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TOTALLY CONTACT UMBILICAL SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KENMOTSU MANIFOLDS

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Abstract. In this paper, we study totally contact umbilical slant lightlike submanifolds of indefinite Kenmotsu manifolds. We prove that there does not exist totally contact umbilical proper slant lightlike submanifold in indefinite Kenmotsu manifolds other than totally contact geodesic proper slant lightlike submanifold. We also prove that there does not exist totally contact umbilical proper slant lightlike submanifold of indefinite Kenmotsu space forms. Finally, we give some characterization theorems on minimal slant lightlike submanifolds of indefinite Kenmotsu manifolds.

1. Introduction

In the theory of submanifolds of semi-Riemannian manifolds it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial. Thus, the study becomes more interesting and remarkably different from the study of non-degenerate submanifolds. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was presented by Duggal and Bejancu in [7]. Chen [5, 6], introduced the notion of slant submanifolds as a generalizing of holomorphic and totally real submanifolds for complex geometry and further extended by Lotta [11] for contact geometry. Cabrerizo et. al. [3, 4] studied slant, semi-slant and bi-slant submanifolds in contact geometry. They all studied the geometry of slant submanifolds with positive definite metric. Therefore this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily definite. Thus the geometry of slant submanifolds with indefinite metric became a topic of chief discussion and Sahin [13] played a very crucial role in this study by introducing the notion of slant lightlike submanifolds of indefinite Hermitian manifolds.

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Recently the notion of slant lightlike submanifolds of indefinite Kenmotsu manifolds is introduced by Gupta et. al. in [9] and obtained necessary and sufficient conditions for their existence.

In this paper, we study totally contact umbilical slant lightlike submanifolds of indefinite Kenmotsu manifolds. We prove that there do not exist totally contact umbilical proper slant lightlike submanifolds in indefinite Kenmotsu manifolds other than totally contact geodesic proper slant lightlike submanifolds. We also prove that there do not exist totally contact umbilical proper slant lightlike submanifolds of indefinite Kenmotsu space forms. Finally, we give characterization theorems on minimal slant lightlike submanifolds.

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi, V, \eta, \bar{g})$, where ϕ is a $(1,1)$ tensor field, V is a vector field called structure vector field, η is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ satisfying

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1, \tag{1}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \quad \bar{g}(X, V) = \eta(X), \tag{2}$$

for $X, Y \in \Gamma(T\bar{M})$, where $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$.

An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Kenmotsu manifold if (see [2]),

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(\phi X, Y)V + \eta(Y)\phi X, \text{ and } \bar{\nabla}_X V = -X + \eta(X)V, \tag{3}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}$.

A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called an r-lightlike submanifold [7] if it admits a degenerate metric g induced from $\bar{g}$, whose radical distribution $RadTM = TM \cap TM^\perp$ is of rank r, where $0 \leq r \leq min\{m, n\}$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is, $TM = RadTM \perp S(TM)$ and $S(TM^\perp)$ be a screen transversal vector bundle, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^\perp$. For any local basis $\{\xi_i\}$ of $RadTM$, there exists a null vector bundle $ltr(TM)$ of $RadTM$ in $(S(TM))^\perp$ such that $\{N_i\}$ is a basis of $ltr(TM)$ satisfying

$$\bar{g}(N_i, N_j) = 0 \quad \text{and} \quad \bar{g}(N_i, \xi_j) = \delta_{ij}, \tag{4}$$

for any $i, j \in \{1, 2, \ldots, r\}$. Let $tr(TM)$ be the complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{5}$$

$$T\bar{M}|_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{6}$$

Let $\bar{\nabla}$ and ∇ denote the linear connections on $\bar{M}$ and M, respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{7}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{8}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, A_U is a linear operator on M, known as the shape operator. Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then Gauss and Weingarten formulae become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{9}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{10}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$. As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{12}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using (9)-(12), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{13}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$. Let $\bar{P}$ is a projection of TM on $S(TM)$ then we have

$$\nabla_X \bar{P} Y = \nabla^*_X \bar{P} Y + h^*(X, \bar{P} Y), \tag{14}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{15}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X \bar{P}Y, A^*_\xi X\}$ and $\{h^*(X,\bar{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$ respectively. Here ∇^* and ∇^{*t}_X are linear connections on $S(TM)$ and $RadTM$ respectively. By using (9)-(10) and (14)-(15), we obtain

