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THE EXTENDED GENERALIZED HALF LOGISTIC DISTRIBUTION BASED ON ORDERED RANDOM VARIABLES

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Abstract. In this paper, we considered an extended generalized half logistic distribution and derived some explicit expressions and recurrence relations for marginal and joint moment generating functions of generalized order statistics from extended generalized half logistic distribution. The results for record values and order statistics are deduced from the result. Finally, we obtained the characterizing result of this distribution on using the conditional expectation of generalized order statistics.

1. Introduction

Kamps [16] introduced and extensively studied the generalized order statistics (gos). The order statistics, sequential order statistics, Stigler's order statistics and record values are special cases of gos. Suppose $X(1,n,m,k),\ldots,X(n,n,m,k)$ are n gos from an absolutely continuous cumulative distribution function (cdf) $F(x)$ with the corresponding (pdf) $f(x)$. Their joint is

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\Big)[1-F(x_n)]^{k-1}f(x_n) \tag{1.1}$$

for $F^{-1}(0+) < x_1 \le x_2 \le \ldots \le x_n > F^{-1}(1)$, $m \ge -1$, $\gamma_r = k+(n-r)(m+1) > 0$, $r = 1,2,\ldots,n-1$, $k \ge 1$ and n is a positive integer.

Choosing the parameters appropriately, models such as ordinary order statistics ($\gamma_i = n-i+1;\ i = 1,2,\ldots,n\ \ i.e\ \ m_1 = m_2 = \ldots = m_{n-1} = 0,\ k = 1$), $k-$ th record values ($\gamma_i = k\ \ i.e\ \ m_1 = m_2 = \ldots = m_{n-1} = -1,\ k \in N$), sequential order statistics ($\gamma_i = (n-i+1)\alpha;\ \alpha_1,\alpha_2,\ldots,\alpha_n > 0$), order statistics with non-integral sample size ($\gamma_i = \alpha-i+1;\ \alpha > 0$), Pfeifer's record values ($\gamma_i = \beta_i;\ \beta_1,\beta_2,\ldots,\beta_n > 0$) and progressive type II censored order statistics ($m_i \in N,\ k \in N$)

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are obtained (Kamps [16], Kamps and Cramer [17]).
The marginal pdf of the $r-$th gos, $X(r,n,m,k)$, $1 \le r \le n$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y), \quad x<y, \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i\,, \quad \bar{F}(x) = 1 - F(x),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

Balakrishnan [15] considers half logistic probability models obtained as the distribution of the absolute value of the standard logistic distribution. Some key references about the half logistic distribution include Balakrishnan and Aggarwala [13], Balakrishnan and Wong [10] and Balakrishnan and Chan [12]. Balakrishnan and Puthenpura [14] obtained the best linear unbiased estimator of location and scale parameters of the half logistic distribution through linear functions of order statistics. Balakrishnan and Wong [10] obtained approximate maximum likelihood estimates for the location and scale parameters of the half logistic distribution with Type-II right-censoring. Torabi and Bagheri [6] are estimate of parameters for an extended generalized half logistic distribution based on complete and censored data.

Ahsanullah and Raqab [7], Raqab and Ahsanullah [8], [9] have established recurrence relations for moment generating functions of record values from Pareto and Gumble, power function and extreme value distributions. Recurrence relations for marginal and joint moment generating functions of gos from Erlang-truncated exponential distribution are derived by Kulshrestha et al. [1]. Kumar [3] have established recurrence relations for marginal and joint moment generating functions of lower generalized order statistics from Marshall-Olkin extended logistic distribution respectively. Al-Hussaini et al. [4], [5] have established recurrence relations for conditional and joint moment generating functions of gos based on mixed population, respectively. Kumar [2] have established explicit expressions and some recurrence relations for moment generating functions of record values from generalized logistic distribution.

In the next Section, we present some exact expressions and recurrence relations for marginal moment generating functions of gos from extended generalized half logistic distribution and results for order statistics and record values are deduced as special cases. In Section 3, we discuss joint moment generating functions of gos from extended generalized half logistic distribution and results for order statistics and record values are deduced as special cases. Finally, we present a characterization result of this distribution by using conditional expectation of gos in Section 4.

