THE EQUIVALENCE OF MANN AND ISHIKAWA ITERATIONS DEALING WITH ψ–UNIFORMLY PSEUDOCONTRACTIVE MAPS WITHOUT BOUNDED RANGE

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Abstract. We prove that Mann and Ishikawa iterations are equivalent models dealing with ψ-uniformly pseudocontractive or d-weakly contractive maps without bounded range.

1. Introduction

In this paper $X$ denotes a real Banach space with $X^*$ strictly convex, $T : X \to X$ a map and let $x_0, u_0 \in X$. We consider the following iteration known as Mann iteration,

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n.$$  \hspace{1cm} (1.1)

The sequence $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. We consider the following iteration known as Ishikawa iteration, (\cite{8})

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n.$$  \hspace{1cm} (1.2)

The sequences $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ satisfy

$$\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0, \sum_{n=1}^{\infty} \alpha_n = +\infty.$$  \hspace{1cm} (1.3)

The duality normalized map $J : X \to 2^{X^*}$ is given by

$$J(x) = \{ f \in X^* : \langle f, x \rangle = \|x\|^2, \|x\| = \|f\| \}.$$  \hspace{1cm} (1.4)

We have

$$\langle f, y \rangle \leq \|f\| \|y\|, \forall y \in X.$$  \hspace{1cm} (1.5)

The following Remark is Proposition 12.3 from [7].
Remark 1.1. ([7]) If $X$ is a real Banach space with $X^*$ strictly convex then $J(\cdot)$ is a single map and uniformly continuous on all the bounded sets of $X$.

The following result is Lemma 1 from [10].

Lemma 1.2. If $X$ is a real normed space, then the following relation is true

$$\|x+y\|^2 \leq \|x\|^2 + 2 \langle y, j(x+y) \rangle; \forall x,y \in X, \forall j(x+y) \in J(x+y).$$

(1.6)

The following definitions are from [3], [5] and [6].

Definition 1.3. Let $X$ be a normed space. A map $T : X \to X$ is called weakly contractive map if for all $x,y \in X$, there exist $\psi : [0, +\infty) \to [0, +\infty)$ a continuous and strictly increasing map such that $\psi$ is positive on $(0, +\infty)$, $\psi(0) = 0$, and the following inequality is satisfied

$$\|Tx - Ty\| \leq \|x-y\| - \psi(\|x-y\|).$$

(1.7)

A map $T : X \to X$ is called d-weakly contractive map if for all $x,y \in X$, there exist $j(x-y) \in J(x-y)$ and $\psi : [0, +\infty) \to [0, +\infty)$ a continuous and strictly increasing map such that $\psi$ is positive on $(0, +\infty)$, $\psi(0) = 0$, and the following inequality is satisfied

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \psi(\|x-y\|).$$

(1.8)

A map $T : X \to X$ is called $\psi$-uniformly pseudocontractive if there exist $j(x-y) \in J(x-y)$ and $\psi : [0, +\infty) \to [0, +\infty)$ a strictly increasing map such that $\psi$ is positive on $(0, +\infty)$, $\psi(0) = 0$ and the following inequality is satisfied

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \psi(\|x-y\|), \forall x,y \in B.$$  

(1.9)

A map $C : X \to X$ is called $\psi$-uniformly accretive if there exist $j(x-y) \in J(x-y)$ and $\psi : [0, +\infty) \to [0, +\infty)$ a strictly increasing map such that $\psi$ is positive on $(0, +\infty)$, $\psi(0) = 0$ and the following inequality is satisfied

$$\langle Cx - Cy, j(x-y) \rangle \geq \psi(\|x-y\|), \forall x,y \in X.$$  

(1.10)

We denote the identity map by $I$.

Remark 1.4. (i) If $T$ is a d-weakly contractive map, then $T$ is a $\psi$-uniformly pseudocontractive map.

(ii) The map $T$ is $\psi$-uniformly pseudocontractive if and only if $C := (I-T)$ is $\psi$-uniformly accretive.