$$\bar{g}(h^l(X,\bar{P}Y),\xi) = g(A^*_\xi X, \bar{P}Y), \tag{16}$$

$$\bar{g}(h^*(X,\bar{P}Y),N) = \bar{g}(A_N X, \bar{P}Y). \tag{17}$$

3. Slant lightlike submanifolds

A lightlike submanifold has two distributions, namely the radical distribution and the screen distribution. The radical distribution is totally lightlike and it is not possible to define angle between two vector fields of the radical distribution where the screen distribution is non-degenerate. There are some definitions for angle between two vector fields in Lorentzian setup [12], but not appropriate for our goal. Therefore to introduce the notion of slant lightlike submanifolds one needs a Riemannian distribution. For such distribution Gupta et. al. [9] proved the following lemmas.

Lemma 3.1. *Let M be an r-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ of index $2q$ with structure vector field V tangent to M. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \bigcap \phi RadTM = 0$. Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi ltr(TM) \bigcap \phi RadTM = \{0\}$.*

Lemma 3.2. *Let M be an r-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ of index $2r$ with structure vector field V tangent to M. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \bigcap \phi RadTM = \{0\}$. Then any complementary distribution to $\phi ltr(TM)$ $\bigoplus \phi RadTM$ in screen distribution S(TM) is Riemannian.*

Definition 3.3 ([9])**.** Let M be an r-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ of index $2r$ with structure vector field V tangent to M. Then we say that M is a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(A) $RadTM$ is a distribution on M such that $\phi RadTM \cap Rad(TM) = \{0\}$.

(B) For all $x \in U \subset M$ and for each non zero vector field X tangent to $\bar{D} = D \perp \{V\}$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $\bar{D}_x$ is constant, where D is complementary distribution to $\phi ltr(TM) \bigoplus \phi RadTM$ in screen distribution $S(TM)$.

The constant angle $\theta(X)$ is called the slant angle of $\bar{D}$. A slant lightlike submanifold M is said to be proper if $D \neq \{0\}$, and $\theta \neq 0, \frac{\pi}{2}$.

Then the tangent bundle TM of M is decomposed as

$$TM = RadTM \perp S(TM) = RadTM \perp (\phi RadTM \oplus \phi ltr(TM)) \perp \bar{D}, \tag{18}$$

where $\bar{D} = D \perp \{V\}$. Therefore for any $X \in \Gamma(TM)$ we write

$$\phi X = TX + FX, \tag{19}$$

where TX is the tangential component of ϕX and FX is the transversal component of ϕX. Similarly for any $U \in \Gamma(tr(TM))$ we write

$$\phi U = BU + CU, \tag{20}$$

where BU is the tangential component of ϕV and CU is the transversal component of ϕV. Using the decomposition in (18), we denote by P_1, P_2, Q_1, Q_2 and $\bar{Q}_2$ be the projections on the distributions $RadTM$, $\phi RadTM$, $\phi ltr(TM)$, D and $\bar{D} = D \perp V$, respectively. Then for any $X \in \Gamma(TM)$, we can write

$$X = P_1X + P_2X + Q_1X + \bar{Q}_2X, \tag{21}$$

where $\bar{Q}_2X = Q_2X + \eta(X)V$. Applying ϕ to (21), we obtain

$$\phi X = \phi P_1X + \phi P_2X + FQ_1X + TQ_2X + FQ_2X. \tag{22}$$

Then using (19) and (20), we get

$$\phi P_1X = TP_1X \in \Gamma(\phi RadTM), \quad \phi P_2X = TP_2X \in \Gamma(RadTM),$$

$$FP_1X = FP_2X = 0, \quad TQ_2X \in \Gamma(D), \quad FQ_1X \in \Gamma(ltr(TM)).$$

Now, differentiating (22) and using (9)-(12), (19) and (20), for any $X, Y \in \Gamma(TM)$, we have

$$(\nabla_X T)Y = A_{FQ_1Y}X + A_{FQ_2Y}X + Bh(X,Y) - g(\phi X, Y)V + \eta(Y)TX, \tag{23}$$

and

$$\begin{aligned} D^s(X, FQ_1Y) + D^l(X, FQ_2Y) = {} & F\nabla_X Y - h(X, TY) + Ch(X,Y) - \nabla^s_X FQ_2Y \\ & - \nabla^l_X FQ_1Y + \eta(Y)FQ_1X + \eta(Y)FQ_2X. \end{aligned} \tag{24}$$