2. An extended generalized half logistic distribution

The pdf of an extended generalized half logistic distribution with the parameters μ, σ, λ and θ is defined by

$$f(x) = \frac{\theta(1+\lambda)^{\theta} e^{-\theta(x-\mu)/\sigma}}{\sigma[1+\lambda e^{-(x-\mu)/\sigma}]^{\theta+1}}, \quad x > \mu, \tag{2.1}$$

where $\theta > 0$, $\lambda > 0$, $\sigma > 0$ and $\mu \in \Re$. The corresponding pdf, the hazard rate function and survival function are

$$F(x) = 1 - \frac{(1+\lambda)^{\theta} e^{-\theta(x-\mu)/\sigma}}{\sigma[1+\lambda e^{-(x-\mu)/\sigma}]^{\theta}}, \quad x > \mu, \tag{2.2}$$

$$\lambda(x) = \frac{\theta}{\sigma[1+\lambda e^{-(x-\mu)/\sigma}]}$$

and

$$\bar{F}(x) = \frac{(1+\lambda)^{\theta} e^{-\theta(x-\mu)/\sigma}}{\sigma[1+\lambda e^{-(x-\mu)/\sigma}]^{\theta}}, \quad x > \mu,$$

respectively. If $\mu = 0$, $\sigma = 1$ and $\lambda = 1$, then the extended generalized half logistic distribution reduces to the generalized half logistic distribution with the parameter θ .
Note that for extended generalized half logistic distribution

$$\theta\bar{F}(x) = \sigma[1+\lambda e^{-(x-\mu)/\sigma}] f(x). \tag{2.3}$$

The relation in (2.3) will be exploited in this paper to derive some recurrence relations for the moment generating functions of gos from the extended generalized half logistic distribution. Let us denote the marginal moment generating functions of $X(r,n,m,k)$ by $M_{X(r,n,m,k)}(t)$ its $j-$th derivative by $M^{(j)}_{X(r,n,m,k)}(t)$ and the joint moment generating functions of $X(r,n,m,k)$ and $X(s,n,m,k)$ by $M_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)$ and its $(i,j)-$th partial derivatives with respect to t_1 and t_2, respectively by $M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)$.
For the extended generalized half logistic distribution as given in (2.1), the moment generating functions of $X(r,n,m,k)$ is given as,

$$\begin{aligned} M_{X(r,n,m,k)}(t) &= \int_{-\infty}^{\infty} e^{tx} f_{X(r,n,m,k)}(x)dx \\ &= \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} e^{tx}[\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))dx. \end{aligned} \tag{2.4}$$

Further, on using the binomial expansion, we can rewrite (2.2) as

$$M_{X(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \times \int_{-\infty}^{\infty} e^{tx} [\bar{F}(x)]^{\gamma_{r-u}-1} f(x)\,dx. \tag{2.5}$$

Now letting $z = [\bar{F}(x)]^{1/\theta}$ in (2.5), we get

$$M_{X(r,n,m,k)}(t) = \frac{e^{\mu t}\, C_{r-1}}{(r-1)!(m+1)^r} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{u+q} \binom{r-1}{u} \frac{\lambda^p\, \Gamma(t\sigma+1)}{p!q!\Gamma(t\sigma+1-p)} \times \frac{\Gamma(p+1)}{\Gamma(p+1-q)} B\Big(\frac{k}{m+1} + n - r + u + \frac{q-t\sigma}{m+1}, 1\Big), \tag{2.6}$$

Since

$$\sum_{a=0}^{b} (-1)^a \binom{b}{a} B(a+k, c) = B(k, c+b) \tag{2.7}$$

where $B(a,b)$ is the complete beta function.
Therefore,

$$\begin{aligned} M_{X(r,n,m,k)}(t) &= \frac{e^{\mu t}\, C_{r-1}}{(m+1)^r} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \frac{\lambda^p\, \Gamma(t\sigma+1)\Gamma(p+1)}{p!q!\Gamma(t\sigma+1-p)\Gamma(p+1-q)} \times \frac{\Gamma\Big(\frac{k+(n-r)(m+1)+q-t\sigma}{m+1}\Big)}{\Gamma\Big(\frac{k+n(m+1)+q-t\sigma}{m+1}\Big)} \\ &= e^{\mu t} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \frac{\lambda^p\, \Gamma(t\sigma+1)\Gamma(p+1)}{p!q!\Gamma(t\sigma+1-p)\Gamma(p+1-q)} \frac{1}{\prod_{a=1}^{r} \Big(1 + \frac{q-t\sigma}{\gamma_a}\Big)}. \end{aligned} \tag{2.8}$$