Proposition 1.5. If $T$ is a weakly contractive map, then $T$ is a $\psi$-uniformly pseudocontractive map.
Proof. Let $j(x - y) \in J(x - y)$. Using (1.5), (1.7) and (1.4) we get

$$
(Tx - Ty, j(x - y)) \leq \|Tx - Ty\| \|j(x - y)\|
= \|Tx - Ty\| \|x - y\| \leq \|x - y\|^2 - \|x - y\| \phi(\|x - y\|) \quad (1.11)
$$

Denote $\psi(a) := a \cdot \phi(a), \forall a \in [0, \infty)$ to obtain that $\psi$ is strictly increasing and positive.

The convergence of Mann iteration for a d-weakly contractive map in Hilbert spaces, was studied in [3]. It was shown in [5] that Mann iteration (1.1) for a d-weakly contractive map without a bounded range, converges in a Banach space more general then a Hilbert space. Also, it was shown in [6] that the same iteration for a $\psi$-uniformly pseudocontractive map without a bounded range, converges in a normed space.

If $T$ is a weakly contractive, then $T$ is a nonexpansive map. In this case the equivalence between Mann and Ishikawa iterations follows from Theorem 3 of the paper [11].

The above two motivations lead us to prove, in this note, the equivalence between Mann and Ishikawa iterations, (1.1) and (1.2), dealing with $\psi$-uniformly pseudocontractive maps without bounded range. As a corollary we obtain the convergence of Ishikawa iteration for the above operatorial classes. Also, we give a positive answer to the following conjecture, (see [11], page 452), "If Mann iteration converges, so does Ishikawa iteration".

For a $\psi$-uniformly pseudocontractive (respectively, $\psi$-uniformly accretive) map, the equivalence between Mann and Ishikawa iterations was shown also in Theorem 2.1 and Corollary 3.1 from [12]. There, in [12], the set $T(X)$ was assumed to be bounded. Removing the boundedness of the range, forces us to pay a price: both $\{\alpha_n\}$ and $\{\beta_n\}$ will depend on $T$ and $x^*$ (see condition (2.1)).

Remark 1.6. Let X be a normed space and $T : X \to X$ a uniformly continuous map. Then $I - T$ is a uniformly continuous map.

The following result is Proposition 2.1.2 from [4].

**Proposition 1.7** ([4]) Let X be a normed space and $T : X \to X$ be a uniformly continuous map. Then T is bounded; i.e. it maps any bounded set into a bounded set.

Remark 1.6 and Proposition 1.7 lead to the following result.

**Remark 1.8.** Let X be a normed space and $T : X \to X$ a uniformly continuous map. Then $I - T$ is bounded; i.e. it maps any bounded set into a bounded set.

The following result, stated below, is Lemma 3.1 from [1]. In [1], the map $\psi$ is assumed to be continuous in order to obtain an estimate for the convergence rate of the sequence $\{\lambda_n\}$. Another proof for the Lemma 3.1 can be found in ([2], pages 12-13). The same lemma, without the continuity assumption on $\psi$, appears in [6].

**Lemma 1.9.** ([1]) Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative numbers and $\{\alpha_n\}$ a sequence of positive numbers satisfying the conditions $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $(\gamma_n/\alpha_n) \to 0$.
as \( n \to +\infty \). Suppose that
\[
\lambda_{n+1} \leq \lambda_n - 2\alpha_n \psi(\lambda_n) + \gamma_n,
\]
is satisfied, where \( \psi : [0, +\infty) \to [0, +\infty) \) is a strictly increasing map such that \( \psi \) is positive on \((0, +\infty)\), with \( \psi(0) = 0 \). Then \( \lim_{n \to \infty} \lambda_n = 0 \).

2. Main Result

Let \( F(T) \) denote the fixed point set of \( T \).