We mention the following corollary for later use:

Corollary 3.4. *([9]) Let M be a slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ with structure vector field V tangent to M. Then we have*

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = \cos^2\theta[g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)] \tag{25}$$

and

$$g(F\bar{Q}_2X, F\bar{Q}_2Y) = \sin^2\theta[g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)] \tag{26}$$

for any $X, Y \in \Gamma(TM)$.

4. Totally contact umbilical slant lightlike submanifolds

Definition 4.1 ([14])**.** If the second fundamental form h of a submanifold tangent to characteristic vector field V, of a Sasakian manifold $\bar{M}$ is of the form

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \tag{27}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called a totally contact umbilical and totally contact geodesic if $\alpha = 0$.

The above definition also holds for a lightlike submanifold M. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_L + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \tag{28}$$

$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_S + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \tag{29}$$

where $\alpha_L \in \Gamma(ltr(TM))$ and $\alpha_S \in \Gamma(S(TM^\perp))$.

Lemma 4.2. *Let M be a slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $FQ_2X \in \Gamma(S(TM^\perp))$, for any $X \in \Gamma(TM)$.*

Proof. Using (4) and (5) it is clear that $FQ_2X \in \Gamma(S(TM^\perp))$ if $g(FQ_2X, \xi) = 0$. Therefore $g(FQ_2X,\xi) = g(\phi Q_2X - TQ_2X, \xi) = g(\phi Q_2X, \xi) = -g(Q_2X, \phi\xi) = 0$. Hence the result follows.

Thus from the Lemma (4.2) it follows that $F(D_p)$ is a subspace of $S(TM^\perp)$. Therefore there exists an *invariant* subspace μ_p of $T_p\bar{M}$ such that

$$S(T_pM^\perp) = F(D_p) \perp \mu_p, \tag{30}$$

therefore

$$T_p\bar{M} = S(T_pM) \perp \{Rad(T_pM) \oplus ltr(T_pM)\} \perp \{F(D_p) \perp \mu_p\}.$$

Theorem 4.3. *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then at least one of the following statements is true*

(i) *M is an anti-invariant submanifold.*

(ii) $D = \{0\}$.

(iii) *If M is a proper slant submanifold, then $\alpha_S \in \Gamma(\mu)$.*

Proof. Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then for any $X = Q_2X \in \Gamma(D)$ and using (27), we have $h(TQ_2X, TQ_2X) = g(TQ_2X, TQ_2X)\alpha$. Using (7) and (25), we get

$$\bar{\nabla}_{TQ_2X}TQ_2X - \nabla_{TQ_2X}TQ_2X = \cos^2\theta[g(Q_2X, Q_2X)]\alpha.$$

Using (19) and the fact that $\bar{M}$ is Kenmotsu manifold, that is (3), we obtain

$$\phi\bar{\nabla}_{TQ_2X}Q_2X - \bar{\nabla}_{TQ_2X}FQ_2X - \nabla_{TQ_2X}TQ_2X = \cos^2\theta[g(Q_2X,Q_2X)]\alpha,$$

then using (9), (12), (19), (20), (28) and (29), we have

$$\begin{aligned}&T\nabla_{TQ_2X}Q_2X + F\nabla_{TQ_2X}Q_2X + g(TQ_2X,X)\phi\alpha^l + g(TQ_2X,X)B\alpha^s\\&+g(TQ_2X),X)C\alpha^s + A_{FQ_2X}TQ_2X - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X,FQ_2X)\\&-\nabla_{TQ_2X}TQ_2X = \cos^2\theta[g(Q_2X,Q_2X)]\alpha.\end{aligned}$$

Equating the transversal components, we get

$$\begin{aligned}&F\nabla_{TQ_2X}Q_2X + g(TQ_2X,X)C\alpha^s - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X,FQ_2X)\\&= \cos^2\theta[g(Q_2X,Q_2X)]\alpha.\end{aligned} \tag{31}$$

