}$ denotes the set of all $a(x,\xi) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le c\big(1+|x|\big)^{\nu-|\alpha|}\big(1+|\xi|\big)^{\mu-|\beta|} \tag{12}$$

for all $(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^n$, for all $\alpha \in (\mathbb{Z}_+)^m, \beta \in (\mathbb{Z}_+)^n$ for some $c = c(\alpha,\beta) > 0$.

The best possible constants c in (11) form a semi-norm system on the space $S^\mu(\Omega \times \mathbb{R}^n)$:

$$a \to \sup_{x\in K, \xi \in \mathbb{R}^n} |D_x^\alpha D_\xi^\beta a(x,\xi)|\big(1+|\xi|\big)^{-\mu+|\beta|}.$$

It is clear that $a(x,\xi)$ belongs to $S^\mu(\Omega \times \mathbb{R}^n)$ just in case the estimates (11) are satisfied for a countable system of compact sets $K \subset \Omega$ and all α, β. It suffices to take all closed balls contained in Ω of rational radii and centers with rational coordinates. It is then easy to verify that $S^\mu(\Omega \times \mathbb{R}^n)$ is a Fréchet space with this semi-norm system.

The best possible constants c in (12) form a semi-norm system in $S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n)$:

$$a \to \sup_{x,\xi \in \mathbb{R}^n} |D_x^\alpha D_\xi^\beta a(x,\xi)|\big(1+|x|\big)^{-\nu+|\alpha|}\big(1+|\xi|\big)^{-\mu+|\beta|}.$$

which turns $S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n)$ to a Fréchet space.

Recall that a $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is called an excision function if

$$\chi(\xi) = \begin{cases} 0 & \text{if} \quad |\xi| < r_0 \\ 1 & \text{if} \quad |\xi| > r_1 \end{cases}$$

with constants $0 < r_0 < r_1 < \infty$.

A function $f(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is called homogeneous of order $\mu \in \mathbb{R}$ if $f(\lambda\xi) = \lambda^\mu f(\xi)$ for all $\lambda > 0$, $\xi \in \mathbb{R}^n \setminus \{0\}$. Let $S^{(\mu)}(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ be the space of all $f(x,\xi) \in C^\infty(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ that are homogeneous in ξ or order μ for all $x \in \Omega$. Then

$$D_x^\alpha D_\xi^\beta S^{(\mu)}(\Omega \times (\mathbb{R}^n \setminus \{0\}) \subseteq S^{\mu-|\beta|}(\Omega \times (\mathbb{R}^n \setminus \{0\}))$$

for all $\alpha \in (\mathbb{Z}_+)^m$, $\beta \in (\mathbb{Z}_+)^n$.

Definition 3.

(i) A symbol $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$ is called classical if there is a sequence $a_{(\mu-k)}(x,\xi) \in S^{(\mu-k)}(\Omega \times (\mathbb{R}^n \setminus \{0\}))$, $k \in \mathbb{Z}_+$, such that for any excision function $\chi(\xi)$

$$a(x,\xi) - \sum_{k=0}^{N} \chi(\xi) a_{(\mu-k)}(x,\xi) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^n) \tag{13}$$

for all $N \in \mathbb{Z}_+$. We denote by $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ the space of all classical symbols of order μ.

(ii) A symbol $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n)$ is called classical in x, ξ if it belongs $S^{\nu}_{\mathrm{cl}}(\mathbb{R}^m_x)\widehat{\otimes}_\pi S^{\mu}_{\mathrm{cl}}(\mathbb{R}^n_\xi) =: S^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^m \times \mathbb{R}^n)$ where $\widehat{\otimes}_\pi$ denotes the projective tensor product between the respective Fréchet spaces of classical symbols with constant coefficients; the first factor concerns symbols in $x \in \mathbb{R}^m$, formally treated as a covariable.

Remark 4. The notion of homogeneous principal symbols also makes sense for $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n)$. First, $a(x,\xi)$ in the representation $S^{\nu}_{\mathrm{cl}}(\mathbb{R}^m_x)\widehat{\otimes}_\pi S^{\mu}_{\mathrm{cl}}(\mathbb{R}^n_\xi)$ has a homogeneous principal symbol in $\xi \neq 0$ of order μ

$$\sigma_\psi(a)(x,\xi) \in S^{\nu}_{\mathrm{cl}}(\mathbb{R}^m_x)\widehat{\otimes}_\pi S^{(\mu)}(\mathbb{R}^n \setminus \{0\}) \tag{14}$$

Moreover, $a(x,\xi)$ has a homogeneous principal symbol in $x \neq 0$ of order ν (also called principal exit symbol)

$$\sigma_{\mathrm{e}}(a)(x,\xi) \in S^{(\nu)}(\mathbb{R}^m \setminus \{0\})\widehat{\otimes}_\pi S^{\mu}_{\mathrm{cl}}(\mathbb{R}^n), \tag{15}$$

and there is also a homogeneous principal symbol of (15) in $\xi \neq 0$, namely,

$$\sigma_{\psi,\mathrm{e}}(a)(x,\xi) \in S^{(\nu)}(\mathbb{R}^m \setminus \{0\})\widehat{\otimes}_\pi S^{(\mu)}(\mathbb{R}^n \setminus \{0\}).$$

Note that the latter is equal to the homogeneous principal symbol of (14) with respect to $x \neq 0$.

This gives us altogether a triple

$$\sigma(a) = \big(\sigma_\psi(a), \sigma_{\mathrm{e}}(a), \sigma_{\psi,\mathrm{e}}(a)\big) \tag{16}$$

of homogeneous principal symbols. The conditions of ellipticity with respect to the components are independent.

Remark 5.

(i) From Definitions 2 (i) we obtain

$$D^\alpha_x D^\beta_\xi S^\mu(\Omega \times \mathbb{R}^n) \subset S^{\mu-|\beta|}(\Omega \times \mathbb{R}^n)$$

for all $\alpha \in (\mathbb{Z}_+)^m$, $\beta \in (\mathbb{Z}_+)^n$, and (with obvious notation)

$$S^\mu(\Omega \times \mathbb{R}^n) \circ S^\nu(\Omega \times \mathbb{R}^n) \subset S^{\mu+\nu}(\Omega \times \mathbb{R}^n)$$

for all $\mu, \nu \in \mathbb{R}$. Analogous relations hold for the classical symbols in the sense of Definition 3 (ii) where the respective homogeneous principal symbols behave multiplicatively.

(ii) From Definition 3 (ii), it follows that

$$D^\alpha_x D^\beta_\xi S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n) \subset S^{\mu-|\beta|;\nu-|\alpha|}(\mathbb{R}^m \times \mathbb{R}^n)$$

for all $\alpha \in (\mathbb{Z}_+)^m$, $\beta \in (\mathbb{Z}_+)^n$ and

$$S^{\mu_1;\nu_1}(\mathbb{R}^m \times \mathbb{R}^n) \cdots S^{\mu_2;\nu_2}(\mathbb{R}^m \times \mathbb{R}^n) \subset S^{\mu_1+\mu_2;\nu_1+\nu_2}(\mathbb{R}^m \times \mathbb{R}^n)$$

for all μ_j, ν_j, $j = 1,2$. Analogous relations hold for classical symbols in the sense of Definition 3 (ii) where the components of the homogeneous principal symbols behave multiplicatively.

The following theorem can be viewed as the converse of (13).

Theorem 6.

(i) *Let $a_j(x,\xi) \in S^{\mu_j}(\Omega \times \mathbb{R}^n)$, $j \in \mathbb{Z}_+$, be an arbitrary sequence, where $\mu_j \to -\infty$ as $j \to \infty$. Then there is a symbol $a(x,\xi) \in S^{\mu}(\Omega \times \mathbb{R}^n)$ with $\mu = \max_{j\in\mathbb{Z}_+}\{\mu_j\}$ such that for every M there is an $N(M)$ such that for all $N \geq N(M)$*

$$a(x,\xi) - \sum_{j=0}^{N} a_j(x,\xi) \in S^{\mu-M}(\Omega \times \mathbb{R}^n).$$

The element $a(x,\xi) \in S^{\mu}(\Omega \times \mathbb{R}^n)$ is uniquely determined by this property modulo $S^{-\infty}(\Omega \times \mathbb{R}^n)$.

(ii) *Let $a_j(x,\xi) \in S^{\mu_j;\nu_j}(\mathbb{R}^m \times \mathbb{R}^n)$, $j \in \mathbb{Z}_+$, for $\mu_j \to -\infty$, $\nu_j \to -\infty$ as $j \to \infty$. Then there is a symbol $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n)$ for $\mu = \max_{j\in\mathbb{Z}_+}\{\mu_j\}$, $\nu = \max_{j\in\mathbb{Z}_+}\{\nu_j\}$ and that for all $N \geq N(M)$*

$$a(x,\xi) - \sum_{j=0}^{N} a_j(x,\xi) \in S^{\mu-M;\nu-M}(\mathbb{R}^m \times \mathbb{R}^n),$$

and $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n)$ is uniquely determined by this property modulo $S^{-\infty;-\infty}(\mathbb{R}^m \times \mathbb{R}^n)$.

