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HARMONIC HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper we try to uncover some of the inequalities associating hypergeometric functions with planer harmonic mappings. Sharp coefficient relations, distortion theorems and neighborhood are given for these functions. Furthermore, convolution products are considered.

1. Introduction

Let U denote the open unit disc and S_H denote the class of functions which are complex-valued, harmonic, univalent, sense-preserving in U normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_H$ can be expressed as $f = h + \overline{g}$, where h and g are analytic in U. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ in U. Thus for $f = h + \overline{g} \in S_H$ we may write

$$h(z) = z + \sum_{k=2}^{\infty} A_k z^k,\ g(z) = \sum_{k=1}^{\infty} B_k z^k,\ |B_1| < 1. \tag{1.1}$$

Clunie and Sheil-Small [4] studied the class S_H with some geometric subclasses of S_H.

Let $HP(\beta)$ denote the subclass of S_H satisfying $\text{Re}\{h'(z) + g'(z)\} > \beta, \quad 0 \le \beta < 1$ which was studied by Yalçin et al. [8]; they also denoted by $HP^*(\beta)$ the subclass of $HP(\beta)$ such that the functions h and g in $f = h + \overline{g}$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |A_k| z^k \ \text{ and } \ g(z) = -\sum_{k=1}^{\infty} |B_k| z^k. \tag{1.2}$$

If $f_j = h_j + \overline{g}_j, \quad j = 1, 2, \ldots$ are in the class S_H, then we define the convolution $f_1 * f_2$ of f_1 and f_2 in the natural way $h_1 * h_2 + \overline{g_1 * g_2}$. If ϕ_1 and ϕ_2 are analytic and $f = h + \overline{g}$ is in S_H, Ahuja and Silverman [6] defined

$$f \,\tilde{*}\, (\phi_1 + \overline{\phi}_2) = h * \phi_1 + \overline{g * \phi_2}. \tag{1.3}$$

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Let a, b, c be complex numbers with $c \neq 0, -1, -2, \ldots$ Then the Gauss hypergeometric function written as ${}_2F_1\,(a, b; c; z)$ or simply as $F(a, b; c; z)$ is defined by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k}\, z^k, \tag{1.4}$$

where $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+k-1) \quad \text{for} \quad k = 1, 2, 3, \ldots, \quad \text{and} \quad (\lambda)_0 = 1. \tag{1.5}$$

Since the hypergeometric series in (1.4) converges absolutely in U, it follows that $F(a, b; c; z)$ defines a function which is analytic in U, provided that c is neither zero nor a negative integer. In terms of Gamma functions, we are led to the well-known Gauss's summation theorem: If $\operatorname{Re}(c - a - b) > 0$, then

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\,, \quad c \neq 0, -1, -2, \ldots \tag{1.6}$$

For further information about hypergeometric functions, one may refer to [1], [2], and [3].

Throughout this paper, let $G(z) := \phi_1(z) + \overline{\phi_2(z)}$ be a function where $\phi_1(z) \equiv \phi_1(a_1, b_1; c_1; z)$ and $\phi_2(z) \equiv \phi_2(a_2, b_2; c_2; z)$ are the hypergeometric functions defined by

$$\phi_1(z) = zF(a_1, b_1; c_1; z) = z + \sum_{k=2}^{\infty} \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}}\, z^k, \tag{1.7}$$

$$\phi_2(z) = zF(a_2, b_2; c_2; z) - 1 = \sum_{k=1}^{\infty} \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k}\, z^k, \quad a_2 b_2 < c_2,\ a_j, b_j, c_j \text{ are positive for } j = 1, 2. \tag{1.8}$$

The purpose of this paper is to uncover some of the connections between the theory of harmonic univalent functions and hypergeometric functions. We will investigate the convolution multipliers $f \,\widetilde{*}\, (\phi_1 + \overline{\phi}_2)$ and the neighborhood of $G = \phi_1 + \overline{\phi}_2$, where ϕ_1, ϕ_2 are as defined by (1.7), (1.8) and $f \in HP(\beta)$. Also, convolution products are considered.

2. Preliminary Results

In order to derive new results, we need the following lemmas due to Yalçin et al. [8].