Special Cases

(i) Putting $m = 0$, $k = 1$, in (2.8), the explicit formula for marginal moment generating functions of order statistics from the extended generalized half logistic distribution can be obtained as

$$M_{X_{r:n}}(t) = \frac{e^{\mu t} n!}{(n-r)!} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \frac{\lambda^p\, \Gamma(t\sigma+1)\Gamma(p+1)\Gamma(n-r+1+q-t\sigma)}{p!q!\Gamma(t\sigma+1-p)\Gamma(p+1-q)\Gamma(n+1+q-t\sigma)},$$

for $r = 1$

$$M_{X_{1:n}}(t) = n e^{\mu t} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \frac{\lambda^p\, \Gamma(t\sigma+1)\Gamma(p+1)}{p!q!\Gamma(t\sigma+1-p)\Gamma(p+1-q)(n+q-t\sigma)}.$$

(ii) Setting $m = -1$ in (2.8), we get the explicit expression for marginal moment generating functions of k-th upper record values from the extended generalized half logistic distribution can be obtained as

$$M_{X(r,n,-1,k)}(t) = e^{\mu t}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \frac{\lambda^p\,\Gamma(t\sigma+1)\Gamma(p+1)}{p!q!\Gamma(t\sigma+1-p)\Gamma(p+1-q)\left(1+\frac{q-t\sigma}{\gamma_a}\right)^r}.$$

A recurrence relation for marginal moment generating functions for gos from cdf (2.2) can be obtained in the following theorem.

Theorem 2.1. *For the distribution given in* (2.1) *and for* $2 \le r \le n$, $n \ge 2$ $k = 1,2,\ldots,$

$$\left(1-\frac{t\sigma}{\theta\gamma_r}\right)M^{(j)}_{X(r,n,m,k)}(t) = M^{(j)}_{X(r-1,n,m,k)}(t)+\frac{j\sigma}{\theta\gamma_r}M^{(j-1)}_{X(r,n,m,k)}(t)$$
$$-\frac{\lambda\sigma e^{\mu/\sigma}}{\theta\gamma_r}[tM^{(j)}_{X(r,n,m,k)}(t-(1/\sigma))+jM^{(j-1)}_{X(r,n,m,k)}(t-(1/\sigma))]. \quad (2.9)$$

Proof. From (1.2), we have

$$M_{X(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} e^{tx}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \quad (2.10)$$

Integrating by parts of (2.10) and using (2.3), we get

$$M_{X(r,n,m,k)}(t) = M_{X(r-1,n,m,k)}(t)+\frac{t\sigma}{\theta\gamma_r}M_{X(r,n,m,k)}(t)$$
$$+\frac{\lambda t\sigma e^{\mu/\sigma}}{\theta\gamma_r}M_{X(r,n,m,k)}(t-(1/\sigma)). \quad (2.11)$$

Differentiating both the sides of (2.11) j times with respect to t, we get get the result given in (2.9).

By differentiating both sides of equation (2.9) with respect to t and then setting $t = 0$, we obtain the recurrence relations for single moments of gos from the extended generalized half logistic distribution in the form

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)]+\frac{j\sigma}{\theta\gamma_r}E[X^{j-1}(r,n,m,k)]$$
$$+\lambda E[\phi(X(r,n,m,k))], \quad (2.12)$$

where $\phi(x) = x^{j-1}e^{-(x-\mu)/\sigma}$.