On the other hand, from (26), we have $g(FQ_2X,FQ_2X) = \sin^2\theta[g(Q_2X,Q_2X)]$, for any $X \in \Gamma(D)$. Taking the covariant derivative of the above equation with respect to TQ_2X, we obtain

$$g(\nabla^s_{TQ_2X}FQ_2X,FQ_2X) = \sin^2\theta g(\nabla_{TQ_2X}Q_2X,Q_2X). \tag{32}$$

Now taking the inner product in (31) with FQ_2X, we obtain

$$g(F\nabla_{TQ_2X}Q_2X,FQ_2X) - g(\nabla^s_{TQ_2X}FQ_2X,FQ_2X) = \cos^2\theta[g(Q_2X,Q_2X)]g(\alpha_S,FQ_2X).$$

Then using (26) and (32), we get $\cos^2\theta[g(Q_2X,Q_2X)]g(\alpha_S,FQ_2X) = 0$, it follows that either $\theta = \pi/2$ or $Q_2X = 0$ or $\alpha_S \in \Gamma(\mu)$. This completes the proof.

Lemma 4.4. *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then $g(\nabla_XX,\phi\xi) = 0$, for any $X \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM))$.*

Proof. Let $X \in \Gamma(D)$ therefore $X = Q_2X$, then using (3), (9) and (12) for a totally contact umbilical slant lightlike submanifold, we have

$$\begin{aligned}g(\nabla_XX,\phi\xi) &= \bar{g}(\bar{\nabla}_XX,\phi\xi) = -\bar{g}(\bar{\nabla}_XTQ_2X,\xi) - \bar{g}(\bar{\nabla}_XFQ_2X,\xi)\\&= -g(h^l(X,TQ_2X),\xi) - \bar{g}(D^l(X,FQ_2X),\xi)\\&= -\bar{g}(D^l(X,FQ_2X),\xi),\end{aligned} \tag{33}$$

since for $X \in \Gamma(D)$, using (2), (19) and (28) we have $h^l(X,TQ_2X) = \{g(X,TQ_2X)\}\alpha_L = 0$. Since $\eta(Q_2X) = 0$ and $\eta(\xi) = 0$ therefore by replacing W by FQ_2X and Y by ξ in (13) and using the fact that M is a totally contact umbilical slant lightlike submanifold, we obtain

$$\bar{g}(D^l(X,FQ_2X),\xi) = -\bar{g}(h^s(X,\xi),FQ_2X) = -g(X,\xi)g(\alpha_S,FQ_2X) = 0. \tag{34}$$

Hence from (33) and (34), the result follows.

Theorem 4.5. *Every totally contact umbilical proper slant lightlike submanifold of an indefinite Kenmotsu manifold is totally contact geodesic.*

Proof. Since M is a totally contact umbilical slant lightlike submanifold therefore for any $X = Q_2X \in \Gamma(D)$, using (27) we have $h(TQ_2X, TQ_2X) = g(TQ_2X, TQ_2X)\alpha$, then using (25), we get

$$h(TQ_2X, TQ_2X) = \cos^2\theta[g(Q_2X, Q_2X) - \eta(Q_2X)\eta(Q_2X)]\alpha$$
$$= \cos^2\theta[g(Q_2X, Q_2X)]\alpha. \tag{35}$$

Using (1) and (24) for any $X \in \Gamma(D)$, we obtain

$$F\nabla_{TQ_2X}X = h(TQ_2X, TQ_2X) - Ch(TQ_2X, X) + \nabla^s_{TQ_2X}FQ_2X$$
$$+D^l(TQ_2X, FQ_2X). \tag{36}$$

Since M is a totally contact umbilical slant lightlike submanifold therefore $Ch(TQ_2X, X) = g(TQ_2X, X)C\alpha = 0$, therefore using (35) and (36), we get

$$\cos^2\theta[g(Q_2X, Q_2X)]\alpha = F\nabla_{TQ_2X}X - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X). \tag{37}$$

Taking the scalar product of both sides of (37) with respect to FQ_2X, we obtain

$$\cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = \bar{g}(F\nabla_{TQ_2X}X, FQ_2X) - \bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X),$$

using (26), we get

$$\cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = \sin^2\theta[g(\nabla_{TQ_2X}X, Q_2X)] - \bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X). \tag{38}$$