Lemma 2.1. *For $f = h + \overline{g}$ with h and g of the form* (1.1), *if*

$$\sum_{k=1}^{\infty} k(|A_k| + |B_k|) \leq 2 - \beta, \tag{2.1}$$

where $A_1 = 1$ *and* $0 \le \beta < 1$, *then* $f \in HP(\beta)$.

Lemma 2.2. *For* $f = h + \overline{g}$ *with* h *and* g *of the form* (1.2). *Then* $f \in HP^*(\beta)$ *if and only if*

$$\sum_{k=1}^{\infty} k(|A_k|) + |B_k|) \le 2 - \beta.$$

where $A_1 = 1$ *and* $0 \le \beta < 1$.

Lemma 2.3. *If* $f \in HP^*(\beta)$, *then*

$$|f(z)| \le (1 + |B_1|)r + \frac{1}{2}(1 - |B_1| - \beta)r^2, \quad |z| = r < 1,$$

and

$$|f(z)| \ge (1 - |B_1|)r - \frac{1}{2}(1 - |B_1| - \beta)r^2, \quad |z| = r < 1.$$

Hence

$$\{\omega : |\omega| < \frac{1}{2}(1 - |B_1| + \beta)\} \subset f(U).$$

3. Main Results

Theorem 3.1. *If* $c_j > a_j + b_j + 1$ *for* $j = 1, 2$, *then a sufficient condition for* $G = \phi_1 + \overline{\phi}_2$ *to be harmonic univalent in* U *and* $G \in HP(\beta)$, $\quad 0 \le \beta < 1$ *is that*

$$\left(1 + \frac{a_1 b_1}{c_1 - a_1 - b_1 - 1}\right) F(a_1, b_1; c_1; 1) + \frac{a_2 b_2}{c_2 - a_2 - b_2 - 1} F(a_2, b_2; c_2; 1) \le 2 - \beta. \quad (3.1)$$

Proof. In order to prove that G is locally univalent and sense-preserving in U, we only need to show that $|\phi_1'(z)| > |\phi_2'(z)|, \quad z \in U$. In view of (1.4), (1.5), (1.6) and (1.7), we have

$$\begin{aligned}
|\phi_1'(z)| &= \left|1 + \sum_{k=2}^{\infty} k \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} \; z^{k-1}\right| \\
&> 1 - \sum_{k=2}^{\infty}(k-1)\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} - \sum_{k=2}^{\infty} \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} \\
&= 1 - \frac{a_1 b_1}{c_1} \sum_{k=2}^{\infty} \frac{(a_1+1)_{k-2}(b_1+1)_{k-2}}{(c_1+1)_{k-2}(1)_{k-2}} - \sum_{k=1}^{\infty} \frac{(a_1)_k (b_1)_k}{(c_1)_k (1)_k} \\
&= 1 - \frac{a_1 b_1}{c_1} \sum_{k=1}^{\infty} \frac{(a_1+1)_{k-1}(b_1+1)_{k-1}}{(c_1+1)_{k-1}(1)_{k-1}} - \sum_{k=1}^{\infty} \frac{(a_1)_k (b_1)_k}{(c_1)_k (1)_k} \\
&= 2 - \frac{a_1 b_1}{c_1} \frac{\Gamma(c_1+1)\Gamma(c_1 - a_1 - b_1 - 1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)} - F(a_1, b_1; c_1; 1) \\
&= 2 - \left(\frac{a_1 b_1}{c_1 - a_1 - b_1 - 1} + 1\right) F(a_1, b_1; c_1; 1).
\end{aligned}$$

Again, using (3.1), (1.4), (1.6) and (1.8) in turn, to the above mentioned inequality, we have

$$
\begin{aligned}
|\phi_1'(z)| &\geq \beta + \frac{a_2 b_2}{c_2 - a_2 - b_2 - 1} F(a_2, b_2; c_2; 1) \\
&\geq \frac{a_2 b_2}{c_2 - a_2 - b_2 - 1} F(a_2, b_2; c_2; 1) \\
&= \frac{a_2 b_2}{c_2} \frac{\Gamma(c_2+1)\Gamma(c_2 - a_2 - b_2 - 1)}{\Gamma(c_2 - a_2)\Gamma(c_2 - b_2)} \\
&= \sum_{k=0}^{\infty} \frac{(a_2)_{k+1}(b_2)_{k+1}}{(c_2)_{k+1}(1)_k} \\
&> \sum_{k=1}^{\infty} k \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} \; |z|^{k-1} \\
&\geq \left| \sum_{k=1}^{\infty} k \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} \; z^{k-1} \right| = |\phi_2'(z)|.
\end{aligned}
$$