**Theorem 2.1.** Let \( X \) be a real Banach space with \( X^\ast \) strictly convex. If \( T : X \to X \) is a \( \psi \)-uniformly pseudocontractive and uniformly continuous map with \( x^\ast \in F(T) \), \( x_0 = u_0 \in X \) and there exists a constant \( d_0 := d_0(T, x^\ast) \in (0,1) \), which depends on \( T \) and \( x^\ast \), such that \( \{\alpha_n\}, \{\beta_n\} \) satisfy
\[
\alpha_n, \beta_n \leq d_0, \forall n \in \mathbb{N},
\]
and (1.3), then the following are equivalent:
(i) the Mann iteration (1.1) converges to the \( x^\ast \in F(T) \),
(ii) the Ishikawa iteration (1.2) converges to the same \( x^\ast \).

**Proof.** The fixed point \( x^\ast \) is unique. If not, then there exists at least another fixed point \( y^\ast \in F(T) \), with \( x^\ast \neq y^\ast \). Relation (1.9) leads to
\[
\langle Tx^\ast - Ty^\ast, J(x^\ast - y^\ast) \rangle \leq \|x^\ast - y^\ast\|^2 - \psi(\|x^\ast - y^\ast\|)
\]
\[
(x^\ast - y^\ast, J(x^\ast - y^\ast)) \leq \|x^\ast - y^\ast\|^2 - \psi(\|x^\ast - y^\ast\|)
\]
\[
\|x^\ast - y^\ast\|^2 \leq \|x^\ast - y^\ast\|^2 - \psi(\|x^\ast - y^\ast\|)
\]
\[
\psi(\|x^\ast - y^\ast\|) \leq 0 \Rightarrow \|x^\ast - y^\ast\| = 0.
\]

The implication (ii)\(\Rightarrow\)(i) is obvious, by setting, in (1.2), \( \beta_n = 0 \), for all \( n \in \mathbb{N} \). We will prove the implication (i)\(\Rightarrow\)(ii). Suppose that \( \lim_{n \to \infty} u_n = x^\ast \). If
\[
\lim_{n \to \infty} \|x_n - u_n\| = 0,
\]
then
\[
0 \leq \|x^\ast - x_n\| \leq \|u_n - x^\ast\| + \|x_n - u_n\|
\]
and it follows that
\[
\lim_{n \to \infty} x_n = x^\ast.
\]
Thus, to complete the proof it suffices to verify relation (2.3).
With $A := (I - T)$ in (1.9), we have
\[
    \langle Ax - Ay, J(x - y) \rangle = \langle (x - Tx) - (y - Ty), J(x - y) \rangle \\
    = \|x - y\|^2 - \langle Tx - Ty, J(x - y) \rangle \\
    \geq \|x - y\|^2 - \|x - y\|^2 + \psi(\|x - y\|) \\
    = \psi(\|x - y\|).  \tag{2.6}
\]

Taking $x := x_n$ and $y := u_n$ in (2.6) we obtain
\[
    \langle Ax_n - Au_n, J(x_n - u_n) \rangle \geq \psi(\|x_n - u_n\|).  \tag{2.7}
\]

Choose $R > 0$ such that $\{u_n : n \in \mathbb{N}\} \subset B_R(x^*)$ and $x_0 \in B_{2R}(x^*)$. Remark 1.8 assures that $A(B_{2R}(x^*))$ is bounded. Denote
\[
    \sigma := \text{diam}(A(B_{2R}(x^*))) + R.  \tag{2.8}
\]

Since the map $J(\cdot)$ is uniformly continuous on bounded subsets of $X$, with
\[
    \varepsilon := \frac{\psi(\frac{R}{2})}{4\sigma} > 0,  \tag{2.9}
\]
there exists a $\delta_1 > 0$ such that $\|x - y\| \leq \delta_1$ implies $\|J(x) - J(y)\| \leq \varepsilon$.

