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# THE EQUIVALENCE OF MANN AND ISHIKAWA ITERATIONS DEALING WITH $\psi$−UNIFORMLY PSEUDOCONTRACTIVE MAPS WITHOUT BOUNDED RANGE

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**Abstract**. We prove that Mann and Ishikawa iterations are equivalent models dealing with $\psi$-uniformly pseudocontractive or d-weakly contractive maps without bounded range.

## 1. Introduction

In this paper $X$ denotes a real Banach space with $X^*$ strictly convex, $T : X \to X$ a map and let $x_0, u_0 \in X$. We consider the following iteration known as Mann iteration, ([9])

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n. \tag{1.1}$$

The sequence $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. We consider the following iteration known as Ishikawa iteration, ([8])

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n. \end{aligned} \tag{1.2}$$

The sequences $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfy

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = +\infty. \tag{1.3}$$

The *duality normalized map* $J : X \to 2^{X^*}$ is given by

$$J(x) = \{f \in X^* : \langle f, x\rangle = \|x\|^2, \|x\| = \|f\|\}. \tag{1.4}$$

We have

$$\langle f, y\rangle \le \|f\| \, \|y\|, \forall y \in X. \tag{1.5}$$

The following Remark is Proposition 12.3 from [7].

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**Remark 1.1.**([7]) If $X$ is a real Banach space with $X^*$ strictly convex then $J(\cdot)$ is a single map and uniformly continuous on all the bounded sets of $X$.

The following result is Lemma 1 from [10].

**Lemma 1.2.** *If $X$ is a real normed space, then the following relation is true*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \forall x, y \in X, \forall j(x+y) \in J(x+y). \tag{1.6}$$

The following definitions are from [3], [5] and [6].

**Definition 1.3.** Let $X$ be a normed space.

A map $T : X \to X$ is called weakly contractive map if for all $x, y \in X$, there exist $\psi : [0, +\infty) \to [0, +\infty)$ a continuous and strictly increasing map such that $\psi$ is positive on $(0, +\infty)$, $\psi(0) = 0$, and the following inequality is satisfied

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|). \tag{1.7}$$

A map $T : X \to X$ is called d-weakly contractive map if for all $x, y \in X$, there exist $j(x-y) \in J(x-y)$ and $\psi : [0, +\infty) \to [0, +\infty)$ a continuous and strictly increasing map such that $\psi$ is positive on $(0, +\infty)$, $\psi(0) = 0$, and the following inequality is satisfied

$$|\langle Tx - Ty, j(x-y)\rangle| \leq \|x-y\|^2 - \psi(\|x-y\|). \tag{1.8}$$

A map $T : X \to X$ is called $\psi$-*uniformly pseudocontractive if there exist* $j(x-y) \in J(x-y)$ and $\psi : [0, +\infty) \to [0, +\infty)$ a strictly increasing map such that $\psi$ is positive on $(0, +\infty)$, $\psi(0) = 0$ and the following inequality is satisfied

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \psi(\|x-y\|), \forall x, y \in B. \tag{1.9}$$

A map $C : X \to X$ is called $\psi$-uniformly *accretive if there exist* $j(x-y) \in J(x-y)$ and $\psi : [0, +\infty) \to [0, +\infty)$ a strictly increasing map such that $\psi$ is positive on $(0, +\infty)$, $\psi(0) = 0$ and the following inequality is satisfied

$$\langle Cx - Cy, j(x-y)\rangle \geq \psi(\|x-y\|), \forall x, y \in X. \tag{1.10}$$

We denote the identity map by $I$.

**Remark 1.4.** (i) If $T$ is a d-weakly contractive map, then $T$ is a $\psi$-uniformly pseudocontractive map.

(ii) The map $T$ is $\psi$-uniformly pseudocontractive if and only if $C := (I - T)$ is $\psi$-uniformly accretive.

**Proposition 1.5.** *If $T$ is a weakly contractive map, then $T$ is a $\psi$-uniformly pseudocontractive map.*

**Proof.** Let $j(x-y) \in J(x-y)$. Using (1.5), (1.7) and (1.4) we get

$$\begin{aligned}\langle Tx - Ty, j(x-y)\rangle &\leq \|Tx - Ty\| \, \|j(x-y)\| \\ &= \|Tx - Ty\| \, \|x-y\| \leq \|x-y\|^2 - \|x-y\| \, \phi(\|x-y\|) \quad . \quad (1.11)\end{aligned}$$

Denote $\psi(a) := a \cdot \phi(a)$, $\forall a \in [0, \infty)$ to obtain that $\psi$ is strictly increasing and positive.