We call $a(x,\xi)$ an *asymptotic sum* of $a_j(x,\xi)$, $j \in \mathbb{Z}_+$, and write

$$a \sim \sum_{j=0}^{\infty} a_j.$$

It can easily be proved that for every sequence $a_j(x,\xi) \in S^{\mu_j}(\Omega \times \mathbb{R}^n)$, $j \in \mathbb{Z}_+$, and any choice of an excision function $\chi(\xi)$ in $\mathbb{R}^n$ there are constants $c_j > 0$, tending to ∞ sufficiently fast as $j \to \infty$, such that

$$a(x,\xi) := \sum_{j=0}^{\infty} \chi\Big(\frac{\xi}{c_j}\Big) a_j(x,\xi)$$

converges in $S^{\mu}(\Omega \times \mathbb{R}^n)$. For the second case, *i.e.*, $a_j(x,\xi) \in S^{\mu_j;\nu_j}(\mathbb{R}^m \times \mathbb{R}^n)$, $j \in \mathbb{Z}_+$, we choose an excision function $\chi(x,\xi)$ in $\mathbb{R}^{m+n}$ and find constants $c_j > 0$, tending to ∞ sufficiently fast as $j \to \infty$, such that

$$a(x,\xi) := \sum_{j=0}^{\infty} \chi\Big(\frac{x,\xi}{c_j}\Big) a_j(x,\xi)$$

converges in $S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^n)$ and represents an element with the asserted properties.

Definition 7. Let $\Omega \subseteq \mathbb{R}^n$ be open and $\mu \in \mathbb{R}$. An operator of the form

$$\mathrm{Op}(a)u(x) = \iint e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\,dy\,đ\xi$$

with a symbol (or amplitude function) $a(x,y,\xi) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)$ is a pseudo-differential operator on Ω of order μ. We set

$$L^\mu(\Omega) = \{\mathrm{Op}(a) : \; a(x,y,\xi) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)\}, \tag{17}$$

$$L^\mu_{\mathrm{cl}}(\Omega) = \{\mathrm{Op}(a) : \; a(x,y,\xi) \in S^\mu_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}^n)\}. \tag{18}$$

The elements of (18) are called classical pseudo-differential operators.

Let us set

$$L^{-\infty}(\Omega) := \bigcap_{\mu\in\mathbb{R}} L^\mu(\Omega).$$

For any $A \in L^\mu(\Omega)$ we have a continuous map

$$A : C_0^\infty(\Omega) \to C^\infty(\Omega). \tag{19}$$

Thus, by virtue of the Schwartz theorem, we have a distributional kernel $K_A(x,y) \in \mathscr{D}'(\Omega\times\Omega)$, where

$$\langle Au, v\rangle = \langle K_A(x,y), u(y)v(x)\rangle$$

for all $u, v \in C_0^\infty(\Omega)$; here $\langle\cdot,\cdot\rangle$ means the bilinear pairing. The distributional kernel K_A is in one-to-one correspondence to the respective operator A.

Definition 8. An operator $A \in L^\mu(\Omega)$ is called properly supported, if both $\mathrm{supp}(K_A) \cap \{M\times\Omega\}$ and $\mathrm{supp}(K_A) \cap \{\Omega\times M'\}$ are compact for arbitrary compact subsets $M, M' \Subset \Omega$.

Theorem 9. *For every $A \in L^\mu(\Omega)$ we have*

$$\mathrm{sing\,supp}(K_A) \subseteq \mathrm{diag}(\Omega\times\Omega). \tag{20}$$

Proof. We employ the identity

$$e^{i(x-y)\cdot\xi} = |x-y|^{-2N}\big(-\Delta_\xi\big)^N e^{i(x-y)\cdot\xi} \tag{21}$$

for any $N \in \mathbb{Z}_+$. This gives us, using integration by parts

$$\begin{aligned} Au(x) &= \int e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\,dy\,đ\xi \\ &= \int |x-y|^{-2N}\big(-\Delta_\xi\big)^N e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\,dy\,đ\xi \\ &= \int e^{i(x-y)\cdot\xi}|x-y|^{-2N}\big(-\Delta_\xi\big)^N a(x,y,\xi)u(y)\,dy\,đ\xi. \end{aligned}$$

Applying Remark 5 (i) we have $\big(-\Delta_\xi\big)^N a(x,y,\xi) \in S^{\mu-2N}(\Omega\times\Omega\times\mathbb{R}^n)$ which gives us

$$K_A(x,y) = \int e^{i(x-y)\cdot\xi}|x-y|^{-2N}\big(-\Delta_\xi\big)^N a(x,y,\xi)u(y)\,đ\xi.$$

This is a convergent integral as long as $x \neq y$ and N is sufficiently large, in fact, $\mu-2N<-n$. Moreover, for every $k\in\mathbb{Z}_+$, we have $K_A(x,y)\in C^k(\Omega\times\Omega\setminus(\text{diag}(\Omega\times\Omega)))$ for any $N=N(k)$ sufficiently large. Since k is arbitrary, it follows that

$$K_A(x,y)\in C^\infty(\Omega\times\Omega\setminus(\text{diag}(\Omega\times\Omega))). \tag{22}$$

This completes the proof of the theorem. □

Corollary 10. *Every $A\in L^\mu(\Omega)$ can be written in the form*

$$A = A_0 + C$$

where $A_0\in L^\mu(\Omega)$ is properly supported and $C\in L^{-\infty}(\Omega)$.

Proof. In fact, looking at the expression

$$Au(x) = \iint e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)\,đ\xi$$

we can always write

$$Au(x) = \iint e^{i(x-y)\cdot\xi}\big[\omega(x,y)+(1-\omega(x,y))\big]a(x,y,\xi)u(y)\,đ\xi$$

for arbitrary $\omega\in C_0^\infty(\Omega\times\Omega)$. Choosing ω in such a way that $\text{supp}(\omega)$ is proper in the sense of Definition 8 and $\omega\equiv 1$ in a neighborhood of $\text{diag}(\Omega\times\Omega)$ that contains $\text{diag}(\Omega\times\Omega)$ in its open interior which is always possible. Then, by virtue of (20), $A_0=\text{Op}(\omega a)$ is properly supported while $C=\text{Op}((1-\omega)a)$ is smoothing. □

Remark 11. Let $A\in L^\mu(\Omega)$; then (7) extends to a linear operators

$$A:\mathscr{E}'(\Omega)\to\mathscr{D}'(\Omega).$$

If A is properly supported then A induces continuous operator

$$A: C_0^\infty(\Omega)\to C_0^\infty(\Omega),\qquad\text{and}\qquad A: C^\infty(\Omega)\to C^\infty(\Omega)$$

which extend to linear operators

$$A:\mathscr{E}'(\Omega)\to\mathscr{E}'(\Omega),\qquad\text{and}\qquad A:\mathscr{D}'(\Omega)\to\mathscr{D}'(\Omega).$$

Example 1. A differential operator (3) is always properly supported. In this case,

$$\text{supp}(K_A)\subseteq\text{diag}(\Omega\times\Omega)$$

which entails that A is local, i.e., $\operatorname{supp}(Au) \subseteq \operatorname{supp} u$. As is well-known the differential operators are the only local operators in $L^\mu(\Omega)$.

Remark 12. From (20), we see that an $A \in L^\mu(\Omega)$ is pseudo-local, i.e.,

$$\operatorname{sing\,supp}(Au) \subseteq \operatorname{sing\,supp}(u)$$

for every $u \in \mathscr{E}'(\Omega)$.

Theorem 13. *Let $A \in L^\mu(\Omega)$ be properly supported. Then there is a unique left symbol $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$ such that $A = \operatorname{Op}(a)$.*

Denoting by $S^\mu(\Omega\times\mathbb{R}^n)_M$ the subspace of all $a(x,\xi) \in S^\mu(\Omega\times\mathbb{R}^n)$ such that $M := \operatorname{supp}(K_{\operatorname{Op}(a)})$, and $L^\mu(\Omega)_M$ the space of all $A \in L^\mu(\Omega)$ such that $\operatorname{supp}(K_A) \subseteq M$, then the above mentioned correspondence $A \to a(x,\xi)$ gives us an isomorphism

$$L^\mu(\Omega)_M \cong S^\mu(\Omega \times \mathbb{R}^n)_M.$$

3. Element of the calculus

For $\operatorname{Op}(\cdot)$ to express a pseudo-differential operator in terms of a symbol we may admit double symbols $a(x,y,\xi) =: a_{\mathrm{D}}(x,y,\xi)$ or left and right symbols $a_{\mathrm{L}}(x,\xi)$ and $a_{\mathrm{R}}(y,\xi)$, respectively. The following theorem tells us that $\operatorname{Op}(a_{\mathrm{D}})$ can always be turned to $\operatorname{Op}(a_{\mathrm{L}})$ or $\operatorname{Op}(a_{\mathrm{R}})$ modulo smoothing operators. For the proof, we need the following result.

Lemma 14. *Let $a(x,y,\xi) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^n)$ be a symbol with the property $|x-y|^{-2N} a(x,y,\xi) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^n)$ for some $N \in \mathbb{Z}_+$. Then there exists an $a_N(x,y,\xi) \in S^{\mu-2N}(\Omega \times \Omega \times \mathbb{R}^n)$ such that*

$$\operatorname{Op}(a) = \operatorname{Op}(a_N).$$

Proof. Using (21):

$$e^{i(x-y)\cdot\xi} = |x-y|^{-2N}\big(-\Delta_\xi\big)^N e^{i(x-y)\cdot\xi}$$

and integration by parts, it follows that $\operatorname{Op}(a) = \operatorname{Op}(a_N)$ with

$$a_N(x,y,\xi) = \big(-\Delta_\xi\big)^N |x-y|^{-2N} a(x,y,\xi) \in S^{\mu-2N}(\Omega \times \Omega \times \mathbb{R}^n).$$ □

Theorem 15. *To every $a(x,y,\xi) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^n)$ there is an $a_{\mathrm{L}} \in S^\mu(\Omega \times \mathbb{R}^n)$ with $\operatorname{Op}(a) = \operatorname{Op}(a_{\mathrm{L}})$ modulo $L^{-\infty}(\Omega)$ and $a_{\mathrm{L}}(x,\xi)$ admits the asymptotic expansion*

$$a_{\mathrm{L}}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha a(x,y,\xi)\Big|_{y=x}. \tag{23}$$