The map $T(\cdot)$ is also uniformly continuous. Thus for the same $\varepsilon$, there exits a $\delta_2 > 0$ such that $\|x - y\| \leq \delta_2$ implies $\|Tx - Ty\| \leq \varepsilon$.

We shall prove by induction that $\{x_n\}$ is bounded. We know that $0 = \|x_0 - u_0\| \leq R$. Suppose that $\|x_k - u_k\| \leq R, \forall k \in \{1, \ldots, n\}$. We shall prove that
\[
    \|x_{n+1} - u_{n+1}\| \leq R.  \tag{2.10}
\]

Assume that $\|x_n - u_n\| \leq R$ and that
\[
    \|x_{n+1} - u_{n+1}\| > R.  \tag{2.11}
\]

From $\|x_k - u_k\| \leq R, \forall k \in \{1, \ldots, n\}$ we know
\[
    \|x_k - x^*\| \leq \|x_k - u_k\| + \|u_k - x^*\| \leq 2R, \forall k \in \{1, \ldots, n\}.  \tag{2.12}
\]

From (2.12), we have $x_n \in B_{2R}(x^*)$ and the following inequality satisfied
\[
    \|x_k\| \leq \|x_k - x^*\| + \|x^*\| \leq 2R + \|x^*\|, \forall k \in \{1, \ldots, n\}.  \tag{2.13}
\]

Using $\text{diam}(A(B_{2R}(x^*))) \leq \sigma$ and $x_n \in B_{2R}(x^*)$, (i. e. $\|Ax_n\| \leq \sigma)$, we get
\[
    \|Ty_n - Tx_n\| \leq \|y_n + Ty_n + x_n - Tx_n\| + \|y_n - x_n\| \\
    = \|Ay_n - Ax_n\| + \|y_n - x_n\| \\
    \leq \|Ay_n\| + \|Ax_n\| + \|y_n - x_n\| \\
    \leq S + \sigma + \beta_n \|x_n - Tx_n\| = S + \sigma + \beta_n \|Ax_n\| \\
    \leq S + \sigma + \beta_n \sigma.  \tag{2.14}
\]
Such a $S > 0$ exists because
\[
\|y_k\| \leq \|x_k\| + \beta_k \|Ax_k\| \leq \|x_k\| + \|Ax_k\| \leq 2R + \|x^*\| + \sigma, \forall k \in \{1, \ldots, n\},
\]
and $A$ is a bounded map.

For all $n \in \mathbb{N}$, we have
\[
\|u_{n+1} - u_n\| = \alpha_n \|(I - T)u_n\| = \alpha_n \|Au_n\| \leq \alpha_n \sigma. \tag{2.15}
\]

Set
\[
\delta := \min\{\delta_1, \delta_2\}. \tag{2.16}
\]

Defining
\[
d_0 := \min\{1, \frac{\delta}{2\sigma}, \frac{R}{2(4\sigma + S)}\}. \tag{2.17}
\]

it follows that, for all $n \in \mathbb{N}$, using (2.1) and (2.16), that
\[
\alpha_n (3\sigma + S + \beta_n \sigma) \leq \alpha_n (4\sigma + S) < \frac{R}{2},
\]
\[
\beta_n < \frac{\delta}{\sigma} \quad \text{and} \quad \alpha_n < \frac{\delta}{2\sigma}. \tag{2.18}
\]

From (1.1) and (1.2),
\[
\|x_{n+1} - u_{n+1}\| = \|(1 - \alpha_n) (x_n - u_n) + \alpha_n (Ty_n - Tu_n)\|
\]
\[
= \|(x_n - u_n) - \alpha_n (Ax_n - Au_n) + \alpha_n (Ty_n - Tx_n)\|
\]
\[
\leq \|x_n - u_n\| + \alpha_n \|Ax_n - Au_n\| + \alpha_n \|Ty_n - Tx_n\|. \tag{2.19}
\]