The convergence of Mann iteration for a d-weakly contractive map in Hilbert spaces, was studied in [3]. It was shown in [5] that Mann iteration (1.1) for a d-weakly contractive map without a bounded range, converges in a Banach space more general then a Hilbert space. Also, it was shown in [6] that the same iteration for a $\psi$-uniformly pseudocontractive map without a bounded range, converges in a normed space.

If $T$ is a weakly contractive, then $T$ is a nonexpansive map. In this case the equivalence between Mann and Ishikawa iterations follows from Theorem 3 of the paper [11].

The above two motivations lead us to prove, in this note, the equivalence between Mann and Ishikawa iterations, (1.1) and (1.2), dealing with $\psi$-uniformly pseudocontractive maps without bounded range. As a corollary we obtain the convergence of Ishikawa iteration for the above operatorial classes. Also, we give a positive answer to the following conjecture, (see [11], page 452), "If Mann iteration converges, so does Ishikawa iteration".

For a $\psi$-uniformly pseudocontractive (respectively, $\psi$-uniformly accretive) map, the equivalence between Mann and Ishikawa iterations was shown also in Theorem 2.1 and Corollary 3.1 from [12]. There, in [12], the set $T(X)$ was assumed to be bounded. Removing the boundedness of the range, forces us to pay a price: both $\{\alpha_n\}$ and $\{\beta_n\}$ will depend on $T$ and $x^*$( see condition (2.1)).

**Remark 1.6.** Let $X$ be a normed space and $T : X \to X$ a uniformly continuous map. Then $I - T$ is a uniformly continuous map.

The following result is Proposition 2.1.2 from [4].

**Proposition 1.7**([4]) *Let $X$ be a normed space and $T : X \to X$ be a uniformly continuous map. Then $T$ is bounded; i.e. it maps any bounded set into a bounded set.*

Remark 1.6 and Proposition 1.7 lead to the following result.

**Remark 1.8.** Let $X$ be a normed space and $T : X \to X$ a uniformly continuous map. Then $I - T$ is bounded; i.e. it maps any bounded set into a bounded set.

The following result, stated below, is Lemma 3.1 from [1]. In [1], the map $\psi$ is assumed to be continuous in order to obtain an estimate for the convergence rate of the sequence $\{\lambda_n\}$. Another proof for the Lemma 3.1 can be found in ([2], pages 12-13). The same lemma, without the continuity assumption on $\psi$, appears in [6].

**Lemma 1.9.**([1]) *Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative numbers and $\{\alpha_n\}$ a sequence of positive numbers satisfying the conditions $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $(\gamma_n/\alpha_n) \to 0$*

*as* $n \to +\infty$. *Suppose that*

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n \psi(\lambda_n) + \gamma_n, \tag{1.12}$$

*is satisfied, where* $\psi : [0, +\infty) \to [0, +\infty)$ *is a strictly increasing map such that* $\psi$ *is positive on* $(0, +\infty)$, *with* $\psi(0) = 0$. *Then* $\lim_{n\to\infty} \lambda_n = 0$.

## 2. Main Result

Let $F(T)$ denote the fixed point set of $T$.

**Theorem 2.1.** *Let* $X$ *be a real Banach space with* $X^*$ *stricly convex. If* $T : X \to X$ *is a* $\psi$*-uniformly pseudocontractive and uniformly continuous map with* $x^* \in F(T)$, $x_0 = u_0 \in X$ *and there exists a constant* $d_0 := d_0(T, x^*) \in (0, 1)$, *which depends on* $T$ *and* $x^*$, *such that* $\{\alpha_n\}, \{\beta_n\}$ *satisfy*

$$\alpha_n, \beta_n \leq d_0, \forall n \in \mathbb{N}, \tag{2.1}$$

*and* (1.3), *then the following are equivalent:*
(i) *the Mann iteration* (1.1) *converges to the* $x^* \in F(T)$,
(ii) *the Ishikawa iteration* (1.2) *converges to the same* $x^*$.