In other words every $A \in L^\mu(\Omega)$ has a left symbol. Moreover, there is a right symbol $a_{\mathrm{R}}(y,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$ with $\operatorname{Op}(a) = \operatorname{Op}(a_{\mathrm{R}})$ modulo $L^{-\infty}(\Omega)$ that has the asymptotic expansion

$$a_{\mathrm{R}}(y,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \big(-D_\xi\big)^\alpha \partial_x^\alpha a(x,y,\xi)\Big|_{x=y}. \tag{24}$$

Proof. Applying the Taylor expansion of $a(x,y,\xi)$ on $\operatorname{diag}(\Omega\times\Omega)$ we obtain

$$a(x,y,\xi) = \sum_{|\alpha|\le M} \frac{1}{\alpha!}(y-x)^\alpha \partial_y^\alpha a(x,y,\xi)\Big|_{y=x} + R_M(x,y,\xi) \tag{25}$$

with a remainder term $R_M(x,y,\xi)\in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)$. For every N we can choose M so large such that

$$|x-y|^{-2N}R_M(x,y,\xi)\in S^\mu(\Omega\times\Omega\times\mathbb{R}^n).$$

Applying Op on both sides of (25) we obtain from Lemma 14, together with the identity $\operatorname{Op}\big((y-x)^\alpha a_{\mathrm{R}}\big)=\operatorname{Op}\big(D_\xi^\alpha a_{\mathrm{R}}\big)$ for any symbol a_{R},

$$\operatorname{Op}(a) = \sum_{|\alpha|\le M}\frac{1}{\alpha!}\operatorname{Op}\Big(D_\xi^\alpha\partial_y^\alpha a(x,y,\xi)\Big|_{y=x}\Big) + \operatorname{Op}(a_N)$$

with some $a_N\in S^{\mu-2N}(\Omega\times\Omega\times\mathbb{R}^n)$. If we form (23) by carrying out the asymptotic sum we obtain immediately

$$\operatorname{Op}(a)-\operatorname{Op}(a_{\mathrm{L}}) = \operatorname{Op}(\tilde{a}_N)$$

for another $\tilde{a}_N\in S^{\mu-2N}(\Omega\times\Omega\times\mathbb{R}^n)$. This is true for every N and hence $\operatorname{Op}(a)-\operatorname{Op}(a_{\mathrm{L}})\in L^{-\infty}(\Omega)$. The second statement can be proved in an analogous manner by interchanging the role of x and y. The proof of the theorem is therefore complete. □

Theorem 16. *Let $A\in L^\mu(\Omega)$ and let A^* be its formal adjoint, defined by*

$$\big(Au,v\big) = \big(u,A^*v\big) \qquad \textit{for all} \quad u,\,v\in C_0^\infty(\Omega)$$

with the L^2 scalar product $(\cdot,\cdot)$. Then $A^*\in L^\mu(\Omega)$. If $A=\operatorname{Op}(a)$ modulo $L^{-\infty}(\Omega)$, for some $a(x,\xi)\in S^\mu(\Omega\times\mathbb{R}^n)$, we have $A^*=\operatorname{Op}(a^*)$ modulo $L^{-\infty}(\Omega)$ for an $a^*(x,\xi)\in S^\mu(\Omega\times\mathbb{R}^n)$ such that

$$a^*(x,\xi)\sim\sum_\alpha\frac{1}{\alpha!}D_\xi^\alpha\partial_x^\alpha\overline{a(x,\xi)}. \tag{26}$$

Proof. The behavior of operators in $L^{-\infty}(\Omega)$ under formal adjoints is obvious. Thus we may assume $A=\operatorname{Op}(a)$ with a left symbol $a(x,\xi)$. Then

$$A^*v(x) = \iint e^{i(x-y)\cdot\xi}\overline{a(y,\xi)}v(y)\,dy\,đ\xi$$

shows $A^*\in L^\mu(\Omega)$ including (26) as a consequence of Theorem 15. The conclusion of the theorem follows immediately. □

Now we come to the most important result in this section.

Theorem 17. *Let $A_1 \in L^{\mu_1}(\Omega)$ and $A_2 \in L^{\mu_2}(\Omega)$ and A_1 or A_2 be properly supported. Then $A_1A_2 \in L^{\mu_1+\mu_2}(\Omega)$. If $A_1 = \mathrm{Op}(a_1)$ modulo $L^{-\infty}(\Omega)$, $a_1(x,\xi) \in S^{\mu_1}(\Omega\times\mathbb{R}^n)$, and $A_2 = \mathrm{Op}(a_2)$ modulo $L^{-\infty}(\Omega)$, $a_2(x,\xi) \in S^{\mu_2}(\Omega\times\mathbb{R}^n)$, then $A_1A_2 = \mathrm{Op}(a)$ modulo $L^{-\infty}(\Omega)$ for a symbol $a(x,\xi) \in S^{\mu_1+\mu_2}(\Omega\times\mathbb{R}^n)$,*

$$a(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\big(D_\xi^\alpha a_1(x,\xi)\big)\big(\partial_x^\alpha a_2(x,\xi)\big). \tag{27}$$

Proof. Let us assume, for instance, that A_1 is properly supported. Write A_2 in the form $A_2 = \mathrm{Op}(\widetilde{a}_2)$ modulo $L^{-\infty}(\Omega)$ for a symbol $\widetilde{a}_2(y,\xi) \in S^{\mu_2}(\Omega\times\mathbb{R}^n)$. Then, according to (24),

$$\widetilde{a}_2(y,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\big(-D_\xi\big)^\alpha\big(\partial_y^\alpha a_2(y,\xi)\big). \tag{28}$$

The relations

$$\mathrm{Op}(\widetilde{a}_2)u(y) = \mathscr{F}^{-1}_{\xi\to y}\mathscr{F}_{y\to\xi}\big\{\widetilde{a}_2(y,\xi)u(y)\big\}$$

and

$$\mathrm{Op}(a_1)v(x) = \mathscr{F}^{-1}_{\xi\to x}a_1(x,\xi)\big\{\mathscr{F}_{y\to\xi}v(y)\big\}$$

yield

$$\mathrm{Op}(a_1)\mathrm{Op}(\widetilde{a}_2)u(x) = \iint e^{i(x-y)\cdot\xi}a_1(x,\xi)\widetilde{a}_2(y,\xi)u(y)\,dy\,đ\xi.$$

In order to obtain an asymptotic formula for $a(x,\xi)$ it suffices to apply (23) to $a_1(x,\xi)\widetilde{a}_2(y,\xi)$, which gives us

$$\begin{aligned} a(x,\xi) &\sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha\big\{a_1(x,\xi)\,D_x^\alpha\widetilde{a}_2(x,\xi)\big\} \\ &\sim \sum_{\alpha,\beta}\frac{1}{\alpha!\beta!}\partial_\xi^\alpha\big\{a_1(x,\xi)\,(-\partial_\xi)^\beta D_x^{\alpha+\beta}a_2(x,\xi)\big\}. \end{aligned}$$

It was used in the second relation (28). Now we employ the Leibniz rule

$$\partial_\xi^\alpha\big(f(\xi)g(\xi)\big) = \sum_{\gamma+\delta=\alpha}\frac{\alpha!}{\gamma!\delta!}\big\{\partial_\xi^\gamma f(\xi)\big\}\big\{\partial_\xi^\delta g(\xi)\big\}$$

and the binomial formula

$$(x+y)^k = \sum_{\beta+\delta=\mathbf{k}}\frac{\mathbf{k}!}{\beta!\delta!}x^\delta y^\beta$$

for arbitrary $x, y \in \mathbb{R}^n$, $\mathbf{k} \in (\mathbb{Z}_+)^n$. Then, in particular, for $x = -y = e = (1,1,\ldots,1)$ we get

$$(e-e)^k = \sum_{\beta+\delta=\mathbf{k}}\frac{\mathbf{k}!}{\beta!\delta!}e^\delta(-e)^\beta = \mathbf{k}!\sum_{\beta+\delta=\mathbf{k}}\frac{(-1)^{|\beta|}}{\beta!\delta!} = \delta_{0,\mathbf{k}}$$

with $\delta_{0,\mathbf{k}} = 1$ for $\mathbf{k} = (0,0,\dots,0)$, $\delta_{0,\mathbf{k}} = 0$ for $\mathbf{k} \neq 0$. Thus

$$\begin{aligned} a(x,\xi) &\sim \sum_{\alpha,\beta,\gamma,\delta,\gamma+\delta=\alpha} \frac{1}{\beta!\gamma!\delta!}\{\partial_\xi^\gamma a_1(x,\xi)\}\{(-\partial_\xi)^\beta \partial_\xi^\delta D_x^{\alpha+\beta} a_2(x,\xi)\} \\ &= \sum_{\beta,\gamma,\delta} \frac{(-1)^{|\beta|}}{\beta!\gamma!\delta!}\{\partial_\xi^\gamma a_1(x,\xi)\}\{\partial_\xi^{\beta+\delta} D_x^{\beta+\gamma+\delta} a_2(x,\xi)\} \\ &= \sum_\gamma \frac{1}{\gamma!}\sum_{\mathbf{k}}\Big\{\sum_{\beta+\delta=\mathbf{k}} \frac{(-1)^{|\beta|}}{\beta!\delta!}\Big\}\big[\partial_\xi^\gamma a_1(x,\xi)\big] \times \big[\partial_\xi^{\mathbf{k}} D_x^{\mathbf{k}+\gamma} a_2(x,\xi)\big\} \end{aligned}$$

This can be simplified to (27) and the proof of the theorem is therefore complete. □

Remark 18. Observe that

$$a_1(x,\xi)a_2(x,\xi) = a_1(x,\xi)\# a_2(x,\xi) \quad \text{mod } S^{\mu_1+\mu_2-1}(\Omega\times\mathbb{R}^n).$$

In other words, the commutator $A_1A_2 - A_2A_1$ belongs to $L^{\mu_1+\mu_2-1}(\Omega\times\mathbb{R}^n)$.