From (2.20), using (2.11), (2.8), (2.14) and the first evaluation from (2.19),
\[
\|x_n - u_n\| \geq \|x_{n+1} - u_{n+1}\| - \alpha_n \|Ax_n - Au_n\| - \alpha_n \|Ty_n - Tx_n\|
\]
\[
\geq R - 2\alpha_n \sigma - \alpha_n (S + \sigma + \beta_n \sigma)
\]
\[
= R - \alpha_n (3\sigma + S + \beta_n \sigma) \geq R - R/2 = R/2. \tag{2.20}
\]

Using the induction assumption,
\[
\|x_{n+1} - u_{n+1}\| = \|(1 - \alpha_n) (x_n - u_n) + \alpha_n (Ty_n - Tu_n)\|
\]
\[
= \|(x_n - u_n) - \alpha_n (x_n - u_n - Tx_n + Tu_n) + \alpha_n (Ty_n - Tx_n)\|
\]
\[
\leq \|x_n - u_n\| + \alpha_n \|Ax_n - Au_n\| + \alpha_n \|Ty_n - Tx_n\|
\]
\[
\leq R + 2\alpha_n \sigma + \alpha_n S + \alpha_n \sigma + \alpha_n \beta_n \sigma = R + \alpha_n S + 3\alpha_n \sigma + \alpha_n \beta_n \sigma
\]
\[
< R + \frac{R}{2} \leq 2R. \tag{2.21}
\]
Thus we get

$$-1 \leq - \frac{\|x_{n+1} - u_{n+1}\|}{2R}.$$  \hfill (2.23)

By setting (1.6),

$$x := (x_n - u_n) - \alpha_n (Ax_n - Au_n),$$
$$y := \alpha_n (Ty_n - Tx_n),$$
$$x + y = x_{n+1} - u_{n+1}. \hfill (2.24)$$

we obtain

$$\|x_{n+1} - u_{n+1}\|^2 = \|(1 - \alpha_n) (x_n - u_n) + \alpha_n (Ty_n - Tu_n)\|^2$$
$$= \|(x_n - u_n) - \alpha_n (x_n - u_n) + \alpha_n (Tx_n - Tu_n) + \alpha_n (Ty_n - Tx_n)\|^2$$
$$= \|(x_n - u_n) - \alpha_n (Ax_n - Au_n) + \alpha_n (Ty_n - Tx_n)\|^2$$
$$\leq \|(x_n - u_n) - \alpha_n (Ax_n - Au_n)\|^2 + 2\alpha_n \langle Ty_n - Tx_n, J (x_{n+1} - u_{n+1}) \rangle.$$  \hfill (2.25)

We again apply (1.6) with

$$x := x_n - u_n,$$
$$y := -\alpha_n (Ax_n - Au_n),$$
$$x + y = (x_n - u_n) - \alpha_n (Ax_n - Au_n),$$  \hfill (2.26)

to obtain,

$$\|(x_n - u_n) - \alpha_n (Ax_n - Au_n)\|^2$$
$$\leq \|x_n - u_n\|^2 - 2\alpha_n \langle Ax_n - Au_n, J ((x_n - u_n) - \alpha_n (Ax_n - Au_n)) \rangle$$
$$\leq \|x_n - u_n\|^2 - 2\alpha_n \langle Ax_n - Au_n, J (x_n - u_n) - \alpha_n (Ax_n - Au_n) - J (x_n - u_n) \rangle$$
$$- 2\alpha_n \langle Ax_n - Au_n, J (x_n - u_n) \rangle$$
$$\leq \|x_n - u_n\|^2 - 2\alpha_n \langle Ax_n - Au_n, J (x_n - u_n) \rangle$$
$$+ 2\alpha_n \|Ax_n - Au_n\| \times \|J ((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J (x_n - u_n)\|.$$  \hfill (2.27)