To show that G is univalent in U, we assume that $z_1, z_2 \in U$ so that $z_1 \neq z_2$. Since U is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2 \in U$, where $0 \leq t \leq 1$. Then we can write

$$
G(z_2) - G(z_1) = \int_0^1 \left[(z_2 - z_1)\phi_1'(z(t)) + \overline{(z_2 - z_1)\phi_2'(z(t))} \right] dt
$$

so that

$$
\begin{aligned}
\text{Re} \frac{G(z_2) - G(z_1)}{z_2 - z_1} &= \int_0^1 \; \text{Re} \left[\phi_1'(z(t)) + \frac{\overline{(z_2 - z_1)}}{z_2 - z_1} \phi_2'(z(t)) \right] dt \\
&> \int_0^1 \left[\text{Re } \phi_1'(z(t)) - |\phi_2'(z(t))| \right] dt.
\end{aligned} \tag{3.2}
$$

On the other hand,

$$
\begin{aligned}
&\text{Re } \phi_1'(z) - |\phi_2'(z)| \\
&\geq 1 - \sum_{k=2}^{\infty} k \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} |z|^{k-1} - \sum_{k=1}^{\infty} k \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} |z|^{k-1} \\
&> 1 - \sum_{k=2}^{\infty} (k-1+1) \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} - \sum_{k=1}^{\infty} k \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} \\
&= 2 - \sum_{k=2}^{\infty} \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-2}} - \sum_{k=0}^{\infty} \frac{(a_1)_k (b_1)_k}{(c_1)_k (1)_k} - \frac{a_2 b_2}{c_2} \sum_{k=1}^{\infty} \frac{(a_2+1)_{k-1}(b_2+1)_{k-1}}{(c_2+1)_{k-1}(1)_{k-1}}
\end{aligned}
$$

$$
\begin{aligned}
&= 2 - \left(1 + \frac{a_1 b_1}{c_1 - a_1 - b_1 - 1}\right) F(a_1, b_1; c_1; 1) - \frac{a_2 b_2}{c_2 - a_2 - b_2 - 1} F(a_2, b_2; c_2; 1) \\
&\geq \beta \\
&\geq 0\,, \quad \text{by (3.1).}
\end{aligned}
$$

Thus (3.2) and the above inequality lead to $G(z_1) \neq G(z_2)$ and hence G is univalent in U. In order to prove that $G \in HP(\beta)$, using Lemma 2.1, we only need to prove that

$$\sum_{k=1}^{\infty} k \left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} + \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} \right) \leq 2 - \beta.$$

or

$$\sum_{k=2}^{\infty} k \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} + \sum_{k=1}^{\infty} k \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} \leq 1 - \beta. \tag{3.3}$$

Writing $k = k - 1 + 1$, the left hand side of (3.3) reduces to

$$
\begin{aligned}
&\frac{a_1 b_1}{c_1} \sum_{k=0}^{\infty} \frac{(a_1 + 1)_k (b_1 + 1)_k}{(c_1 + 1)_k (1)_k} + \left[\sum_{k=0}^{\infty} \frac{(a_1)_k (b_1)_k}{(c_1)_k (1)_k} - 1 \right] \\
&\qquad + \frac{a_2 b_2}{c_2} \sum_{k=0}^{\infty} \frac{(a_2 + 1)_k (b_2 + 1)_k}{(c_2 + 1)_k (1)_k} \\
&\qquad = \left(\frac{a_1 b_1}{c_1 - a_1 - b_1 - 1} + 1 \right) F(a_1, b_1; c_1; 1) + \frac{a_2 b_2}{c_2 - a_2 - b_2 - 1} F(a_2, b_2; c_2; 1) - 1.
\end{aligned}
$$