Now, for any $X = Q_2X \in \Gamma(D)$, (26) implies that

$$g(FQ_2X, FQ_2X) = \sin^2\theta[g(Q_2X, Q_2X)],$$

taking covariant derivative with respect to $\bar{\nabla}_{TQ_2X}$, we get

$$\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = \sin^2\theta[g(\nabla_{TQ_2X}Q_2X, Q_2X)]. \tag{39}$$

Using (39) in (38), we obtain $\cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = 0$. Since M is a proper slant lightlike submanifold and g is a Riemannian metric on D therefore we have $\bar{g}(\alpha_s, FQ_2X) = 0$. Thus using the Lemma (4.2) and the equation (30), we obtain $\alpha_S \in \Gamma(\mu)$. Let $X, Y \in \Gamma(D)$ then using the Kenmotsu property of $\bar{M}$, we have $\bar{\nabla}_X\phi Y = \phi\bar{\nabla}_XY - g(TX, Y)V$, then using (27), we obtain

$$\nabla_X TQ_2Y + g(X, TQ_2Y)\alpha - A_{FQ_2Y}X + \nabla^s_X FQ_2Y + D^l(X, FQ_2Y)$$

$$= T\nabla_X Y + F\nabla_X Y + g(X,Y)\phi\alpha - g(TX,Y)V. \tag{40}$$

Taking the scalar product of both sides of (40) with respect to $\phi\alpha_S$ and using the fact that μ is an invariant subbundle of $T\bar{M}$, we obtain

$$\bar{g}(\nabla^s_X FQ_2X, \phi\alpha_S) = g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S). \tag{41}$$

Again using the Kenmotsu character of $\bar{M}$, we have $\bar{\nabla}_X\phi\alpha_S = \phi\bar{\nabla}_X\alpha_S$, this implies that

$$-A_{\phi\alpha_S}X + \nabla^s_X\phi\alpha_S + D^l(X,\phi\alpha_S) = -TA_{\alpha_S}X - FA_{\alpha_S}X + B\nabla^s_X\alpha_S + C\nabla^s_X\alpha_S + \phi D^l(X,\alpha_S), \tag{42}$$

taking the scalar product of both sides of above equation with respect to FQ_2Y and using invariant character of μ, that is, $C\nabla^s_X\alpha_S \in \Gamma(\mu)$ and using (1) and (26), we get

$$\bar{g}(\nabla^s_X\phi\alpha_S, FQ_2Y) = -g(FA_{\alpha_S}X, FQ_2Y) = -\sin^2\theta[g(A_{\alpha_S}X, Q_2Y)]. \tag{43}$$

Since $\bar{\nabla}$ is a metric connection therefore $(\bar{\nabla}_X g)(FQ_2Y, \phi\alpha_S) = 0$, this further implies that $\bar{g}(\nabla^s_X FQ_2Y, \phi\alpha_S) = \bar{g}(\nabla^s_X\phi\alpha_S, FQ_2Y)$, therefore using (43), we obtain

$$\bar{g}(\nabla^s_X FQ_2Y, \phi\alpha_S) = -\sin^2\theta[g(A_{\alpha_S}X, Q_2Y)]. \tag{44}$$

From (41) and (44), we have $g(Q_2X,Q_2Y)g(\alpha_S,\alpha_S) = -\sin^2\theta g[(A_{\alpha_S}X, Q_2Y)]$, then using (13), we obtain $g(Q_2X,Q_2Y)g(\alpha_S,\alpha_S) = -\sin^2\theta[g(Q_2X,Q_2Y)]g(\alpha_S,\alpha_S)$, this implies that $(1+\sin^2\theta)[g(Q_2X,Q_2Y)]g(\alpha_S,\alpha_S) = 0$. Since M is a proper slant lightlike submanifold therefore $\sin^2\theta \neq -1$ and g is a Riemannian metric on D therefore we obtain