**Proof.** The fixed point $x^*$ is unique. If not, then there exists at least another fixed point $y^* \in F(T)$, with $x^* \neq y^*$. Relation (1.9) leads to

$$\begin{aligned} \langle Tx^* - Ty^*, J(x^* - y^*)\rangle &\leq \|x^* - y^*\|^2 - \psi(\|x^* - y^*\|) \\ \langle x^* - y^*, J(x^* - y^*)\rangle &\leq \|x^* - y^*\|^2 - \psi(\|x^* - y^*\|) \\ \|x^* - y^*\|^2 &\leq \|x^* - y^*\|^2 - \psi(\|x^* - y^*\|) \\ \psi(\|x^* - y^*\|) &\leq 0 \Rightarrow \|x^* - y^*\| = 0. \end{aligned} \tag{2.2}$$

The implication (ii)⇒(i) is obvious, by setting, in (1.2), $\beta_n = 0$, for all $n \in \mathbb{N}$. We will prove the implication (i)⇒(ii). Suppose that $\lim_{n\to\infty} u_n = x^*$. If

$$\lim_{n\to\infty} \|x_n - u_n\| = 0, \tag{2.3}$$

then

$$0 \leq \|x^* - x_n\| \leq \|u_n - x^*\| + \|x_n - u_n\| \tag{2.4}$$

and it follows that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.5}$$

Thus, to complete the proof it suffices to verify relation (2.3).

With $A := (I - T)$ in (1.9), we have

$$\begin{aligned} \langle Ax - Ay, J(x-y)\rangle &= \langle (x - Tx) - (y - Ty), J(x-y)\rangle \\ &= \|x-y\|^2 - \langle Tx - Ty, J(x-y)\rangle \\ &\geq \|x-y\|^2 - \|x-y\|^2 + \psi(\|x-y\|) \\ &= \psi(\|x-y\|). \end{aligned} \tag{2.6}$$

Taking $x := x_n$ and $y := u_n$ in (2.6) we obtain

$$\langle Ax_n - Au_n, J(x_n - u_n)\rangle \geq \psi(\|x_n - u_n\|). \tag{2.7}$$

Choose $R > 0$ such that $\{u_n : n \in \mathbb{N}\} \subset B_R(x^*)$ and $x_0 \in B_{2R}(x^*)$. Remark 1.8 assures that $A(B_{2R}(x^*))$ is bounded. Denote

$$\sigma := diam\,(A(B_{2R}(x^*))) + R. \tag{2.8}$$

Since the map $J(\cdot)$ is uniformly continuous on bounded subsets of $X$, with

$$\varepsilon := \frac{\psi\left(\frac{R}{2}\right)}{4\sigma} > 0, \tag{2.9}$$

there exists a $\delta_1 > 0$ such that $\|x-y\| \leq \delta_1$ implies $\|J(x) - J(y)\| \leq \varepsilon$.

The map $T(\cdot)$ is also uniformly continuous. Thus for the same $\varepsilon$, there exits a $\delta_2 > 0$ such that $\|x-y\| \leq \delta_2$ implies $\|Tx - Ty\| \leq \varepsilon$.

We shall prove by induction that $\{x_n\}$ is bounded. We know that $0 = \|x_0 - u_0\| \leq R$. Suppose that $\|x_k - u_k\| \leq R, \forall k \in \{1, \ldots, n\}$. We shall prove that

$$\|x_{n+1} - u_{n+1}\| \leq R. \tag{2.10}$$

Assume that $\|x_n - u_n\| \leq R$ and that

$$\|x_{n+1} - u_{n+1}\| > R. \tag{2.11}$$

From $\|x_k - u_k\| \leq R$, $\forall k \in \{1, \ldots, n\}$ we know

$$\|x_k - x^*\| \leq \|x_k - u_k\| + \|u_k - x^*\| \leq 2R,\ \forall k \in \{1, \ldots, n\}. \tag{2.12}$$

From (2.12), we have $x_n \in B_{2R}(x^*)$ and the following inequality satisfied

$$\|x_k\| \leq \|x_k - x^*\| + \|x^*\| \leq 2R + \|x^*\|,\ \forall k \in \{1, \ldots, n\}. \tag{2.13}$$