The last expression is bounded above by $1 - \beta$ so that (3.1) is satisfied. This completes the proof

Theorem 3.2. *Let $c_j > a_j + b_j + 1$, for $j = 1, 2$ and $a_2 b_2 < c_2$. If $G = \phi_1(z) + \overline{\phi_2(z)}$ with*

$$
\begin{aligned}
\phi_1(z) &= z - \sum_{k=2}^{\infty} \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} z^k \\
\phi_2(z) &= 1 - zF(a_2, b_2; c_2; z) = -\sum_{k=1}^{\infty} \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k}\, z^k,
\end{aligned}
\tag{3.4}
$$

then $G \in HP^*(\beta)$, $\quad 0 \leq \beta < 1$ if and only if (3.1) holds.

Proof. We observe that $HP^*(\beta) \subset HP(\beta)$. In view of Theorem 3.1, we need only to show the necessary condition for G to be in $HP(\beta)$. If $G \in HP^*(\beta)$, then G satisfies the inequality in Lemma 2.2, and the result follows.

Theorem 3.3. *Let $a_2 b_2 < c_2$ and G of the form (3.4). If $G \in HP^*(\beta)$, then*

$$|G(z)| \leq \left(1 + \frac{a_2 b_2}{c_2}\right) r + \frac{1}{2}\left(1 - \frac{a_2 b_2}{c_2} - \beta\right) r^2, \quad |z| = r < 1$$

and

$$|G(z)| \geq \left(1 - \frac{a_2 b_2}{c_2}\right) r - \frac{1}{2}\left(1 - \frac{a_2 b_2}{c_2} - \beta\right) r^2, \quad |z| = r < 1.$$

Hence

$$\left\{\omega : |\omega| < \frac{1}{2}\left(1 - \frac{a_2 b_2}{c_2} + \beta\right)\right\} \subset G(U).$$

Proof. The result follows from Lemma 2.3.

Theorem 3.4. *Let $c_j > a_j + b_j + 1$ for $j = 1, 2$ and $a_2 b_2 < c_2$. A necessary and sufficient condition such that $f \,\tilde{*}\, (\phi_1 + \overline{\phi_2}) \in HP^*(\beta)$ for $f \in HP^*(\beta)$ is that*

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leq 3, \tag{3.5}$$

where ϕ_1, ϕ_2 are as defined, respectively, by (1.7) *and* (1.8)

Proof. Let $f = h + \overline{g} \in HP^*(\beta)$, where h and g are given by (1.2). Then

$$\begin{aligned}(f \,\tilde{*}\, (\phi_1 + \overline{\phi}_2))(z) &= h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)} \\ &= z - \sum_{k=2}^{\infty} \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} |A_k| z^k \\ &\quad - \overline{\sum_{k=1}^{\infty} \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} |B_k| z^k}.\end{aligned}$$

In view of Lemma 2.2, we need to prove that $f \,\tilde{*}\, (\phi_1 + \overline{\phi}_2) \in HP^*(\beta)$ if and only if

$$\sum_{k=1}^{\infty} k \left[\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} |A_k| + \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} |B_k|\right] \leq 2 - \beta$$

or

$$\sum_{k=2}^{\infty} k \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} |A_k| + \sum_{k=1}^{\infty} k \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} |B_k| \leq 1 - \beta. \tag{3.6}$$

As an application of Lemma 2.2, we have

$$|A_k| \leq \frac{1-\beta}{k}\,, \quad |B_k| \leq \frac{1-\beta}{k}.$$

Therefore, the left hand side of (3.6) is bounded above by

$$\begin{aligned}&\sum_{k=2}^{\infty} (1-\beta) \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} + \sum_{k=1}^{\infty} (1-\beta) \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k} \\ &\qquad = (1-\beta)\left[F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) - 2\right].\end{aligned}$$

The last expression is bounded above by $(1-\beta)$ if and only if (3.5) is satisfied. This proves (3.6) and the result follows.