$$\alpha_S = 0. \tag{45}$$

Next, for $X \in \Gamma(D)$, using the Kenmotsu character of $\bar{M}$, we have $\bar{\nabla}_X\phi X = \phi\bar{\nabla}_X X$, this implies that $\nabla_X TQ_2X + h(X,TQ_2X) - A_{FQ_2X}X + \nabla^s_X FQ_2X + D^l(X,FQ_2X) = T\nabla_X X + F\nabla_X X + Bh(X,X) + Ch(X,X)$. Since M is totally contact umbilical slant lightlike submanifold therefore using $h(X,TQ_2X) = 0$, in above equation and then comparing the tangential components, we obtain $\nabla_X TQ_2X - A_{FQ_2X}X = T\nabla_X X + Bh(X,X)$. Taking the scalar product of both sides with respect to $\phi\xi \in \Gamma(\phi Rad(TM))$ and then using the Lemma (4.4), we get

$$g(A_{FQ_2X}X, \phi\xi) + \bar{g}(h^l(Q_2X,Q_2X),\xi) = 0. \tag{46}$$

Now using (11), we have $\bar{g}(h^s(X,\phi\xi), FQ_2X) + \bar{g}(\phi\xi, D^l(X,FQ_2X)) = g(A_{FQ_2X}X,\phi\xi)$, since M is a totally contact umbilical slant lightlike submanifold therefore using (29) and (45), we obtain

$$g(A_{FQ_2X}X, \phi\xi) = 0. \tag{47}$$

Using (47) in (46), we obtain that $\bar{g}(h^l(Q_2X,Q_2X),\xi)=0$, then using (28), we obtain $g(Q_2X,Q_2X)\bar{g}(\alpha_L,\xi)=0$. Since g is a Riemannian metric on D therefore $\bar{g}(\alpha_L,\xi)=0$, then using (4), we obtain that

$$\alpha_L=0. \tag{48}$$

Thus from (45) and (48), the proof is complete.

Next, denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively, then using (9)-(12), we have

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y\\ &\quad -A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z)\\ &\quad +D^l(X,h^s(Y,Z)) - D^l(Y,h^s(X,Z)) + (\nabla_X h^s)(Y,Z)\\ &\quad -(\nabla_Y h^s)(X,Z) + D^s(X,h^l(Y,Z)) - D^s(Y,h^l(X,Z)),\end{aligned} \tag{49}$$

where

$$(\nabla_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y,Z) - h^s(Y,\nabla_X Z),$$

and

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y,Z) - h^l(Y,\nabla_X Z). \tag{50}$$

An indefinite Kenmotsu space form is a connected indefinite Kenmotsu manifold of constant holomorphic sectional curvature c and denoted by $\bar{M}(c)$. Then the curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by (see [10])

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{c-3}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\\ &\quad +\bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V + \bar{g}(\phi Y,Z)\phi X + \bar{g}(\phi Z,X)\phi Y\\ &\quad -2\bar{g}(\phi X,Y)\phi Z\}.\end{aligned} \tag{51}$$

for X,Y,Z vector fields on $\bar{M}$.

Theorem 4.6. *There does not exist a totally contact umbilical proper slant lightlike submanifold of an indefinite Kenmotsu space form $\bar{M}(c)$ such that $c\neq -1$.*

Proof. Suppose M be a totally contact umbilical proper lightlike submanifold of $\bar{M}(c)$ such that $c\neq -1$. Then for any $X\in\Gamma(D)$, $Z\in\Gamma(\phi ltr(TM))$ and $\xi\in\Gamma(Rad(TM))$, using (2) and (51), we obtain

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -\frac{c+1}{2}g(Q_2X,Q_2X)g(\phi Z,\xi). \tag{52}$$

On the other hand using (27) and (49), we get

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = \bar{g}((\nabla_X h^l)(\phi X,Z),\xi) - \bar{g}((\nabla_{\phi X} h^l)(X,Z),\xi), \tag{53}$$

where using (28) and (50), we have

$$(\nabla_X h^l)(\phi X, Z) = -g(\nabla_X \phi X, Z)\alpha_L - g(TQ_2 X, \nabla_X Z)\alpha_L, \tag{54}$$

and

$$(\nabla_{\phi X} h^l)(X, Z) = -g(\nabla_{\phi X} X, Z)\alpha_L - g(X, \nabla_{\phi X} Z)\alpha_L. \tag{55}$$