Remark 2.1. Putting $m = 0$, $k = 1$ in (2.9) and (2.12), we can get the relations for marginal moment generating functions and single moments of order statistics

$$\left(1-\frac{t\sigma}{\theta(n-r+1)}\right)M^{(j)}_{X_{r:n}}(t) = M^{(j)}_{X_{r-1:n}}(t)+\frac{j\sigma}{\theta(n-r+1)}M^{(j-1)}_{X_{r:n}}(t)$$

$$-\frac{\lambda\sigma e^{\mu/\sigma}}{\theta(n-r+1)}[tM^{(j)}_{X_{r:n}}(t-(1/\sigma))+jM^{(j-1)}_{X_{r:n}}(t-(1/\sigma))]$$

and

$$E[X^j_{r:n}]Z = E[X^j_{r-1:n}]+\frac{j\sigma}{\theta(n-r+1)}\{E[X^{j-1}_{r:n}]+\lambda E[\phi(X_{r:n})]\}.$$

Remark 2.2. Setting $m=-1$ and $k\geq 1$, in (2.9) and (2.12), relations for k-th record values can be obtained as

$$\Big(1-\frac{t\sigma}{\theta k}\Big)M^{(j)}_{Z^{(k)}_r}(t) = M^{(j)}_{Z^{(k)}_{r-1}}(t)+\frac{j\sigma}{\theta k}M^{(j-1)}_{Z^{(k)}_r}(t)$$
$$+\frac{\lambda\sigma e^{\mu/\sigma}}{\theta k}[tM^{(j)}_{Z^{(k)}_r}(t-(1/\sigma))+jM^{(j-1)}_{Z^{(k)}_r}(t-(1/\sigma))]$$

and

$$E[(Z^{(k)}_r)^j] = E[(Z^{(k)}_{r-1})^j]+\frac{j\sigma}{\theta k}\{E[(Z^{(k)}_r)^{j-1}]+\lambda E[\phi(Z_r)]\}.$$

For $k=1$

$$\Big(1-\frac{t\sigma}{\theta}\Big)M^{(j)}_{X_{U(r)}}(t) = M^{(j)}_{X_{U(r-1)}}(t)+\frac{j\sigma}{\theta}M^{(j-1)}_{X_{U(r)}}(t)$$
$$+\frac{\lambda\sigma e^{\mu/\sigma}}{\theta}[tM^{(j)}_{X_{U(r)}}(t-(1/\sigma))+jM^{(j-1)}_{X_{U(r)}}(t-(1/\sigma))]$$

and

$$E[X^j_{U(r)}] = E[X^j_{U(r-1)}]+\frac{j\sigma}{\theta}\{E[X^{j-1}_{U(r)}]+\lambda E[\phi(X_{U(r)})]\}.$$

3. Relations for joint moment generating functions

For extended generalized half logistic distribution, the joint moment generating functions of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given as

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) = \int_{-\infty}^{\infty}\int_x^{\infty} e^{t_1x+t_2y}f_{X(r,n,m,k)X(s,n,m,k)}(x,y)dxdy.$$

On using (1.3) and binomial expansion, we have

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)$$
$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$
$$\times\binom{r-1}{u}\binom{s-r-1}{v}\int_0^{\mu} e^{t_1x}[\bar F(x)]^{(s-r+u-v)(m+1)-1}f(x)G(x)dx, \tag{3.1}$$

where

$$G(x) = \int_x^{\mu} e^{t_2y}[\bar F(y)]^{\gamma_{s-v}-1}f(y)dy. \tag{3.2}$$

By setting $z = [\bar{F}(y)]^{1/\theta}$ in (3.2), we obtain

$$G(x) = e^{\mu t_2} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \frac{\lambda^p \Gamma(t_2\sigma+1)\Gamma(p+1)[\bar{F}(x)]^{\gamma_{s-v}+q-t_2\sigma}}{p!q!\Gamma(t_2\sigma+1-p)\Gamma(p+1-q)(\gamma_{s-v}+q-t_2\sigma)}.$$

On substituting the above expression of $G(x)$ in (3.1), and simplifying the resulting equation, we get

$$\begin{aligned} &M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) \\ &= \frac{e^{\mu(t_1+t_2)} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^s} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty} (-1)^{q+w} \\ &\times \frac{\lambda^{p+l}\Gamma(t_2\sigma+1)\Gamma(p+1)\Gamma(t_1\sigma+1)\Gamma(l+1)}{p!q!l!w!\Gamma(t_2\sigma+1-p)\Gamma(p+1-q)\Gamma(t_1\sigma+1-l)\Gamma(l+1-w)} \\ &\times \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} B\Big(\frac{k}{m+1}+n-r+u+\frac{q+w-\sigma(t_1+t_2)}{m+1}, 1\Big) \\ &\times \sum_{v=0}^{s-r-1} (-1)^v \binom{s-r-1}{v} B\Big(\frac{k}{m+1}+n-s+v+\frac{q-t_2\sigma}{m+1}, 1\Big), \end{aligned} \tag{3.3}$$