Theorem 3.5. *If $c_j > a_j + b_j$ for $j = 1, 2$, then a sufficient condition for a function*

$$G_1(z) = \int_0^z F(a_1, b_1; c_1; t)dt + \overline{\int_0^z [F(a_2, b_2; c_2; t) - 1]\, dt}$$

to be in $HP(\beta)$ is that

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leq 3 - \beta, \quad 0 \leq \beta < 1.$$

Proof. In view of Lemma 2.1, the function

$$G_1(z) = z + \sum_{k=2}^{\infty} \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_k} z^k + \overline{\sum_{k=2}^{\infty} \frac{(a_2)_{k-1}(b_2)_{k-1}}{(c_2)_{k-1}(1)_k} z^k}$$

is in $HP(\beta)$ if

$$\sum_{k=2}^{\infty} k \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_k} + \sum_{k=2}^{\infty} k \frac{(a_2)_{k-1}(b_2)_{k-1}}{(c_2)_{k-1}(1)_k} \leq 1 - \beta.$$

That is, if

$$\sum_{k=1}^{\infty} \frac{(a_1)_k(b_1)_k}{(c_1)_k(1)_k} + \sum_{k=1}^{\infty} \frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} \leq 1 - \beta.$$

Equivalently, $G_1 \in HP(\beta)$ if

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leq 3 - \beta.$$

In the next theorem we give a necessary and sufficient convolution condition for $G = \phi_1 + \overline{\phi}_2$ to be in $HP(\beta)$.

Theorem 3.6. *Let $a_2 b_2 < c_2$. If $G = \phi_1(z) + \overline{\phi}_2(z)$ with ϕ_1, ϕ_2 are as defined, respectively, by* (1.7) *and* (1.8). *Then $G \in HP(\beta)$ if and only if*

$$\left((\phi_1 + \phi_2) * \frac{(\xi + 1)}{(1 - z)^2} \right) + \frac{a_2 b_2 + c_2}{c_2}(1 - 2\beta - \xi) \neq 0, \quad 0 \leq \beta < 1,\ |\xi| = 1,\ \ 0 < |z| < 1.$$

Proof. Let $G \in HP(\beta)$, then

$$\mathrm{Re}\{\phi_1' + \phi_2'\} > \beta.$$

Since $\frac{c_2}{a_2b_2+c_2}(\phi_1'(z)+\phi_2'(z))=1$ at $z=0$, therefore we can write $G\in HP(\beta)$ if and only if

$$\frac{1}{1-\beta}\left[\frac{c_2}{a_2b_2+c_2}(\phi_1'(z)+\phi_2'(z))-\beta\right]\neq\frac{\xi-1}{\xi+1}\,,\quad |\xi|=1,\ \xi\neq-1,\ 0<|z|<1.$$

By a simple algebraic manipulation, we get

$$\begin{aligned}0&\neq(\xi+1)\frac{c_2}{a_2b_2+c_2}(\phi_1'(z)+\phi_2'(z))-\beta(\xi+1)-(1-\beta)(\xi-1)\\&=\frac{c_2}{a_2b_2+c_2}(\xi+1)(\phi_1'(z)+\phi_2'(z))+(1-2\beta-\xi)\\&=\frac{c_2}{a_2b_2+c_2}(\phi_1(z)+\phi_2(z))*\frac{(\xi+1)}{(1-z)^2}+(1-2\beta-\xi)\end{aligned}$$

which is the condition required by Theorem 3.6.

Theorem 3.7. *Let $f\in HP^*(\beta)$, $\quad 0\leq\beta<1$. Then the function $H=f\,\tilde{*}\,(\phi_1+\overline{\phi}_2)$ is starlike of order $\gamma(0\leq\gamma<1)$ in $|z|<R$, where*

$$R=\inf_k\left[\frac{2k(1-\gamma)(c_1)_{k-1}(c_2)_k\ k!}{(1-\beta)[k(k-\gamma)(a_1)_{k-1}(b_1)_{k-1}(c_2)_k+(k+\gamma-2)(a_2)_k(b_2)_k(c_1)_{k-1}]}\right]^{\frac{1}{k-1}}$$

ϕ_1,ϕ_2 are as defined by (1.7) and (1.8).