Using $diam\,(A(B_{2R}(x^*))) \leq \sigma$ and $x_n \in B_{2R}(x^*)$, (i. e. $\|Ax_n\| \leq \sigma$), we get

$$\begin{aligned} \|Ty_n - Tx_n\| &\leq \|-y_n + Ty_n + x_n - Tx_n\| + \|y_n - x_n\| \\ &= \|Ay_n - Ax_n\| + \|y_n - x_n\| \\ &\leq \|Ay_n\| + \|Ax_n\| + \|y_n - x_n\| \\ &\leq S + \sigma + \beta_n \|x_n - Tx_n\| = S + \sigma + \beta_n \|Ax_n\| \\ &\leq S + \sigma + \beta_n \sigma. \end{aligned} \tag{2.14}$$

Such a $S > 0$ exists because

$$\begin{aligned} \|y_k\| &\le \|x_k\| + \beta_k \|Ax_k\| \le \|x_k\| + \|Ax_k\| \\ &\le 2R + \|x^*\| + \sigma, \ \forall k \in \{1, \ldots, n\}, \end{aligned} \tag{2.15}$$

and $A$ is a bounded map.

For all $n \in \mathbb{N}$, we have

$$\|u_{n+1} - u_n\| = \alpha_n \|(I - T) u_n\| = \alpha_n \|Au_n\| \le \alpha_n \sigma. \tag{2.16}$$

Set

$$\delta := \min\{\delta_1, \delta_2\}. \tag{2.17}$$

Defining

$$d_0 := \min\{1, \delta, \frac{\delta}{2\sigma}, \frac{R}{2(4\sigma + S)}\}. \tag{2.18}$$

it follows that, for all $n \in \mathbb{N}$, using (2.1) and (2.16), that

$$\begin{aligned} \alpha_n (3\sigma + S + \beta_n \sigma) &\le \alpha_n (4\sigma + S) < \frac{R}{2}, \\ \beta_n &< \frac{\delta}{\sigma} \text{ and} \\ \alpha_n &< \frac{\delta}{2\sigma}. \end{aligned} \tag{2.19}$$

From (1.1) and (1.2),

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|(1 - \alpha_n)(x_n - u_n) + \alpha_n (Ty_n - Tu_n)\| \\ &= \|x_n - u_n - \alpha_n (Ax_n - Au_n) + \alpha_n (Ty_n - Tx_n)\| \\ &\le \|x_n - u_n\| + \alpha_n \|Ax_n - Au_n\| + \alpha_n \|Ty_n - Tx_n\|. \end{aligned} \tag{2.20}$$

From (2.20), using (2.11), (2.8), (2.14) and the first evaluation from (2.19),

$$\begin{aligned} \|x_n - u_n\| &\ge \|x_{n+1} - u_{n+1}\| - \alpha_n \|Ax_n - Au_n\| - \alpha_n \|Ty_n - Tx_n\| \\ &\ge R - 2\alpha_n \sigma - \alpha_n (S + \sigma + \beta_n \sigma) \\ &= R - \alpha_n (3\sigma + S + \beta_n \sigma) \ge R - R/2 = R/2. \end{aligned} \tag{2.21}$$

Using the induction assumption,

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|(1 - \alpha_n)(x_n - u_n) + \alpha_n (Ty_n - Tu_n)\| \\ &= \|(x_n - u_n) - \alpha_n (x_n - u_n - Tx_n + Tu_n) + \alpha_n (Ty_n - Tx_n)\| \\ &\le \|x_n - u_n\| + \alpha_n \|Ax_n - Au_n\| + \alpha_n \|Ty_n - Tx_n\| \\ &\le R + 2\alpha_n \sigma + \alpha_n S + \alpha_n \sigma + \alpha_n \beta_n \sigma = R + \alpha_n S + 3\alpha_n \sigma + \alpha_n \beta_n \sigma \\ &< R + R/2 \le 2R. \end{aligned} \tag{2.22}$$