which after simplification yields

$$\begin{aligned} &M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) \\ &= e^{\mu(t_1+t_2)} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty} (-1)^{q+w} \\ &\times \frac{\lambda^{p+l}\Gamma(t_2\sigma+1)\Gamma(p+1)\Gamma(t_1\sigma+1)\Gamma(l+1)}{p!q!l!w!\Gamma(t_2\sigma+1-p)\Gamma(p+1-q)\Gamma(t_1\sigma+1-l)\Gamma(l+1-w)} \\ &\times \frac{1}{\prod_{a=1}^{r}\Big(1+\frac{q+w-\sigma(t_1+t_2)}{\gamma_a}\Big)\prod_{b=r+1}^{s}\Big(1+\frac{q-t_2\sigma}{\gamma_b}\Big)}. \end{aligned} \tag{3.4}$$

Special Cases

(i) Putting $m = 0$, $k = 1$ in (3.4), the explicit formula for joint moment generating functions of order statistics can be obtained as

$$\begin{aligned} M_{X_{r:n}X_{s:n}}(t_1,t_2) = &\frac{e^{\mu(t_1+t_2)} n!}{(n-s)!} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty} (-1)^{q+w} \\ &\times \frac{\lambda^{p+l}\Gamma(t_2\sigma+1)\Gamma(p+1)\Gamma(t_1\sigma+1)\Gamma(l+1)}{p!q!l!w!\Gamma(t_2\sigma+1-p)\Gamma(p+1-q)\Gamma(t_1\sigma+1-l)\Gamma(l+1-w)} \\ &\times \frac{\Gamma[n-r+1+q+w-\sigma(t_1+t_2)]\Gamma[n-s+1+q-t_2\sigma]}{\Gamma[n+1+q+w-\sigma(t_1+t_2)]\Gamma[n-r+1+q-t_2\sigma]}. \end{aligned}$$

(ii) Setting $m = -1$ in (3.4), we deduce the explicit expression for joint moment generating functions of k-th upper record values in the form

$$M_{X_{U(r):k}X_{U(s):k}}(t_1,t_2) = e^{\mu(t_1+t_2)} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty} (-1)^{q+w}$$

$$\times \frac{\lambda^{p+l}\Gamma(t_2\sigma+1)\Gamma(p+1)\Gamma(t_1\sigma+1)\Gamma(l+1)}{p!q!l!w!\Gamma(t_2\sigma+1-p)\Gamma(p+1-q)\Gamma(t_1\sigma+1-l)\Gamma(l+1-w)}$$
$$\times \frac{1}{\left(1+\frac{q+w-\sigma(t_1+t_2)}{k}\right)^r\left(1+\frac{q-t_2\sigma}{k}\right)^{s-r}}.$$

Making use of (2.3), we can derive the recurrence relations for joint moment generating functions of gos.

Theorem 3.1. *For the distribution given in* (2.1) *and for* $1 \le r < s \le n,\ \ n \ge 2$ *and* $k = 1, 2, \ldots$

$$\begin{aligned}\Big(1-\frac{\sigma t_2}{\theta\gamma_s}\Big)&M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)\\ &= M^{(j)}_{X(r,n,m,k),X(s-1,n,m,k)}(t_1,t_2)\\ &\quad+\frac{j\sigma}{\theta\gamma_s}M^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)+\frac{\lambda\sigma e^{\mu/\sigma}}{\theta\gamma_s}\\ &\quad\times[t_2M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2-(1/\sigma))\\ &\quad+jM^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2-(1/\sigma))].\end{aligned} \tag{3.5}$$

Proof. Using (1.3), the joint moment generating functions of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given by

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))I(x)\,dx \tag{3.6}$$

where

$$I(x) = \int_x^{\infty} e^{t_1x+t_2y}[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)\,dy.$$