Proof. It is sufficient to show that

$$\left|\frac{zH'}{H}-1\right|<1-\gamma\quad\text{in}\quad |z|<R. \tag{3.7}$$

For the left hand side of (3.7), we have

$$\left|\frac{zH'}{H}-1\right|\leq\frac{\sum_{k=2}^{\infty}\frac{(k-1)(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}}|A_k|\ |z|^{k-1}+\sum_{k=1}^{\infty}\frac{(k+1)(a_2)_k(b_2)_k}{(c_2)_k(1)_k}|B_k|\ |z|^{k-1}}{1-\sum_{k=2}^{\infty}\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}}|A_k|\ |z|^{k-1}-\sum_{k=1}^{\infty}\frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k}|B_k|\ |z|^{k-1}}.$$

The last expression is less than $1-\gamma$ if

$$\sum_{k=2}^{\infty}\frac{(k-\gamma)}{(1-\gamma)}\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}}|A_k|\ |z|^{k-1}+\sum_{k=1}^{\infty}\frac{(k+2-\gamma)}{(1-\gamma)}\frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k}\ |B_k|\ |z|^{k-1}<1. \tag{3.8}$$

Using the fact that $f\in HP^*(\beta)$ if and only if

$$\sum_{k=2}^{\infty}k\ |A_k|+\sum_{k=1}^{\infty}k\ |B_k|\leq1-\beta,$$

we can say (3.8) is true if

$$\left\{\frac{k-\gamma}{1-\gamma}\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}}+\frac{k+2-\gamma}{1-\gamma}\frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k}\right\}\ |z|^{k-1}<\frac{2k}{1-\beta}.$$

Or equivalently,

$$|z|^{k-1} < \frac{2k(1-\gamma)}{1-\beta}\left\{\frac{(c_1)_{k-1}(1)_{k-1}(c_2)_k(1)_k}{(k-\gamma)(a_1)_{k-1}(b_1)_{k-1}(c_2)_k(1)_k+(k+2-\gamma)(a_2)_k(b_2)_k(c_1)_{k-1}(1)_{k-1}}\right\}$$
$$= \frac{2k(1-\gamma)}{1-\beta}\left\{\frac{k!(c_1)_{k-1}(c_2)_k}{k(k-\gamma)(a_1)_{k-1}(b_1)_{k-1}(c_2)_k+(k+2-\gamma)(a_2)_k(b_2)_k(c_1)_{k-1}}\right\}.$$

If h, g, H, G are of the form (1.1) and as we know if $f = h + \overline{g}$ and $F = H + \overline{G}$, then the convolution of f and F is defined to be the function

$$f * F(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k}$$

while the integral convolution is defined by

$$f \diamond F(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k + \overline{\sum_{k=1}^{\infty} \frac{b_k B_k}{k} z^k}.$$

The δ-neighborhood of f is the set

$$N_\delta(f) = \left\{F = z + \sum_{k=2}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k} : \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + |b_1 - B_1| \le \delta\right\}$$

(see [5], [7]). In this case, let us define the generalized δ-neighborhood of $G = \phi_1 + \overline{\phi}_2$, where ϕ_1, ϕ_2 of the form (1.7) and (1.8) to be the set

$$N_\delta(G) = \left\{F = z + \sum_{k=2}^{\infty} \frac{(A_1)_{k-1}(B_1)_{k-1}}{(C_1)_{k-1}(1)_{k-1}} z^k + \overline{\sum_{k=1}^{\infty} \frac{(A_2)_k(B_2)_k}{(C_2)_k(1)_k} z^k} :\right.$$
$$\sum_{k=2}^{\infty} k\left\{\left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} - \frac{(A_1)_{k-1}(B_1)_{k-1}}{(C_1)_{k-1}(1)_{k-1}}\right) + \left(\frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} - \frac{(A_2)_k(B_2)_k}{(C_2)_k(1)_k}\right)\right\}$$
$$\left. + \left(\frac{a_2 b_2}{c_2} - \frac{A_2 B_2}{C_2}\right) \le (1-\beta)\delta.\right.$$

Let P_H^0 denote the class of functions F complex and harmonic in U, $F = h + \overline{g}$ such that Re $F(z) > 0$, $\quad z \in U$ and

$$h(z) = 1 + \sum_{k=1}^{\infty} A_k z^k, \quad g(z) = \sum_{k=2}^{\infty} B_k z^k.$$

It is known [9, Theorem 3] that the sharp inequalities $|A_k| \le k+1$, $\quad |B_k| \le k-1$ are true.