The asymptotic sum in (27) will also be called the *Leibniz product* between a_1 and a_2, written

$$a(x,\xi) = a_1(x,\xi)\sharp a_2(x,\xi). \tag{29}$$

$a(x,\xi)$ is unique modulo $S^{-\infty}(\Omega\times\mathbb{R}^n)$. Let us set $e_\xi(x) = e^{ix\cdot\xi}$ and consider a properly supported operator $A \in L^\mu(\Omega)$. Then

$$\widetilde{a}(x,\xi) := e_{-\xi}(x)Ae_\xi \tag{30}$$

is a smooth function in (x,ξ).

Theorem 19. *Let $A \in L^\mu(\Omega)$ be properly supported. Then (30) is a double symbol of A satisfying $A = \mathrm{Op}(\widetilde{a})$. If the operator is given as $A = \mathrm{Op}(a)$ for an $a(x,y,\xi) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)$ then (30) has the asymptotic expansion*

$$\widetilde{a}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha a(x,y,\xi)\Big|_{y=x}.$$

Proof. Let us apply A to $u \in C_0^\infty(\Omega)$ and use the Fourier inversion formula to write

$$u(x) = \int_{\mathbb{R}^n} e_\xi(x)\widehat{u}(\xi)\,đ\xi.$$

Then

$$Au(x) = \int_{\mathbb{R}^n} \big(Ae_\xi\big)(x)\widehat{u}(\xi)\,đ\xi,$$

because the integrals converge in $C^\infty(\Omega)$. Thus

$$\begin{aligned} Au(x) &= \int e^{ix\cdot\xi}\{e_{-\xi}(x)(Ae_\xi)(x)\}\widehat{u}(\xi)\,đ\xi \\ &= \int e^{ix\cdot\xi}\widetilde{a}(x,\xi)\Big\{\int e^{-iy\cdot\xi}u(y)dy\Big\}đ\xi. \end{aligned}$$

We will show $\widetilde{a}(x,\xi) \in S^\mu(\Omega\times\mathbb{R}^n)$. Then $\widetilde{a}(x,\xi)$ is a left symbol of A. Since A is properly supported, we may start with $A = \mathrm{Op}(a)$ for an $a(x,y,\xi) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)$ which has proper support in (x,y). We have

$$\widetilde{a}(x,\xi) := \iint e^{-ix\cdot\xi} e^{i(x-y)\cdot\eta} a(x,y,\eta) e^{iy\cdot\xi}\, dy\, đ\eta \tag{31}$$

as an iterated integral, where the integration in y is taken over a compact set when $x\in K$ for any $K \Subset \Omega$. The expression (31) can be regarded as an oscillatory integral dependent on the parameter $x \in K$. Substituting $z = x-y$, $\zeta = \eta - \xi$ we obtain

$$\widetilde{a}(x,\xi) := \iint a(x,x+z,\xi+\zeta) e^{-iz\cdot\zeta}\, dz\, đ\zeta.$$

Taylor expansion of $a(x,x+z,\xi+\zeta)$ in ζ at $\zeta=0$ gives us

$$a(x,x+z,\xi+\zeta) = \sum_{|\alpha|\le N-1} \frac{1}{\alpha!}\big(\partial_\xi^\alpha a\big)(x,x+z,\xi)\zeta^\alpha + R_N(x,x+z,\xi,\zeta)$$

with

$$R_N(x,x+z,\xi,\zeta) = \sum_{|\alpha|=N} \frac{N\zeta^\alpha}{\alpha!}\int_0^1 (1-t)^{N-1}\big(\partial_\xi^\alpha a\big)(x,x+z,\xi+t\zeta)\,dt.$$

From the Fourier inversion formula it follows that

$$\iint \big(\partial_\xi^\alpha a\big)(x,x+z,\xi)\zeta^\alpha e^{-iz\cdot\zeta}\, dz\, đ\zeta = \partial_\xi^\alpha D_z^\alpha a(x,x+z,\xi)\Big|_{z=0}.$$

Thus

$$\widetilde{a}(x,\xi) := b_N(x,\xi) + \iint R_N(x,x+z,\xi,\zeta) e^{-iz\cdot\zeta}\, dz\, đ\zeta \tag{32}$$

for

$$b_N(x,\xi) = \sum_{|\alpha|\le N-1}\frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha a(x,y,\xi)\Big|_{y=x} \in S^\mu(\Omega\times\mathbb{R}^n).$$

If we show that the second summand on the right hand side of (32) is a symbol $C_N(x,\xi)$ of order μ_N tending to $-\infty$ as $N\to\infty$, then we obtain $\widetilde{a}(x,\xi) \in S^\mu(\Omega\times\mathbb{R}^n)$ and at the same time the asserted asymptotic expansion. We have

$$C_N(x,\xi) = \sum_{|\alpha|=N}\frac{N}{\alpha!}\int_0^1 (1-t)^{N-1}\Big\{\iint e^{-iz\cdot\zeta}\zeta^\alpha \partial_\xi^\alpha a(x,x+z,\xi+t\zeta)\, dz\, đ\zeta\Big\}dt.$$

Let us consider

$$r_{\alpha,t}(x,\xi) = \iint e^{-iz\cdot\zeta}\zeta^\alpha\partial_\xi^\alpha a(x,x+z,\xi+t\zeta)\, dz\, đ\zeta.$$

Integrating by parts yields

$$r_{\alpha,t}(x,\xi) = \iint e^{-iz\cdot\zeta}\partial_\xi^\alpha D_z^\alpha a(x,x+z,\xi+t\zeta)\, dz\, đ\zeta.$$

Choosing an $m \in \mathbb{N}$ and applying again integration by parts, using

$$\langle\zeta\rangle^{-2m}\langle D_z\rangle^{2m}e^{iz\cdot\zeta} = e^{-iz\cdot\zeta}$$

it follows that $r_{\alpha,t}(x,\xi)$ equals a finite sum of expressions of the form

$$r_{\alpha,\beta,t}(x,\xi) = \iint e^{-iz\cdot\zeta}\langle\zeta\rangle^{-2m}\partial_\xi^\alpha D_z^{\alpha+\beta}a(x,x+z,\xi+t\zeta)\,dz\,đ\zeta,$$

where $|\beta| \le 2m$. Let us write $r_{\alpha,\beta,t} = p_{\alpha,\beta,t} + q_{\alpha,\beta,t}$, where $p_{\alpha,\beta,t}$ is defined as the integral over $\{(z,\zeta) : |\zeta| \le |\xi|/2\}$ and $q_{\alpha,\beta,t}$ as the integral over the complement. If $|\zeta| \le |\xi|/2$ then

$$\frac{|\xi|}{2} \le |\xi + t\zeta| \le \frac{3|\xi|}{2}.$$

Since the integration in $p_{\alpha,\beta,t}$ with respect to ζ is taken over a domain of measure $\le c|\xi|^n$ for a constant c, we obtain

$$|p_{\alpha,\beta,t}(x,\xi)| \le c\langle\zeta\rangle^{\mu-N+n} \tag{33}$$

for another constant c which is independent of ξ and t. For $|\zeta| > |\xi|/2$ we have

$$\left|\partial_\xi^\alpha D_z^{\alpha+\beta}a(x,x+z,\xi+t\zeta)\right| \le c\langle\zeta\rangle^{\mu-N}$$

for $\mu - N \ge 0$ and $\le$ constant for $\mu - N < 0$. Thus, if $h := \max\{\mu - N, 0\}$, we get

$$|q_{\alpha,\beta,t}(x,\xi)| \le c\int_{|\zeta|>|\xi|/2}\langle\zeta\rangle^{-2m}\langle\zeta\rangle^{\ell}\,d\zeta.$$

For m so large that $\ell - 2m + n + 1 < 0$ it follows that

$$|q_{\alpha,\beta,t}(x,\xi)| \le c\langle\xi\rangle^{\ell-2m+n+1}\int\langle\zeta\rangle^{-n-1}d\zeta \le c\langle\xi\rangle^{\ell-2m+n+1} \tag{34}$$

with different constants c that are independent of ξ and t, $0 \le t \le 1$. Since m is arbitrary, (33) and (34) imply the required estimate for the remainder, with $\mu_N = \mu - N + n \to -\infty$ and $N \to \infty$. The proof of the theorem is therefore complete. ☐

Theorem 20. *For $\Omega \subseteq \mathbb{R}^n$ and $\mu \in \mathbb{R}$, the map*

$$\mathrm{Op} : S^\mu(\Omega\times\mathbb{R}^n) \to L^\mu(\Omega) \tag{35}$$

induces an (*algebraic*) *isomorphism*

$$S^\mu(\Omega\times\mathbb{R}^n)/S^{-\infty}(\Omega\times\mathbb{R}^n) \cong L^\mu(\Omega)/L^{-\infty}(\Omega).$$

Proof. The map (35) induces an linear operator

$$\mathrm{Op}: S^{-\infty}(\Omega\times\mathbb{R}^n) \to L^{-\infty}(\Omega). \tag{36}$$

Therefore, (35) together with (36) induces a map between the respective quotient spaces

$$S^{\mu}(\Omega\times\mathbb{R}^n)/S^{-\infty}(\Omega\times\mathbb{R}^n) \to L^{\mu}(\Omega)/L^{-\infty}(\Omega). \tag{37}$$

This map is surjective, since $L^{\mu}(\Omega)$ can be equivalently characterized as

$$L^{\mu}(\Omega) = \left\{\mathrm{Op}(a)+C:\ a(x,\xi)\in S^{\mu}(\Omega\times\mathbb{R}^n),\ C\in L^{-\infty}(\Omega)\right\},$$

cf. relation (17), *i.e.*, it suffices to take left symbols, in order to reach $L^{\mu}(\Omega)/L^{-\infty}(\Omega)$.