Solving the integral in $I(x)$ by parts and using (2.3), substituting the resulting expression in (3.6), we get

$$\begin{aligned}M_{X(r,n,m,k)X(s,n,m,k)}&(t_1,t_2)\\ &= M_{X(r,n,m,k)X(s-1,n,m,k)}(t_1,t_2)\\ &\quad+\frac{\sigma t_2}{\theta\gamma_s}[M_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)+\lambda e^{\mu/\sigma}M_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2-(1/\sigma))].\end{aligned} \tag{3.7}$$

Differentiating both the sides of (3.7) i times with respect to t_1 and then j times with respect to t_2, we get

$$\begin{aligned}M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}&(t_1,t_2)\\ &= M^{(i,j)}_{X(r,n,m,k)X(s-1,n,m,k)}(t_1,t_2)+\frac{\sigma}{\theta\gamma_s}\\ &\quad\times[t_2M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)+jM^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)]+\frac{\lambda\sigma e^{\mu/\sigma}}{\theta\gamma_s}\end{aligned}$$

$$\times [t_2 M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1, t_2 - (1/\sigma)) + j M^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1, t_2 - (1/\sigma))],$$

which, when rewritten gives the recurrence relation in (3.5).

One can also note that Theorem 2.1 can be deduced from Theorem 3.1 by letting t_1 tends to zero.

By differentiating both sides of equation (3.5) with respect to t_1, t_2 and then setting $t_1 = t_2 = 0$, we obtain the recurrence relations for product moments of gos from extended generalized half logistic distribution in the form

$$\begin{aligned} &E[X^i(r,n,m,k)X^j(s,n,m,k)] \\ &\quad = E[X^i(r,n,m,k)X^j(s-1,n,m,k)] \\ &\quad + \frac{j\sigma}{\theta\gamma_s} E[X^i(r,n,m,k)X^{j-1}(s,n,m,k)] + \lambda E[\phi(X(r,n,m,k)X(s,n,m,k))], \end{aligned} \tag{3.8}$$

where

$\phi(x,y) = x^i y^{j-1} e^{-(y-\mu)/\sigma}$.

Remark 3.1. Putting $m = 0$, $k = 1$ in (3.5) and (3.8), we obtain the recurrence relations for joint moment generating functions and product moments of order statistics in the form

$$\begin{aligned} &\left(1 - \frac{\sigma t_2}{\theta(n-s+1)}\right) M^{(i,j)}_{X_{r,s:n}}(t_1,t_2) \\ &\quad = M^{(i,j)}_{X_{r,s-1:n}}(t_1,t_2) + \frac{j\sigma}{\theta(n-s+1)} M^{(i,j-1)}_{X_{r,s:n}}(t_1,t_2) + \frac{\lambda\sigma e^{\mu/\sigma}}{\theta(n-s+1)} \\ &\quad \times [t_2 M^{(i,j)}_{X_{r,s:n}}(t_1, t_2 - (1/\sigma)) + j M^{(i,j-1)}_{X_{r,s:n}}(t_1, t_2 - (1/\sigma))] \end{aligned}$$

and

$$E[X^{(i,j)}_{r,s:n}] = E[X^{(i,j)}_{r,s-1:n}] + \frac{j\sigma}{\theta(n-s+1)}\{E[X^{(i,j-1)}_{r,s:n}] + \lambda E[\phi(X_{r,s:n})]\}.$$

Remark 3.2. Substituting $m = -1$ and $k \geq 1$, in (3.5) and (3.8), we get recurrence relation for joint moment generating functions and product moments of k-th upper record values for extended generalized half logistic distribution.

4. Characterization

Let $X(r,n,m,k)$, $r = 1,2,\ldots,n$ be gos, then from a continuous population with cdf $F(x)$ and pdf $f(x)$, then the conditional pdf of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \leq r < s \leq n$, in view of (1.2) and (1.3), is

$$\begin{aligned} f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) &= \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \\ &\quad \times \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1}}{[\bar{F}(x)]^{\gamma_{r+1}}} f(y). \quad x < y \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then*

$$E[e^{tX(s,n,m,k)}|X(r,n,m,k)=x]$$
$$=e^{\mu t}(-\lambda)^{t/\sigma}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma\left(\frac{t}{\sigma}+1\right)}{p!\,\Gamma\left(\frac{t}{\sigma}+1-p\right)}\left(\frac{\lambda+e^{(x-\mu)/\sigma}}{\lambda}\right)^p\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\theta}\right), \tag{4.2}$$

if and only if

$$F(x)=1-\frac{(1+\lambda)^{\theta}e^{-\theta(x-\mu)/\sigma}}{\sigma[1+\lambda e^{-(x-\mu)/\sigma}]^{\theta}},\quad x>\mu.$$