Thus we get

$$-1 \leq -\frac{\|x_{n+1} - u_{n+1}\|}{2R}. \tag{2.23}$$

By setting (1.6),

$$\begin{aligned} x &:= (x_n - u_n) - \alpha_n (Ax_n - Au_n), \\ y &:= \alpha_n (Ty_n - Tx_n), \\ x + y &= x_{n+1} - u_{n+1}, \end{aligned} \tag{2.24}$$

we obtain

$$\begin{aligned} \|x_{n+1} - u_{n+1}\|^2 &= \|(1 - \alpha_n)(x_n - u_n) + \alpha_n (Ty_n - Tu_n)\|^2 \\ &= \|(x_n - u_n) - \alpha_n (x_n - u_n) + \alpha_n (Tx_n - Tu_n) + \alpha_n (Ty_n - Tx_n)\|^2 \\ &= \|(x_n - u_n) - \alpha_n (Ax_n - Au_n) + \alpha_n (Ty_n - Tx_n)\|^2 \\ &\leq \|(x_n - u_n) - \alpha_n (Ax_n - Au_n)\|^2 + 2\alpha_n \langle Ty_n - Tx_n, J(x_{n+1} - u_{n+1})\rangle. \end{aligned} \tag{2.25}$$

We again apply (1.6) with

$$\begin{aligned} x &:= x_n - u_n, \\ y &:= -\alpha_n (Ax_n - Au_n), \\ x + y &= (x_n - u_n) - \alpha_n (Ax_n - Au_n), \end{aligned} \tag{2.26}$$

to obtain,

$$\begin{aligned} &\|(x_n - u_n) - \alpha_n (Ax_n - Au_n)\|^2 \\ &\leq \|x_n - u_n\|^2 - 2\alpha_n \langle Ax_n - Au_n, J((x_n - u_n) - \alpha_n (Ax_n - Au_n))\rangle \\ &\leq \|x_n - u_n\|^2 - 2\alpha_n \langle Ax_n - Au_n, J((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J(x_n - u_n)\rangle \\ &- 2\alpha_n \langle Ax_n - Au_n, J(x_n - u_n)\rangle \\ &\leq \|x_n - u_n\|^2 - 2\alpha_n \psi(\|x_n - u_n\|) \\ &\quad + 2\alpha_n \|Ax_n - Au_n\| \times \|J((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J(x_n - u_n)\|. \end{aligned} \tag{2.27}$$

Substituting (2.27) into (2.25) and using (2.21) we have

$$\begin{aligned} &\|x_{n+1} - u_{n+1}\|^2 \\ &\quad \leq \|x_n - u_n\|^2 - 2\alpha_n \psi(\|x_n - u_n\|) \\ &\qquad +2\alpha_n \|Ax_n - Au_n\| \times \|J((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J(x_n - u_n)\| \\ &\qquad +2\alpha_n \langle Ty_n - Tx_n, J(x_{n+1} - u_{n+1})\rangle \\ &\quad \leq \|x_n - u_n\|^2 - 2\alpha_n \psi(\|x_n - u_n\|) \\ &\qquad +2\alpha_n \|Ax_n - Au_n\| \times \|J((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J(x_n - u_n)\| \\ &\qquad +2\alpha_n \|Ty_n - Tx_n\| \|x_{n+1} - u_{n+1}\| \end{aligned}$$

$$
\begin{aligned}
&\leq \|x_n - u_n\|^2 - 2\alpha_n \psi\left(\frac{R}{2}\right) \\
&\quad +2\alpha_n \|Ax_n - Au_n\| \times \|J\left((x_n - u_n) - \alpha_n\left(Ax_n - Au_n\right)\right) - J\left(x_n - u_n\right)\| \\
&\quad +2\alpha_n \|Ty_n - Tx_n\| \|x_{n+1} - u_{n+1}\| \\
&\leq \|x_n - u_n\|^2 - 2\alpha_n \psi\left(\frac{R}{2}\right) + 4\alpha_n \sigma \tau_n + 2\alpha_n \zeta_n \|x_{n+1} - u_{n+1}\|. \qquad (2.28)
\end{aligned}
$$

Setting

$$\tau_n := \|J\left((x_n - u_n) - \alpha_n\left(Ax_n - Au_n\right)\right) - J\left(x_n - u_n\right)\| \qquad (2.29)$$

and

$$\zeta_n := \|Ty_n - Tx_n\|, \qquad (2.30)$$

and using (2.8) and (2.23),

$$
\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&\leq \|x_n - u_n\|^2 - 2\alpha_n \psi\left(\frac{R}{2}\right)\frac{\|x_{n+1} - u_{n+1}\|}{2R} + 4\alpha_n \sigma \tau_n + 2\alpha_n \zeta_n \|x_{n+1} - u_{n+1}\|. \qquad (2.31)
\end{aligned}
$$