On the other hand, (37) is surjective, because of the fact , that $A=\mathrm{Op}(a)$ can be written as $A = A_0+C$ for a properly supported $A_0\in L^{\mu}(\Omega)$ and $C\in L^{-\infty}(\Omega)$.

Writing with notation from Theorem 19

$$\begin{aligned} L^{\mu}(\Omega) &= L^{\mu}(\Omega)_M + L^{-\infty}(\Omega),\\ S^{\mu}(\Omega\times\mathbb{R}^n) &= S^{\mu}(\Omega\times\mathbb{R}^n)_M + S^{-\infty}(\Omega\times\mathbb{R}^n)\end{aligned} \tag{38}$$

where $M\Subset\Omega$ is an arbitrary compact. We see at the same time that the map

$$L^{\mu}(\Omega)_M \to S^{\mu}(\Omega\times\mathbb{R}^n)_M$$

induces a map

$$L^{-\infty}(\Omega)_M \to S^{-\infty}(\Omega\times\mathbb{R}^n)_M.$$

Thus, since

$$L^{\mu}(\Omega)/L^{-\infty}(\Omega) = L^{\mu}(\Omega)_M/L^{-\infty}(\Omega)_M,$$

and

$$S^{\mu}(\Omega\times\mathbb{R}^n)/S^{-\infty}(\Omega\times\mathbb{R}^n) = S^{\mu}(\Omega\times\mathbb{R}^n)_M/S^{-\infty}(\Omega\times\mathbb{R}^n)_M,$$

the map (37) is an isomorphism. □

Remark 21. Pseudo-differential operators are known to be meaningful also on smooth manifolds. We skip for the moment the details and only note that a diffeomorphism $\chi:\Omega\to\widetilde{\Omega}$ between open sets $\Omega,\widetilde{\Omega}$ in $\mathbb{R}^n$ induces an operator push forward

$$\chi_*: L^{\mu}(\Omega)\to L^{\mu}(\widetilde{\Omega}) \tag{39}$$

defined by $\chi_*A=(\chi^*)^{-1}A\chi^*$ with χ^* denoting the function pull backs. (39) is an isomorphism and it restricts to an isomorphism between the respective spaces of classical operators. In addition there is a transformation rule via (37) between corresponding left symbols.

4. Operator-valued symbols

The *standard* concept of pseudo-differential operators admits various generalizations, motivated by corresponding applications. By *standard* we understand operators with symbols as in Definition 2 (i), also denoted by $S^{\mu}_{1,0}(\Omega \times \mathbb{R}^n)$. There are also the well-known classes $S^{\mu}_{\rho,\delta}(\Omega \times \mathbb{R}^n)$, e.g., for $0 < \delta \leq \rho \leq 1$, where the symbolic estimates (11) are generalized to replacing $(1 + |\xi|)^{\mu-|\beta|}$ by $(1 + |\xi|)^{\mu+\delta|\alpha|-\rho|\beta|}$. There is another generalization which appears in boundary value problems and also in operators on a manifold with singularities, say, edges, where the values of the symbols belong to $\mathscr{L}(H, \widetilde{H})$ for some Hilbert spaces H, $\widetilde{H}$ (or Fréchet spaces) which are endowed with groups of isomorphisms. We need a few aspects of this concept for boundary value problems below, and later on also for operators on manifolds with edge and corner singularity.

An elementary approach is to iterate pseudo-differential operators with symbols

$$p(x, y, \xi, \eta) \in S^{\mu}(\mathbb{R}^{n+q}_{x,y} \times \mathbb{R}^{n+q}_{\xi,\eta})$$

by applying first Op_x, *i.e.,* consider

$$\mathrm{Op}_x(p)(y, \eta) =: a(y, \eta)$$

which is a $(y, \eta)-$dependent family of operators in $L^{\mu}(\mathbb{R}^n)$, *i.e.,* operator-valued, and then pass to $\mathrm{Op}_y(a)$. Such an iteration makes sense, in particular, in boundary and transmission problems.

Assume for convenience that the symbol p is independent of x. Then

$$\mathrm{Op}_x(p)(y, \eta) : H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n)$$

is a family of continuous operators for every $s \in \mathbb{R}$.

Let us consider the group of isomorphisms

$$\kappa_{\lambda} : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n), \qquad \lambda \in \mathbb{R}_+,$$

defined by

$$\big(\kappa_{\lambda} u\big)(x) := \lambda^{\frac{n}{2}} u(\lambda x).$$

Lemma 22. *We have*

$$\|\kappa^{-1}_{\langle\eta\rangle} \mathrm{Op}_x(p)(y, \eta) \kappa_{\langle\eta\rangle}\|_{\mathscr{L}(H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n))} \leq c\langle\eta\rangle^{\mu}$$

for all $(y, \eta) \in M \times \mathbb{R}^n$, $M \Subset \mathbb{R}^n$, *and some constant* $c = c(M) > 0$. *Here* $\kappa^{-1}_{\langle\eta\rangle} = \big(\kappa_{\langle\eta\rangle}\big)^{-1}$.

Proof. First observe that for any $f(\xi) \in S^{\mu}(\mathbb{R}^n_\xi)$ and $s \in \mathbb{R}$ we have

$$\|\mathrm{Op}_x(f)u\|^2_{H^{s-\mu}(\mathbb{R}^n)} = \int \langle\xi\rangle^{2(s-\mu)}|f(\xi)\hat{u}(\xi)|^2 đ\xi$$
$$\leq \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-2\mu}|f(\xi)|^2 \|u\|^2_{H^s(\mathbb{R}^n)},$$

i.e.,

$$\|\mathrm{Op}_x(f)u\|^2_{\mathscr{L}(H^s(\mathbb{R}^n),H^{s-\mu}(\mathbb{R}^n))} \leq \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-2\mu}|f(\xi)|^2.$$

Moreover, we have

$$\kappa^{-1}_{\langle\eta\rangle}\mathrm{Op}_x(p)(y,\eta)\kappa_{\langle\eta\rangle}u(x) = \iint e^{i(\langle\eta\rangle^{-1}x-x')\cdot\xi} p(y,\xi,\eta)u(\langle\eta\rangle x)\,dx'\,đ\xi$$
$$= \iint e^{i(x-\tilde{x})\cdot\langle\eta\rangle^{-1}\xi} p(y,\xi,\eta)u(\tilde{x})\,d\tilde{x}\,\langle\eta\rangle^{-n}đ\xi$$
$$= \iint e^{i(x-\tilde{x})\cdot\tilde{\xi}} p(y,\langle\eta\rangle\tilde{\xi},\eta)u(\tilde{x})\,d\tilde{x}đ\tilde{\xi}.$$

Thus, for $f(y,\tilde{\xi},\eta) = p(y,\langle\eta\rangle\tilde{\xi},\eta)$ from the first part of the proof we obtain

$$\|\kappa^{-1}_{\langle\eta\rangle}\mathrm{Op}_x(p)(y,\eta)\kappa_{\langle\eta\rangle}\|^2_{\mathscr{L}(H^s(\mathbb{R}^n),H^{s-\mu}(\mathbb{R}^n))} = \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-2\mu}|p(y,\langle\eta\rangle\xi,\eta)|^2$$
$$\leq c \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-2\mu}\langle\langle\eta\rangle\xi,\eta\rangle^{2\mu}$$
$$\leq c\langle\eta\rangle^{2\mu}.$$

In the latter estimate we employed the relation

$$\langle\langle\eta\rangle\xi,\eta\rangle = \langle\xi\rangle\langle\eta\rangle.$$

□

Corollary 23. *Because of*

$$D^\alpha_y D^\beta_\eta p(y,\xi,\eta) \in S^{\mu-|\beta|}(\Omega\times\mathbb{R}^{n+q}_{\xi,\eta}),$$

it follows that

$$\left\|\kappa^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta \mathrm{Op}_x(p)(y,\eta)\}\kappa_{\langle\eta\rangle}\right\|^2_{\mathscr{L}(H^s(\mathbb{R}^n),H^{s-\mu}(\mathbb{R}^n))} \leq c\langle\eta\rangle^{\mu-|\beta|}$$

for all $(y,\eta) \in M\times\mathbb{R}^n$, $M \Subset \Omega$, *and* α, $\beta \in (\mathbb{Z}_+)^n$, *for constants* $c = c(\alpha,\beta,M) > 0$.

Corollary 23 is a motivation for introducing operator-valued symbols with twisted symbolic estimates. Those play an important role in the pseudo-differential calculus on manifolds with edges and higher singularities, but also as boundary amplitude functions in boundary value problems, to be studied in Section 5 below.

The spaces $H := H^s(\mathbb{R}^n)$ are one example of Hilbert spaces with group action in the following sense.

Definition 24. A (separable) Hilbert space H is said to be endowed with a group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ if

$$\kappa_\lambda : H \to H$$

is a family of isomorphisms,

$$\kappa_\lambda \kappa_\delta = \kappa_{\lambda\delta} \qquad \text{for every} \qquad \lambda, \delta \in \mathbb{R}_+,$$

and $\lambda \to \kappa_\lambda h$ defines an element of $C(\mathbb{R}_+, H)$ for every $h \in H$.

Theorem 25. *If H is a Hilbert space with group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, then there are constants $c, N > 0$ such that*

$$\|\kappa_\lambda\|_{\mathscr{L}(H)} \leq c\big(\max\{\lambda, \lambda^{-1}\}\big)^N$$

for all $\mathbb{R}_+$.

Concerning the proof, see the paper [9].