Using (54) and (55) in (53), we obtain

$$\begin{aligned}\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -g(\nabla_X \phi X, Z)\bar{g}(\alpha_L,\xi) - g(\phi X, \nabla_X Z)\bar{g}(\alpha_L,\xi)\\ +g(\nabla_{\phi X} X, Z)\bar{g}(\alpha_L,\xi) + g(X, \nabla_{\phi X} Z)\bar{g}(\alpha_L,\xi).\end{aligned} \tag{56}$$

Now using (27), we have $g(\phi X, \nabla_X Z) = -\bar{g}(\bar{\nabla}_X \phi X, Z) = -g(\nabla_X \phi X, Z)$ and $g(X, \nabla_{\phi X} Z) = -\bar{g}(\bar{\nabla}_{\phi X} X, Z) = -g(\nabla_{\phi X} X, Z)$. Hence (56) becomes $\bar{g}(\bar{R}(X,\phi X)Z,\xi) = 0$, using this in (52), we have $(c+1)g(Q_2X, Q_2X)g(\phi Z,\xi) = 0$. Since g is a Riemannian metric on D and $g(\phi Z,\xi) \neq 0$, therefore $c = -1$. This contradiction completes the proof.

5. Minimal slant lightlike submanifolds

In [7], a minimal lightlike submanifold M is defined when M is a hypersurface of a 4-dimensional Minkowski space. Then in [1], a general notion of minimal lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ is introduced as follows:

Definition 5.1. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if

(i) $h^s = 0$ on $Rad(TM)$ and

(ii) $trace\ h = 0$, where trace is written with respect to g restricted to $S(TM)$.

We use the quasi orthonormal basis of M given by

$$\{\xi_1, \ldots, \xi_r, \phi\xi_1, \ldots, \phi\xi_r, V, e_1, \ldots, e_q, \phi N_1, \ldots, \phi N_r\},$$

such that $\{\xi_1, \ldots, \xi_r\}$, $\{\phi\xi_1, \ldots, \phi\xi_r\}$, $\{e_1, \ldots, e_q\}$ and $\{\phi N_1, \ldots, \phi N_r\}$ form a basis of $Rad(TM)$, $\phi(Rad(TM))$, D and $\phi(ltr(TM))$ respectively.

Theorem 5.2. *A totally contact umbilical proper slant lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is minimal if and only if $trace\ A_{W_k} = 0$ and $trace\ A^*_{\xi_i} = 0$ on D, where $\{W_k\}_{k=1}^{l}$ is a basis of $S(TM^\perp)$ and $\{\xi_i\}_{i=1}^{r}$ is a basis of $Rad(TM)$.*

Proof. Using (1) and (3) we have $\bar{\nabla}_V V = 0$ therefore using (9), we get $h^l(V,V) = 0$ and $h^s(V,V) = 0$. Thus from the definition of minimal submanifold and (18), a slant lightlike submanifold is minimal if and only if

$$\sum_{i=1}^{r} h(\phi\xi_i, \phi\xi_i) + \sum_{i=1}^{r} h(\phi N_i, \phi N_i) + \sum_{j=1}^{q} h(e_j, e_j) = 0$$

and $h^s = 0$ on $Rad(TM)$. Since M is totally contact umbilical therefore from (27), we have $h(\phi\xi_i, \phi\xi_i) = 0$ and $h(\phi N_i, \phi N_i) = 0$. Similarly $h^s = 0$ on $Rad(TM)$. Thus M is minimal submanifold if and only if $\sum_{j=1}^{q} h(e_j, e_j) = 0$, where

$$\begin{aligned}\sum_{j=1}^{q} h(e_j, e_j) &= \sum_{j=1}^{q} h^l(e_j, e_j) + h^s(e_j, e_j) \\ &= \sum_{j=1}^{q} \{\frac{1}{r}\sum_{i=1}^{r} g(h^l(e_j, e_j), \xi_i)N_i + \frac{1}{l}\sum_{k=1}^{l} g(h^s(e_j, e_j), W_k)W_k\},\end{aligned}$$

where $\{W_1, \ldots, W_l\}$ is an orthonormal basis of $S(TM^{\perp})$. Using (13) and (16) we obtain

$$\sum_{j=1}^{q} h(e_j, e_j) = \sum_{j=1}^{q} \{\frac{1}{r}\sum_{i=1}^{r} g(A^*_{\xi_i} e_j, e_j)N_i + \frac{1}{l}\sum_{k=1}^{l} g(A_{W_k} e_j, e_j)W_k\}. \tag{57}$$

Thus our assertion follows from (57).