Substituting (2.27) into (2.25) and using (2.21) we have

$$\|x_{n+1} - u_{n+1}\|^2$$
$$\leq \|x_n - u_n\|^2 - 2\alpha_n \langle Ax_n - Au_n, J (x_n - u_n) \rangle$$
$$+ 2\alpha_n \|Ty_n - Tx_n\| \times \|x_{n+1} - u_{n+1}\|.$$
\[
\leq \|x_n - u_n\|^2 - 2\alpha_n \psi \left(\frac{R}{2}\right) + 2\alpha_n \|Ax_n - Au_n\| \times \|J((x_n - u_n) - \alpha_n(Ax_n - Au_n)) - J(x_n - u_n)\| + 2\alpha_n \|Ty_n - Tx_n\| \|x_{n+1} - u_{n+1}\| \\
\leq \|x_n - u_n\|^2 - 2\alpha_n \psi \left(\frac{R}{2}\right) + 4\alpha_n \sigma \tau_n + 2\alpha_n \zeta_n \|x_{n+1} - u_{n+1}\|. \tag{2.28}
\]

Setting
\[
\tau_n := \|J((x_n - u_n) - \alpha_n(Ax_n - Au_n)) - J(x_n - u_n)\| \tag{2.29}
\]
and
\[
\zeta_n := \|Ty_n - Tx_n\|, \tag{2.30}
\]
and using (2.8) and (2.23),
\[
\|x_{n+1} - u_{n+1}\|^2 \\
\leq \|x_n - u_n\|^2 - 2\alpha_n \psi \left(\frac{R}{2}\right) \frac{\|x_{n+1} - u_{n+1}\|}{2R} + 4\alpha_n \sigma \tau_n + 2\alpha_n \zeta_n \|x_{n+1} - u_{n+1}\|. \tag{2.31}
\]

Using (2.14) and (2.19) we obtain
\[
\|(x_n - u_n) - \alpha_n(Ax_n - Au_n) - (x_n - u_n)\| \\
= \|\alpha_n(Ax_n - Au_n)\| \leq 2\alpha_n \sigma < \delta. \tag{2.32}
\]

From the uniform continuity of \(J(\cdot)\),
\[
\tau_n \leq \epsilon. \tag{2.33}
\]
Relation (2.19) leads to
\[
\|y_n - x_n\| = \|-\beta_n x_n + \beta_n Tx_n\| = \beta_n \|Ax_n\| \leq \beta_n \sigma < \delta. \tag{2.34}
\]
Since \(T\) is uniformly continuous,
\[
\zeta_n < \epsilon. \tag{2.35}
\]
Substituting (2.33), (2.35) (with $\varepsilon$ given by (2.9)), and (2.23) in (2.31) we obtain

$$\|x_{n+1} - u_{n+1}\|^2 \leq \|x_n - u_n\|^2 - 2\alpha_n \psi \left( \frac{R}{2} \right) \frac{\|x_{n+1} - u_{n+1}\|}{2R} + 4\alpha_n \sigma \psi \left( \frac{R}{4\sigma} \right) + 2\alpha_n \zeta \|x_{n+1} - u_{n+1}\|$$

Substituting (2.33), (2.35) (with $\varepsilon$ given by (2.9)), and (2.23) in (2.31) we obtain

$$\|x_{n+1} - u_{n+1}\|^2 \leq \|x_n - u_n\|^2 - 2\alpha_n \psi \left( \frac{R}{2} \right) \frac{\|x_{n+1} - u_{n+1}\|}{2R} + 4\alpha_n \sigma \psi \left( \frac{R}{4\sigma} \right) + 2\alpha_n \zeta \|x_{n+1} - u_{n+1}\|$$

Thus there exists an $\varepsilon > 0$ such that

$$\|x_{n+1} - u_{n+1}\| \leq \varepsilon.$$

Relation (2.36) is in contradiction with $\|x_{n+1} - u_{n+1}\| > R$.