Using (2.14) and (2.19) we obtain

$$
\begin{aligned}
&\|(x_n - u_n) - \alpha_n\left(Ax_n - Au_n\right) - (x_n - u_n)\| \qquad (2.32) \\
&= \|\alpha_n\left(Ax_n - Au_n\right)\| \leq 2\alpha_n \sigma < \delta.
\end{aligned}
$$

From the uniform continuity of $J(\cdot)$,

$$\tau_n \leq \varepsilon. \qquad (2.33)$$

Relation (2.19) leads to

$$\|y_n - x_n\| = \|-\beta_n x_n + \beta_n T x_n\| = \beta_n \|Ax_n\| \leq \beta_n \sigma < \delta. \qquad (2.34)$$

Since $T$ is uniformly continuous,

$$\zeta_n < \varepsilon. \qquad (2.35)$$

Substituting (2.33), (2.35) (with $\varepsilon$ given by (2.9)), and (2.23) in (2.31) we obtain

$$
\begin{aligned}
&\|x_{n+1}-u_{n+1}\|^2 \\
&\leq \|x_n-u_n\|^2 - 2\alpha_n\psi\left(\frac{R}{2}\right)\frac{\|x_{n+1}-u_{n+1}\|}{2R} + 4\alpha_n\sigma\frac{\psi\left(\frac{R}{2}\right)}{4\sigma} + 2\alpha_n\zeta_n\|x_{n+1}-u_{n+1}\| \\
&\leq \|x_n-u_n\|^2 - \alpha_n\psi\left(\frac{R}{2}\right)\frac{\|x_{n+1}-u_{n+1}\|}{R} + \alpha_n\psi\left(\frac{R}{2}\right) + \frac{1}{2}\alpha_n\frac{\psi\left(\frac{R}{2}\right)}{\sigma}\|x_{n+1}-u_{n+1}\| \\
&\leq \|x_n-u_n\|^2 - \alpha_n\psi\left(\frac{R}{2}\right)\frac{\|x_{n+1}-u_{n+1}\|}{R} + \alpha_n\psi\left(\frac{R}{2}\right)\frac{\|x_{n+1}-u_{n+1}\|}{2R} + \\
&\quad +\frac{1}{2}\alpha_n\psi\left(\frac{R}{2}\right)\frac{\|x_{n+1}-u_{n+1}\|}{\sigma} \\
&= \|x_n-u_n\|^2 - \alpha_n\psi\left(\frac{R}{2}\right)\frac{\|x_{n+1}-u_{n+1}\|}{R} + \frac{1}{2}\alpha_n\psi\left(\frac{R}{2}\right)\frac{\|x_{n+1}-u_{n+1}\|}{R} + \\
&\quad +\frac{1}{2}\alpha_n\psi\left(\frac{R}{2}\right)\frac{\|x_{n+1}-u_{n+1}\|}{R} \\
&= \|x_n-u_n\|^2 \leq R^2.
\end{aligned}
\tag{2.36}
$$

Relation (2.36) is in contradiction with $\|x_{n+1}-u_{n+1}\| > R$.
Thus there exists an $R>0$ such that

$$
\|x_n-u_n\| \leq R, \forall n \in \mathbb{N}. \tag{2.37}
$$

Relations (2.28) and (2.37) lead to

$$
\begin{aligned}
\|x_{n+1}-u_{n+1}\|^2 &\leq \|x_n-u_n\|^2 - 2\alpha_n\psi\left(\|x_n-u_n\|\right) \\
&\quad +2\alpha_n\|Ax_n-Au_n\| \times \|J\left((x_n-u_n)-\alpha_n\left(Ax_n-Au_n\right)\right)-J\left(x_n-u_n\right)\| \\
&\quad +2\alpha_n\|Ty_n-Tx_n\|\,\|x_{n+1}-u_{n+1}\| \\
&\leq \|x_n-u_n\|^2 - 2\alpha_n\psi\left(\|x_n-u_n\|\right) \\
&\quad +4\alpha_n\sigma\|J\left((x_n-u_n)-\alpha_n\left(Ax_n-Au_n\right)\right)-J\left(x_n-u_n\right)\| \\
&\quad +2\alpha_nR\|Ty_n-Tx_n\|.
\end{aligned}
\tag{2.38}
$$