Definition 26. Let H and $\widetilde{H}$ be Hilbert spaces with group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\widetilde{\kappa} = \{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Then for any open $\Omega \subseteq \mathbb{R}^p$ and $\mu \in \mathbb{R}$ by

$$S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H}) \tag{40}$$

we denote the set of all $a(y,\eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathscr{L}(H, \widetilde{H}))$ such that

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathscr{L}(H,\widetilde{H})} \leq c\langle\eta\rangle^{\mu-|\beta|} \tag{41}$$

for all $(y,\eta) \in M \times \mathbb{R}^q, M \Subset \Omega, \alpha \in (\mathbb{Z}_+)^p, \beta \in (\mathbb{Z}_+)^q$, for constants $c = c(\alpha, \beta, M) > 0$.

The estimates (41) are also referred to as twisted symbolic estimates. Clearly for $H = \widetilde{H} = \mathbb{C}$ and $\kappa_\lambda = \mathrm{id}_{\mathbb{C}}, \tilde{\kappa}_\lambda = \mathrm{id}_{\mathbb{C}}$ for all $\lambda \in \mathbb{R}_+$ we recover Definition 2 (i). The space (40) depends on the choice of κ and $\tilde{\kappa}$. If necessary we also write $S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})_{\kappa,\tilde{\kappa}}$ rather than $S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$.

We also have subspaces of classical symbols. Those are based on twisted homogeneity. Let

$$S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H})$$

be the set of all $a_{(\mu)}(y,\eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathscr{L}(H, \widetilde{H}))$ such that

$$a_{(\mu)}(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a_{(\mu)}(y,\eta)\kappa^{-1}_\lambda$$

for all $\lambda \in \mathbb{R}_+$.

Definition 27. $S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;H,\widetilde{H})$ is the subspace of all $a(y,\eta)\in S^{\mu}(\Omega\times\mathbb{R}^q;H,\widetilde{H})$ such that there are $a_{(\mu-k)}(y,\eta)\in S^{(\mu-k)}(\Omega\times(\mathbb{R}^q\setminus\{0\});H,\widetilde{H})$, such that for any excision function $\chi(\eta)$

$$a(y,\eta)-\sum_{k=0}^{N}\chi(\eta)a_{(\mu-k)}(y,\eta)\in S^{\mu-(N+1)}(\Omega\times\mathbb{R}^q;H,\widetilde{H})$$

for all $N\in\mathbb{Z}_+$.

Example 2.

(i) Let $\mathbf{e}^+:L^2(\mathbb{R}_+)\to L^2(\mathbb{R})$ be the operator of extension by zero to $\mathbb{R}_-$, *i.e.*,

$$\mathbf{e}^+u=\begin{cases}u & \text{on} \quad \mathbb{R}_+,\\ 0 & \text{on} \quad \mathbb{R}_-.\end{cases}$$

Then, for

$$(\kappa_\lambda u)(r)=\lambda^{\frac{1}{2}}u(\lambda r),\qquad \lambda\in\mathbb{R}_+,$$

$u\in L^2(\mathbb{R}_+)$ (or $u\in L^2(\mathbb{R})$) we have

$$\mathbf{e}^+u\in S^{(0)}(\mathbb{R}^q;L^2(\mathbb{R}_+),L^2(\mathbb{R}))$$

for any $q\in\mathbb{N}$. At the same time we have $\mathbf{e}^+\in S^0_{\mathrm{cl}}(\mathbb{R}^q;L^2(\mathbb{R}_+),L^2(\mathbb{R}))$.

(ii) Let $\mathbf{r}^+:H^s(\mathbb{R})\to H^s(\mathbb{R}_+)$ be the operator of restriction to $\mathbb{R}_+$. Then, again for $\kappa_\lambda u(r)=\lambda^{\frac{1}{2}}u(\lambda r)$, $\lambda\in\mathbb{R}_+$, we have

$$\mathbf{r}^+\in S^{(0)}(\mathbb{R}^q\setminus\{0\};H^s(\mathbb{R}),H^s(\mathbb{R}_+))\qquad\text{and}\qquad \mathbf{r}^+\in S^0_{\mathrm{cl}}(\mathbb{R}^q;H^s(\mathbb{R}),H^s(\mathbb{R}_+))$$

for every $s\in\mathbb{R}$.

(iii) Let $\mathbf{r}':H^s(\mathbb{R})\to\mathbb{C}$ for $s>\frac{1}{2}$ be the operator of restriction $u\to u(0)$. Then for $\widetilde{\kappa}_\lambda$ on $H^s(\mathbb{R})$ as in (ii) and $\kappa_\lambda=\mathrm{id}_{\mathbb{C}}$ we have

$$\mathbf{r}'\in S^{(\frac{1}{2})}(\mathbb{R}^q\setminus\{0\};H^s(\mathbb{R}),\mathbb{C})\qquad\text{and}\qquad \mathbf{r}'\in S^{\frac{1}{2}}_{\mathrm{cl}}(\mathbb{R}^q;H^s(\mathbb{R}),\mathbb{C}).$$

More generally, the operator

$$\mathbf{r}'\frac{d^j}{dr^j}:H^s(\mathbb{R})\to\mathbb{C}\qquad\text{for}\qquad s-j>\frac{1}{2}$$

defines elements

$$\mathbf{r}'\frac{d^j}{dr^j}\in S^{(j+\frac{1}{2})}(\mathbb{R}^q\setminus\{0\};H^s(\mathbb{R}),\mathbb{C})\qquad\text{and}\qquad \mathbf{r}'\frac{d^j}{dr^j}\in S^{j+\frac{1}{2}}_{\mathrm{cl}}(\mathbb{R}^q;H^s(\mathbb{R}),\mathbb{C}).$$

Remark 28. Example 2 shows that twisted homogeneity of some order is possible even when the mappings in question do not depend on η at all.

In our calculus we will employ many other examples of operator-valued symbols.

An important generalization concerns Fréchet spaces E, $\widetilde{E}$ rather than Hilbert spaces with group action.

A Fréchet space E written as a projective limit

$$E = \varprojlim_{j\in\mathbb{Z}_+} E^j$$

of Hilbert spaces E^j, where we assume continuous embeddings $E^j \hookrightarrow E^0$ for all j is said to be endowed with a group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ if κ is a group action on E^0 and $\kappa|_{E^j}$ a group action on E^j for every j.

Example 3. The space $\mathscr{S}(\mathbb{R}_+) := \mathscr{S}(\mathbb{R})|_{\mathbb{R}_+}$ is Fréchet as a projective limit

$$\mathscr{S}(\mathbb{R}_+) = \varprojlim_{j\in\mathbb{Z}_+} H^{j,j}(\mathbb{R}_+)$$

where $H^{j,j}(\mathbb{R}_+) := \langle r\rangle^{-j} H^j(\mathbb{R}_+)$, $H^s(\mathbb{R}_+) := H^s(\mathbb{R})|_{\mathbb{R}_+}$, and the above-mentioned $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a group action on $\mathscr{S}(\mathbb{R}_+)$.

In connection with boundary value problems in a domain of dimension n with smooth boundary, locally described $\mathbb{R}^{n-1}\times\overline{\mathbb{R}}_+$, the following kind of operator-valued symbols is of interest. In the following for abbreviation we often write $q := n-1$.

Definition 29. A $g(y,\eta) \in C^\infty(\mathbb{R}^q\times\mathbb{R}^q, \mathscr{L}(L^2(\mathbb{R}_+), L^2(\mathbb{R}_+)))$ is called a Green symbol of order $\mu\in\mathbb{R}$ if g and g^* (the (y,η)-wise adjoint) are symbols $g(y,\eta)$, $g^*(y,\eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q\times\mathbb{R}^q; L^2(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$. In this notation

$$S^\mu_{\mathrm{cl}}(\mathbb{R}^q\times\mathbb{R}^q; L^2(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+)) := \varprojlim_{j\in\mathbb{Z}_+} S^\mu_{\mathrm{cl}}(\mathbb{R}^q\times\mathbb{R}^q; L^2(\mathbb{R}_+), H^{j,j}(\mathbb{R}_+)).$$

Observe that for Green symbols $g_j(y,\eta)$, $j = 0,\dots,d$, of order $\mu - j$ we have

$$g(y,\eta) = \sum_{j=0}^{d} g_j(y,\eta)\Big(\frac{\partial}{\partial r}\Big)^j \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q\times\mathbb{R}^q; H^s(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+)) \tag{42}$$

for every $j\in\mathbb{Z}_+$, $s > d - \frac{1}{2}$.

Pseudo-differential boundary value problems with the transmission property at the boundary locally in the half-space $\overline{\mathbb{R}}^n_+ \ni (y,r)$, $y\in\mathbb{R}^q$, $r\in\overline{\mathbb{R}}_+$, are formulated in terms of operator-valued symbols

$$a(y,\eta) := \mathbf{r}^+\mathrm{Op}_r(p)(y,\eta)\mathbf{e}^+ + g(y,\eta) \tag{43}$$

where $p(x,\xi)$ is a classical symbol in $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$ of order $\mu \in \mathbb{Z}$ with the transmission property at the boundary. Here we employ the splitting of variables $x = (y,r)$ and coverables $\xi = (\eta,\rho)$. The symbols (43) belong to $S^{\mu}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+))$ for every $s > d - \frac{1}{2}$. Boundary conditions of trace and potential type are encoded by symbols $t(y,\eta)$, $k(y,\eta)$, $q(y,\eta)$ where $t(y,\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; H^s(\mathbb{R}_+), \mathbb{C})$, for $s > d - \frac{1}{2}$,

$$t(y,\eta)u = \sum_{j=0}^{d-1} b_j(y,\eta)\gamma^j u \tag{44}$$

for $\gamma^j u := \left(\frac{\partial}{\partial r}\right)^j u\Big|_{r=0}$, $b_j(y,\eta) \in S^{\mu-j-\frac{1}{2}}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q)$, $k(y,\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; \mathbb{C}, \mathscr{S}(\mathbb{R}_+))$, and $g(y,\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$.