Proof. From (4.1), we have

$$E[e^{tX(s,n,m,k)}|X(r,n,m,k)=x]$$
$$=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_x^{\infty}e^{ty}\left[1-\left(\frac{\bar F(y)}{\bar F(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{\bar F(y)}{\bar F(x)}\right)^{\gamma_s-1}\frac{f(y)}{\bar F(x)}dy. \tag{4.3}$$

By setting $u=\frac{\bar F(y)}{\bar F(x)}$ from (2.2) in (4.3), we obtain

$$E[e^{tX(s,n,m,k)}|X(r,n,m,k)=x]$$
$$=\frac{e^{\mu t}(-\lambda)^{t/\sigma}C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma\left(\frac{t}{\sigma}+1\right)}{p!\,\Gamma\left(\frac{t}{\sigma}+1-p\right)}\left(\frac{\lambda+e^{(x-\mu)/\sigma}}{\lambda}\right)^p$$
$$\times\int_0^1 u^{\gamma_s+(p/\theta)-1}(1-u^{m+1})^{s-r-1}du. \tag{4.4}$$

Again by setting $t=u^{m+1}$ in (4.4) and simplifying the resulting expression, we derive the relation given in (4.2).

To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_x^{\infty}e^{ty}[(\bar F(x))^{m+1}-(\bar F(y))^{m+1}]^{s-r-1}[\bar F(y)]^{\gamma_s-1}f(y)dy$$
$$=[\bar F(x)]^{\gamma_{r+1}}H_r(x), \tag{4.5}$$

where

$$H_r(x)=e^{\mu t}(-\lambda)^{t/\sigma}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma\left(\frac{t}{\sigma}+1\right)}{p!\,\Gamma\left(\frac{t}{\sigma}+1-p\right)}\left(\frac{\lambda+e^{(x-\mu)/\sigma}}{\lambda}\right)^p\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\theta}\right).$$

Differentiating (4.5) both sides with respect to x and rearranging the terms, we get

$$-\frac{C_{s-1}[\bar F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_x^{\infty}e^{ty}[(\bar F(x))^{m+1}-(\bar F(y))^{m+1}]^{s-r-2}[\bar F(y)]^{\gamma_s-1}f(y)dy$$
$$=H_r'(x)[\bar F(x)]^{\gamma_{r+1}}-\gamma_{r+1}H_r(x)[\bar F(x)]^{\gamma_{r+1}-1}f(x)$$

or

$$-\gamma_{r+1}H_{r+1}(x)[\bar{F}(x)]^{\gamma_{r+2}+m}f(x) = H_r'(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1}H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1}f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]} = \frac{\theta e^{(x-\mu)/\sigma}}{\sigma[1+\lambda e^{(x-\mu)/\sigma}]}$$

which proves that

$$F(x) = 1 - \frac{(1+\lambda)^{\theta} e^{-\theta(x-\mu)/\sigma}}{\sigma[1+\lambda e^{-(x-\mu)/\sigma}]^{\theta}}, \quad x > \mu.$$

Remark 4.1. For $m = 0,\ k = 1$ and $m = -1,\ k = 1$, we obtain the characterization results of the extended generalized half logistic distribution based on order statistics and record values, respectively.

5. Concluding Remarks

(i) In this paper, we propose a new explicit expressions and recurrence relations for marginal and joint moment generating functions of gos from the extended generalized half logistic distribution. Further, characterization of this distribution has also been obtained on using the conditional expectation gos.

(ii) The recurrence relations for moments of ordered random variables are important because they reduce the amount of direct computations for moments, evaluate the higher moments and they can be used to characterize distributions.

iii) The recurrence relations of higher joint moments enable us to derive single, product, triple and quadruple moments which can be used in Edgeworth approximate inference.

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