Recalling that $\lim_{n\to\infty}\|u_n-x^*\|=0$, then $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$, and using (2.32) one obtains using (1.3),

$$
\begin{aligned}
&\|(x_n-u_n)-\alpha_n\left(Ax_n-Au_n\right)-(x_n-u_n)\| \\
&=\|\alpha_n\left(Ax_n-Au_n\right)\| \leq 2\alpha_n\sigma \to 0 \text{ as } n\to\infty.
\end{aligned}
\tag{2.39}
$$

The uniformly continuity of $J(\cdot)$ implies that

$$
\lim_{n\to\infty}\|J\left((x_n-u_n)-\alpha_n\left(Ax_n-Au_n\right)\right)-J\left(x_n-u_n\right)\|=0. \tag{2.40}
$$

Also, from (2.34) and (1.3), we have

$$\begin{aligned}\|y_n - x_n\| &= \|-\beta_n x_n + \beta_n T x_n\| \\ &= \beta_n \|Ax_n\| \leq \beta_n \sigma \to 0, \text{ as } n \to \infty.\end{aligned} \tag{2.41}$$

The uniformly continuity of $T(\cdot)$ leads to

$$\lim_{n\to\infty} \|Ty_n - Tx_n\| = 0. \tag{2.42}$$

Relations (2.38), (2.40) and (2.42) with

$$\begin{aligned}&\lambda_n := \|x_n - u_n\|^2, \\ &\gamma_n := \alpha_n(4\sigma \|J((x_n - u_n) - \alpha_n (Ax_n - Au_n)) - J(x_n - u_n)\| \\ &\quad + 2R\|Ty_n - Tx_n\|),\end{aligned} \tag{2.43}$$

lead to (1.12). Using now Lemma 1.9 one obtains $\lim_{n\to\infty} \|x_n - u_n\|^2 = 0$.

Using Remark 1.4 (i), Proposition 1.5, and Theorem 2.1 one obtains the following corollary.

**Corollary 2.2.** *Let $X$ be a real Banach space with $X'$ strictly convex. If $T : X \to X$ is a d−weakly contractive (respectively weakly contractive) and uniformly continuous map with $x^* \in F(T)$, $x_0 = u_0 \in X$ and there exists a constant $d_0 = d_0(T, x^*) \in (0,1)$, which depends on $T$ and $x^*$, such that $\{\alpha_n\}, \{\beta_n\}$ satisfy $\alpha_n, \beta_n \leq d_0, \forall n \in \mathbb{N}$ and* (1.3), *then the following are equivalent:*

(i) *the Mann iteration* (1.1) *converges to the $x^* \in F(T)$,*
(ii) *the Ishikawa iteration* (1.2) *converges to the same $x^*$.*

Let $C$ be a $\psi$-uniformly accretive map. Suppose the equation $Cx = f$ has a solution for a given $f$. Remark 1.4 (ii) ensures that

$$Tx := f + x - Cx, \forall x \in X, \tag{2.44}$$

is a $\psi$-uniformly pseudocontractive map. A fixed point for $T$ is a solution for $Cx = f$ and conversely.

Theorem 2.1 also implies the following corollary.

**Corollary 2.3.** *Let $X$ be a real Banach space with $X'$ strictly convex. If $C : X \to X$ is a $\psi$-uniformly accretive and uniformly continuous map with $x^* \in F(T)$, $x_0 = u_0 \in X$ and there exists a constant $d_0 = d_0(T, x^*) \in (0,1)$, which depends on $T$ and $x^*$, such that $\{\alpha_n\}$, $\{\beta_n\}$ satisfy $\alpha_n$, $\beta_n \leq d_0$, $\forall n \in \mathbb{N}$ and (1.3), then the following are equivalent:*

(i) *the Mann iteration* (1.1), *with $T$ given by* (2.44), *converges to the solution of $Cx = f$,*
(ii) *the Ishikawa iteration* (1.2), *with $T$ given by* (2.44), *converges to the solution of $Cx = f$.*

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