More generally operator-valued boundary symbols of pseudo-differential boundary value problems with the transmission property are block matrix symbols of the form

$$\mathbf{a}(y,\eta) := \begin{pmatrix} a(y,\eta) & k(y,\eta) \\ t(y,\eta) & q(y,\eta) \end{pmatrix} \in S^{\mu}\big(\mathbb{R}^q \times \mathbb{R}^q; H^s(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}, H^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C}^{j_+}\big) \tag{45}$$

where $k = (k_1, \ldots, k_{j_-})$, $t = {}^{\mathrm{t}}(t_1, \ldots, t_{j_+})$ $q = (q_{ij})_{i=1,\ldots,j_+,\, j=1,\ldots,j_-}$, where the components k_i, t_j, q_{ij}, are of the above-mentioned type.

The associated operators $\mathrm{Op}_y(\cdot)$ then describe the corresponding boundary value problems themselves, locally near the boundary. Far from the boundary they contain local contributions from symbols in $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)$. Globally the operators are described by such local operators, combined with charts and a subordinate partition of unity. This global aspect is not the main issue of the present discussion. Instead we point out the nature of $\overline{\mathbb{R}}^n_+ \ni x$ as a non-compact manifold with boundary $\partial\overline{\mathbb{R}}^n_+ = \mathbb{R}^{n-1} \ni x'$, and with conical exit to infinity, *i.e.*, $|x| \to \infty$ and $|x'| \to \infty$, respectively. Similarly as in Definition 2 (ii) and Definition 3 (ii) we need a variant of the operator-valued boundary symbols for an extra weight $\nu \in \mathbb{R}$ at infinity. For $p(x,\xi)$ with the transmission property with control of weight $\nu \in \mathbb{R}$ at infinity we impose the condition

$$p(x,\xi) \in S^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)\Big|_{\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n}$$

and we have analogue of the three principal symbolic components (16), namely,

$$\sigma(p) = \big(\sigma_\psi(p), \sigma_{\mathrm{e}}(p), \sigma_{\psi,\mathrm{e}}(p)\big).$$

Similar triples appear for the operator-valued boundary symbols, namely, for (45)

$$\sigma'(\mathbf{a}) = \big(\sigma_\partial(\mathbf{a}), \sigma_{\mathrm{e}'}(\mathbf{a}), \sigma_{\partial,\mathrm{e}'}(\mathbf{a})\big). \tag{46}$$

where $\sigma_\partial(\mathbf{a})$ is the twisted homogeneous principal part of $\mathbf{a}$ in the standard sense. Moreover, in order to describe the remaining two components of (46), we first look at $g(y,\eta)$ in Definition

29. In this case we first consider (42) in the exit variant of weight $\nu \in \mathbb{R}$ by assuming $g_j(y,\eta) \in S^{\mu-j;\nu}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; H^s(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$, $j = 0,\dots,d$, and then, analogously as in the scalar case we have the symbols

$$\sigma_{\mathrm{e}'}(g_j)(y,\eta), \qquad \sigma_{\partial,\mathrm{e}'}(g_j)(y,\eta)$$

which yield $\sigma_{\mathrm{e}'}(g)$ and $\sigma_{\partial,\mathrm{e}'}(g)$, respectively. For $t(y,\eta)$, $k(y,\eta)$ and $q(y,\eta)$ we have an easier relationship to notion of (16). First for the exit analogue of (44) we assume $b_j(y,\eta) \in S^{\mu-j-\frac{1}{2};\nu}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q)$. Then according to (16) we have the a corresponding triple of symbols, now, because of the relationship to the boundary denoted by

$$\sigma'(b_j) = \big(\sigma_\partial(b_j), \sigma_{\mathrm{e}'}(b_j), \sigma_{\partial,\mathrm{e}'}(b_j)\big)$$

where, for instance, $\sigma_\partial(b_j)$ is the homogeneous principal symbol of b_j in $(y,\eta) \in \mathbb{R}^q \times (\mathbb{R}^q \setminus \{0\})$ of order $\mu - j - 1/2$, etc., and then

$$\sigma'(t) = \big(\sigma_\partial(t), \sigma_{\mathrm{e}'}(t), \sigma_{\partial,\mathrm{e}'}(t)\big)$$

is defined by

$$\begin{aligned}
\sigma_\partial(t)(y,\eta) &= \sum_{j=0}^{d-1} \sigma_\partial(b_j)(y,\eta)\gamma^j, \\
\sigma_{\mathrm{e}'}(t)(y,\eta) &= \sum_{j=0}^{d-1} \sigma_{\mathrm{e}'}(b_j)(y,\eta)\gamma^j, \\
\sigma_{\partial,\mathrm{e}'}(t)(y,\eta) &= \sum_{j=0}^{d-1} \sigma_{\partial,\mathrm{e}'}(b_j)(y,\eta)\gamma^j.
\end{aligned}$$

Moreover, the exit version of the potential symbols $k(y,\eta)$ of weight $\nu \in \mathbb{R}$ has the form

$$k(y,\eta) \in S^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; \mathbb{C}, \mathscr{S}(\overline{\mathbb{R}}_+))$$

where $\sigma_\partial(k)(y,\eta)$ is the twisted homogeneous principal symbol of order μ for $(y,\eta) \in \mathbb{R}^q \times (\mathbb{R}^q \setminus \{0\})$, while $\sigma_{\mathrm{e}'}(k)(y,\eta)$ and $\sigma_{\partial,\mathrm{e}'}(k)(y,\eta)$ are defined in an analogous manner as before. Concerning the symbol $q(y,\eta)$ in the lower right corner we assume

$$q(y,\eta) \in S^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q),$$

and the components coming from (16) are now denoted by

$$\sigma'(q) = \big(\sigma_\partial(q), \sigma_{\mathrm{e}'}(q), \sigma_{\partial,\mathrm{e}'}(q)\big).$$

5. Boundary value problems

In this section, we are going to discuss pseudo-differential calculus for boundary value problems. Let us start with a special example of an elliptic boundary value problem, namely, a Dirichlet problem

$$\mathscr{A}_1 = \begin{pmatrix} d-\Delta \\ \mathbf{r}' \end{pmatrix} : H^s(\mathbb{R}^n_+) \to \begin{matrix} H^{s-2}(\mathbb{R}^n_+) \\ \oplus \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix},$$

with a fixed constant $d > 0$. Compared with the operator $1-\Delta$ at the very beginning we admit here $d-\Delta$, $d > 0$. For convenience we pass to

$$\mathscr{A}_2 = \begin{pmatrix} d-\Delta \\ Q\mathbf{r}' \end{pmatrix} : H^s(\mathbb{R}^n_+) \to \begin{matrix} H^{s-2}(\mathbb{R}^n_+) \\ \oplus \\ H^{s-2}(\mathbb{R}^{n-1}) \end{matrix}, \tag{47}$$

where Q is an order reduction on the boundary that we take of the form $Q = \mathrm{Op}_y(\langle\eta\rangle^{\frac{3}{2}})$ such that

$$Q : H^s(\mathbb{R}^{n-1}) \to H^{s-\frac{3}{2}}(\mathbb{R}^{n-1})$$

is an isomorphism for all $s \in \mathbb{R}$.

In contrast to operators in $\mathbb{R}^n$ (or on an open subset) in boundary value problems we take into account from the very beginning additional trace operators, e.g., extra entries in a column matrix. Compositions, ellipticity, and parametrices give rise to general block 2×2 matrix operators of the form

$$\mathscr{A} = \begin{pmatrix} A+G & K \\ T & Q \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \oplus \\ H^s(\mathbb{R}^{n-1}, \mathbb{C}^{j_1}) \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}^n_+) \\ \oplus \\ H^{s-\mu}(\mathbb{R}^{n-1}, \mathbb{C}^{j_2}) \end{matrix}, \tag{48}$$

where we admit classical pseudo-differential operators A of order $\mu \in \mathbb{Z}$ with the transmission property at the boundary and trace operators T, potential operators K and $j_2 \times j_1$ block matrices of classical pseudo-differential operators Q in the lower left corner. In addition, as we shall see below, the class of such 2×2 block matrix operators also produces so-called Green operators G in the upper left corner. Those arise in compositions and also in forming inverses or parametrices of elliptic elements.

Another essential effect is that the operators acquire from the boundary a second principal symbolic component, namely, apart from σ_ψ the so-called boundary symbol σ_∂. Finally, if we are interested in ellipticity in the half-space $\mathbb{R}^n_+$ with a control of solvability up to $|x| \to \infty$, we again observe effects from the conical exit of the configuration up to infinity. Similarly as in the introduction, this causes the presence of extra exit symbols, not only the ones in the interior, namely, σ_ψ, σ_{e}, $\sigma_{\psi,\mathrm{e}}$, but also on the boundary σ_∂, $\sigma_{\mathrm{e}'}$, $\sigma_{\partial,\mathrm{e}'}$, as we see below. In any case, on $\overline{\mathbb{R}}^n_+$ as a special manifold with boundary, here with conical exit to infinity, we will talk about B^μ as a substitute of L^μ, used for the spaces of pseudo-differential operators in the case without boundary.