Theorem 3.8. (i) *If* $G = \phi_1(z) + \overline{\phi_2(z)} \in HP(\beta)$ *where* ϕ_1, ϕ_2 *are as defied by* (1.7) *and* (1.8), $(0 \le \beta < 1)$, *then for* $\frac{3}{2} \le |A_1| \le 2$, $\frac{1}{A_1} G \diamond F \in HP(\beta)$.

(ii) *If* G *satisfies the condition*

$$\sum_{k=2}^{\infty} k^2 \left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} + \frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} \right) \le 1 - \beta, \tag{3.9}$$

then

$$\frac{1}{A_1} G * F \in HP(\beta).$$

(iii) *If* G *satisfies (3.9) and* $\delta \le \frac{1}{2} - \frac{1}{1-\beta}\frac{a_2 b_2}{c_2}$, *then* $N_\delta(G) \subset HP(\beta)$, *where* $0 \le \beta < 1, \quad a_2 b_2 < c_2$.

Proof. We justify the case (ii). Since

$$\begin{aligned}
&\sum_{k=2}^{\infty} k \left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} \left| \frac{A_k}{A_1} \right| + \frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} \left| \frac{B_k}{A_1} \right| \right) \\
&\le \sum_{k=2}^{\infty} k^2 \left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} \frac{k+1}{|A_1|k} + \frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} \frac{k-1}{|A_1|k} \right) \\
&\le \sum_{k=2}^{\infty} k^2 \left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} + \frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} \right) \\
&\le 1 - \beta,
\end{aligned}$$

then $\frac{1}{A_1} G * F \in HP(\beta)$.

(iii) Let $G(z) = z + \overline{\frac{a_2 b_2}{c_2} z} + \sum_{k=2}^{\infty} \left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} z^k + \overline{\frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} z^k} \right)$ is a member of $HP(\beta)$ and $F(z) = z + \overline{\frac{A_2 B_2}{C_2} z} + \sum_{k=2}^{\infty} \left(\frac{(A_1)_{k-1}(B_1)_{k-1}}{(C_1)_{k-1}(1)_{k-1}} z^k + \overline{\frac{(A_2)_k(B_2)_k}{(C_2)_k(1)_k} z^k} \right)$ belong to $N_\delta(G)$. We have

$$\begin{aligned}
&\frac{A_2 B_2}{C_2} + \sum_{k=2}^{\infty} k \left[\frac{(A_1)_{k-1}(B_1)_{k-1}}{(C_1)_{k-1}(1)_{k-1}} + \frac{(A_2)_k(B_2)_k}{(C_2)_k(1)_k} \right] \\
&\quad \le \left(\frac{A_2 B_2}{C_2} - \frac{a_2 b_2}{c_2} \right) + \frac{a_2 b_2}{c_2} + \sum_{k=2}^{\infty} k \left[\left(\frac{(A_1)_{k-1}(B_1)_{k-1}}{(C_1)_{k-1}(1)_{k-1}} - \frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} \right) \right. \\
&\qquad \left. + \left(\frac{(A_2)_k(B_2)_k}{(C_2)_k(1)_k} - \frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} \right) \right] \\
&\qquad + \sum_{k=2}^{\infty} k \left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} + \frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} \right)
\end{aligned}$$

$$\leq (1-\beta)\delta + \frac{a_2b_2}{c_2} + \frac{1}{2}\sum_{k=2}^{\infty} k^2 \left(\frac{(a_1)_{k-1}(b_1)_{k-1}}{(c_1)_{k-1}(1)_{k-1}} + \frac{(a_2)_k(b_2)_k}{(c_2)_k(1)_k} \right)$$
$$\leq (1-\beta)\delta + \frac{a_2b_2}{c_2} + \frac{1}{2}(1-\beta)$$
$$\leq 1-\beta.$$

Hence, for $\delta \leq \dfrac{1}{2} - \dfrac{1}{1-\beta}\dfrac{a_2b_2}{c_2}$, $F(z) \in HP(\beta)$.

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