Definition 5.3 ([8])**.** A lightlike submanifold is called irrotational if and only if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for all $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.

Theorem 5.4. *Let M be an irrotational slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then M is minimal if and only if $trace\, A_{W_k}|_{S(TM)} = 0$, $trace\, A^*_{\xi_i}|_{S(TM)} = 0$, where $\{W_k\}_{k=1}^{l}$ is a basis of $S(TM^{\perp})$ and $\{\xi_i\}_{i=1}^{r}$ is a basis of $Rad(TM)$.*

Proof. Since $\bar{\nabla}_V V = 0$ using (1) and (3) therefore using (9), we get $h^l(V, V) = 0$ and $h^s(V, V) = 0$. Moreover, M irrotational implies $h^s(X, \xi) = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Thus h^s vanishes on $Rad(TM)$. Hence M is minimal if and only if trace $h = 0$ on $S(TM)$, that is, M is minimal if and only if

$$\sum_{i=1}^{r} h(\phi\xi_i, \phi\xi_i) + \sum_{i=1}^{r} h(\phi N_i, \phi N_i) + \sum_{j=1}^{q} h(e_j, e_j) = 0.$$

Using (13) and (16) we obtain

$$\sum_{i=1}^{r} h(\phi\xi_i, \phi\xi_i) = \sum_{i=1}^{r} \{\frac{1}{r}\sum_{a=1}^{r} g(A^*_{\xi_a}\phi\xi_i, \phi\xi_i)N_a + \frac{1}{l}\sum_{k=1}^{l} g(A_{W_k}\phi\xi_i, \phi\xi_i)W_k\}. \tag{58}$$

Similarly, we have

$$\sum_{i=1}^{r} h(\phi N_i, \phi N_i) = \sum_{i=1}^{r} \{\frac{1}{r}\sum_{a=1}^{r} g(A^*_{\xi_a}\phi N_i, \phi N_i)N_a + \frac{1}{l}\sum_{k=1}^{l} g(A_{W_k} e_j, e_j)W_k\}, \tag{59}$$

and

$$\sum_{j=1}^{q} h(e_j, e_j) = \sum_{j=1}^{q} \{\frac{1}{r}\sum_{i=1}^{r} g(A^*_{\xi_i} e_j, e_j)N_i + \frac{1}{l}\sum_{k=1}^{l} g(A_{W_k} e_j, e_j)W_k\}. \tag{60}$$

Thus our assertion follows from (58)-(60).

References

[1] C. L. Bejan and K. L. Duggal, *Global lightlike manifolds and harmonicity*, Kodai Math. J., **28**(2005), 131–145.

[2] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, New York, Birkhauser, 2002.

[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J., **42**(2000), 125–138.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata. **78**(1999), 183–199.

[5] B. Y. Chen, *Slant Immersions*, Bull. Austral. Math. Soc., **41**(1990), 135–147.

[6] B. Y. Chen, Geometry of Slant Submanifolds, Leuven, Katholieke Universiteit, 1990.

[7] K. L. Duggal and A. Bejancu, Lightlike Submanifolds of semi-Riemannian Manifolds and Applications, Vol. 364 of Mathematics and its Applications, The Netherlands, Kluwer Academic Publishers, 1996.

[8] K. L. Duggal and B. Sahin, *Screen Cauchy Riemann lightlike submanifolds*, Acta Math. Hungar., **106**(2005), 125–153.

[9] R. S. Gupta and A. Sharfuddin, *Slant lightlike submanifolds of indefinite Kenmotsu manifolds*, Turk J. Math., **35**(2011), 115–127.

[10] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., **24**(1972), 93–103.

[11] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Soc. Sci. Math. Roumanie., **39**(1996), 183–198.

[12] B. O'Neill, Semi-Riemannian Geometry with applications to relativity, New York, Academic Press, 1983.

[13] B. Sahin, *Slant lightlike submanifolds of indefinite Hermitian manifolds*, Balkan J. Geom. Appl., **13**(2008), 107–119.

[14] K. Yano and M. Kon, Stuctures on Manifolds. Vol. 3 of Series in Pure Mathematics, Singapore, World Scientific, 1984.

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