Definition 30. By $B^\mu(\overline{\mathbb{R}}^n_+)$ we denote the space of all operators (48) where the operator A in the upper left corner is an element of $L^\mu_{\mathrm{cl}}(\mathbb{R}^n_+)$ with the transmission property at the boundary, G is a Green operator, T a trace and K a potential operator in Boutet de Monvel's calculus. Concerning details we refer to [1] or [8]. For any case G and T are asked to be of type $d \in \mathbb{Z}_+$, see the explanations below. Moreover, since $\overline{\mathbb{R}}^n_+$ is regarded as a manifold with boundary and conical exit to infinity, we assume that the symbols of the involved operators are in weighted classes, e.g., $A \in L^\mu(\mathbb{R}^n_+)$ has a symbol in $S^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)|_{\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n}$ for some $\nu \in \mathbb{R}$, and also the other operators in (48) contain the weight ν (which is kept in mind but suppressed in notation).

Then we have $\mathscr{A}_2 \in B^2(\overline{\mathbb{R}}^n_+)$ for $\nu = 0$. It is the goal of this section is to show that (47) is an isomorphism for all $s > \frac{3}{2}$ and to construct the inverse. We have for $x = (y, r)$ and $\xi = (\eta, \rho)$

$$\sigma_\psi(\mathscr{A}_2) = |\xi|^2, \qquad \sigma_e(\mathscr{A}_2) = d + |\xi|^2, \qquad \sigma_{\psi,e}(\mathscr{A}_2) = |\xi|^2,$$

cf., notation in the introduction, in particular formula (5), or Remark 4;

$$\sigma_\partial(\mathscr{A}_2) = \begin{pmatrix} |\eta|^2 - \partial_r^2 \\ |\eta|^{\frac{3}{2}}\mathbf{r}' \end{pmatrix}, \quad \sigma_{e'}(\mathscr{A}_2) = \begin{pmatrix} d + |\eta|^2 - \partial_r^2 \\ \langle\eta\rangle^{\frac{3}{2}}\mathbf{r}' \end{pmatrix}, \quad \sigma_{\partial,e'}(\mathscr{A}_2) = \begin{pmatrix} |\eta|^2 - \partial_r^2 \\ |\eta|^{\frac{3}{2}}\mathbf{r}' \end{pmatrix}.$$

The latter triple of symbols coming from the boundary have been defined in general form at the end of the preceding section, here for the case $\nu = 0$. Hence $\mathscr{A}_2$ is elliptic in the sense of the symbolic triples $(\sigma_\psi, \sigma_e, \sigma_{\psi,e})$ and $(\sigma_\partial, \sigma_{e'}, \sigma_{\partial,e'})$. First we invert the operator family

$$\begin{pmatrix} \alpha^2 - \partial_r^2 \\ \beta\mathbf{r}' \end{pmatrix} : \mathscr{S}(\mathbb{R}_+) \to \begin{matrix} \mathscr{S}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}, \tag{49}$$

where $\alpha^2 := (d + |\eta|^2)^{\frac{1}{2}}$, $\beta := \langle\eta\rangle^{\frac{3}{2}}$. Let us write

$$\ell_\pm(\rho) = \alpha \pm i\rho.$$

Then we have $\ell_-(\rho)\ell_+(\rho) = \alpha^2 + \rho^2$ and

$$\alpha^2 - \partial_r^2 = \mathrm{Op}^+\big(\ell_-\ell_+\big) = \mathrm{Op}^+\big(\ell_-\big)\mathrm{Op}^+\big(\ell_+\big)$$

where

$$\mathrm{Op}^+(b) := \mathbf{r}^+\mathrm{Op}_r(b)\mathbf{e}^+$$

for any symbol $b(r,\rho) \in S^{\mu}(\mathbb{R}\times\mathbb{R})$. The latter identity is true since $\ell_-(\rho)$ is a minus function;

$$\mathrm{Op}^+\big(\ell_-\big) : \mathscr{S}(\mathbb{R}_+) \to \mathscr{S}(\mathbb{R}_+)$$

is an isomorphism. Thus in order to invert (49), it suffices to consider

$$\begin{pmatrix} \mathrm{Op}^+\big(\ell_+\big) \\ \beta\mathbf{r}' \end{pmatrix} : \mathscr{S}(\mathbb{R}_+) \to \begin{matrix} \mathscr{S}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}$$

which is an isomorphism, because $\mathrm{Op}^+\big(\ell_+\big) : \mathscr{S}(\mathbb{R}_+) \to \mathscr{S}(\mathbb{R}_+)$ is surjective, and $\beta\mathbf{r}'$ induces an isomorphism of kernel: $\ker\Big(\mathrm{Op}^+\big(\ell_+\big)\Big) = \big\{\gamma e^{\alpha r} : \gamma \in \mathbb{C}\big\}$ to $\mathbb{C}$.

Now let us form the potential $v = v(\alpha) : \mathbb{C} \to \mathscr{S}(\mathbb{R}_+)$, defined by $v\gamma = \gamma\beta^{-1}e^{-\alpha r}$, $\gamma \in \mathbb{C}$. Then

$$\begin{pmatrix} \mathrm{Op}^+\big(\ell_+\big) \\ \beta\mathbf{r}' \end{pmatrix} \Big(\mathrm{Op}^+\big(\ell_+^{-1}\big) \;\; v \Big) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

because $\mathbf{r}' \circ \mathrm{Op}^+\big(\ell_+^{-1}\big) = 0$, and hence

$$\begin{pmatrix} \mathrm{Op}^+\big(\ell_+\big) \\ \beta\mathbf{r}' \end{pmatrix}^{-1} = \Big(\mathrm{Op}^+\big(\ell_+^{-1}\big) \;\; v \Big).$$

Consider now $a(\rho) = \alpha^2 + \rho^2$. The operator in (49) can be written in the form

$$\begin{pmatrix} \mathrm{Op}^+\big(a\big) \\ \beta\mathbf{r}' \end{pmatrix} = \begin{pmatrix} \mathrm{Op}^+\big(\ell_-\big) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Op}^+\big(\ell_+\big) \\ \beta\mathbf{r}' \end{pmatrix}.$$

This implies

$$\begin{pmatrix} \mathrm{Op}^+\big(a\big) \\ \beta\mathbf{r}' \end{pmatrix}^{-1} = \Big(\mathrm{Op}^+\big(\ell_+^{-1}\big) \;\; v \Big) \begin{pmatrix} \mathrm{Op}^+\big(\ell_-^{-1}\big) & 0 \\ 0 & 1 \end{pmatrix} = \Big(\mathrm{Op}^+\big(\ell_+^{-1}\big)\mathrm{Op}^+\big(\ell_-^{-1}\big) \;\; v \Big),$$

and we have

$$\mathrm{Op}^+\big(\ell_+^{-1}\big)\mathrm{Op}^+\big(\ell_-^{-1}\big) = \mathrm{Op}^+\big(a^{-1}\big) + g$$

with a certain Green's operator g. We obtain for the inverse of (49)

$$\begin{pmatrix} \mathrm{Op}^+\big(a\big) \\ \beta\mathbf{r}' \end{pmatrix}^{-1} = \Big(\mathrm{Op}^+\big(a^{-1}\big) + g \;\; v \Big).$$

Inserting now the expressions for $\alpha = \alpha(\eta)$, $\beta = \beta(\eta)$, we easily see that the ingredients of

$$\sigma_{e'}(\mathscr{A}_2)^{-1}(\eta) = \Big(\mathrm{Op}^+\big(a^{-1}\big)(\eta) + g(\eta) \;\; v(\eta) \Big) \tag{50}$$

belong to the symbolic structure of $B^{-2}(\overline{\mathbb{R}}^n_+)$ for $v = 0$ (they are, of course, independent of y), and we obviously have

$$\mathscr{A}_2^{-1} = \mathrm{Op}\big(\sigma_{e'}(\mathscr{A}_2)^{-1}\big) =: \begin{pmatrix} P & K \end{pmatrix} \in B^{-2}(\overline{\mathbb{R}}^n_+).$$

The method of calculating (50) gives us explicit expressions also for $\sigma_\partial(\mathscr{A}_2)^{-1}$ and $\sigma_{\partial,e'}(\mathscr{A}_2)^{-1}$, where

$$\sigma(\mathscr{A}_2)^{-1} = \big(|\xi|^{-2}, (d+|\xi|^2)^{-1}, |\xi|^{-2}; \sigma_\partial(\mathscr{A}_2)^{-1}, \sigma_{e'}(\mathscr{A}_2)^{-1}, \sigma_{\partial,e'}(\mathscr{A}_2)^{-1}\big).$$

Finally, we obtain

$$\mathscr{A}_1^{-1} = \begin{pmatrix} P & KQ^{-1} \end{pmatrix} \in B^{-2}(\overline{\mathbb{R}}^n_+).$$

Remark 31. Similar arguments apply to the Neumann problem for $d - \Delta$ in Ω, with $\mathbf{r}'\partial_r$ in place of $\mathbf{r}'$. To obtain an element with unified orders we can pass to the boundary operator $R\mathbf{r}'\partial_r$ for $R = \mathrm{Op}_y(\langle\eta\rangle^{\frac{1}{2}})$. We see that

$$\begin{pmatrix} d - \Delta \\ R\mathbf{r}'\partial_r \end{pmatrix} \in B^2(\overline{\mathbb{R}}^n_+) \tag{51}$$

is also elliptic in the sense of the symbolic triples $(\sigma_\psi, \sigma_e, \sigma_{\psi,e})$ and $(\sigma_\partial, \sigma_{e'}, \sigma_{\partial,e'})$, and even invertible as an operator $H^s(\mathbb{R}^n_+) \to H^{s-2}(\mathbb{R}^n_+) \oplus H^{s-2}(\mathbb{R}^n_+)$ for $s > \frac{3}{2}$. The inverse belongs to $B^{-2}(\overline{\mathbb{R}}^n_+)$.

Remark 32. Of course we may discuss the Dirichlet and Neumann problems also on a bounded domain $\Omega \Subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Then the aspects at conical exits to infinity disappear, and the symbols occurring in the description of operators as well of parametrices are those as in the first part of Section 4.

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