Thus there exists an $R > 0$ such that

$$\|x_n - u_n\| \leq R, \forall n \in \mathbb{N}. \quad (2.37)$$

Relations (2.28) and (2.37) lead to

$$\|x_{n+1} - u_{n+1}\|^2 \leq \|x_n - u_n\|^2 - 2\alpha_n \psi \left( \|x_n - u_n\| \right) + 2\alpha_n \|Ax_n - Au_n\| \times \|J \left( (x_n - u_n) - \alpha_n (Ax_n - Au_n) \right) - J (x_n - u_n)\|$$

$$+ 2\alpha_n \|Ty_n - Tx_n\| \|x_{n+1} - u_{n+1}\|$$

Recalling that $\lim_{n \to \infty} \|u_n - x^*\| = 0$, then $\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0$, and using (2.32) one obtains using (1.3),

$$\|J((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J(x_n - u_n)\| \leq 2\alpha_n \sigma \to 0 \text{ as } n \to \infty. \quad (2.39)$$

The uniformly continuity of $J(\cdot)$ implies that

$$\lim_{n \to \infty} \|J((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J(x_n - u_n)\| = 0. \quad (2.40)$$
Also, from (2.34) and (1.3), we have
\[
\|y_n - x_n\| = \| -\beta_n x_n + \beta_n Tx_n \| \leq \beta_n \sigma \to 0, \quad n \to \infty.
\]
(2.41)
The uniformly continuity of \(T(\cdot)\) leads to
\[
\lim_{n \to \infty} \|Ty_n - Tx_n\| = 0.
\]
(2.42)
Relations (2.38), (2.40) and (2.42) with
\[
\lambda_n := \|x_n - u_n\|^2,
\gamma_n := \alpha_n (4\sigma \|J((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J(x_n - u_n)\| + 2R \|Ty_n - Tx_n\|),
\]
lead to (1.12). Using now Lemma 1.9 one obtains \(\lim_{n \to \infty} \|x_n - u_n\|^2 = 0\).

Using Remark 1.4 (i), Proposition 1.5, and Theorem 2.1 one obtains the following corollary.

**Corollary 2.2.** Let \(X\) be a real Banach space with \(X^\prime\) strictly convex. If \(T : X \to X\) is a \(d\)-weakly contractive (respectively weakly contractive) and uniformly continuous map with \(x^* \in F(T), x_0 = u_0 \in X\) and there exists a constant \(d_0 = d_0(T, x^*) \in (0, 1)\), which depends on \(T\) and \(x^*\), such that \(\{\alpha_n\}, \{\beta_n\}\) satisfy \(\alpha_n, \beta_n \leq d_0, \forall n \in \mathbb{N}\) and (1.3), then the following are equivalent:

(i) the Mann iteration (1.1) converges to the solution of \(Cx = f\),

(ii) the Ishikawa iteration (1.2) converges to the same \(x^*\).

Let \(C\) be a \(\psi\)-uniformly accretive map. Suppose the equation \(Cx = f\) has a solution for a given \(f\). Remark 1.4 (ii) ensures that
\[
Tx := f + x - Cx, \forall x \in X,
\]
(2.44)
is a \(\psi\)-uniformly pseudocontractive map. A fixed point for \(T\) is a solution for \(Cx = f\) and conversely.

Theorem 2.1 also implies the following corollary.

**Corollary 2.3.** Let \(X\) be a real Banach space with \(X^\prime\) strictly convex. If \(C : X \to X\) is a \(\psi\)-uniformly accretive and uniformly continuous map with \(x^* \in F(T), x_0 = u_0 \in X\) and there exists a constant \(d_0 = d_0(T, x^*) \in (0, 1)\), which depends on \(T\) and \(x^*\), such that \(\{\alpha_n\}, \{\beta_n\}\) satisfy \(\alpha_n, \beta_n \leq d_0, \forall n \in \mathbb{N}\) and (1.3), then the following are equivalent:

(i) the Mann iteration (1.1), with \(T\) given by (2.44), converges to the solution of \(Cx = f\),

(ii) the Ishikawa iteration (1.2), with \(T\) given by (2.44), converges to the solution of \(Cx = f\).
References


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