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A USEFUL ORTHONORMAL BASIS ON BI-SLANT SUBMANIFOLDS OF ALMOST HERMITIAN MANIFOLDS

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Abstract. In this paper, we study bi-slant submanifolds of an almost Hermitian manifold for different cases. We introduce a new orthonormal basis on bi-slant submanifold, semi-slant submanifold and hemi-slant submanifold of an almost Hermitian manifold to compute Chen's main inequalities. We investigate these inequalities for semi-slant submanifolds, hemi-slant submanifolds and slant submanifolds of a generalized complex space form. We obtain some characterizations on such submanifolds of a complex space form.

1. Introduction

The theory of submanifolds of an almost Hermitian manifold or a Kaehlerian manifold began as a separate field of study in the last century with the investigation of algebraic curves and algebraic surfaces in classical algebraic geometry. In the early 1950s, invariant submanifolds of an almost Hermitian manifold were defined by E. Calabi [2, 3] and in the early 1970s, anti-invariant submanifolds were defined by B. Y. Chen and K. Ogiue [6] as follows:

Let M be a submanifold of an almost Hermitian manifold $(\widetilde{M}, J, \widetilde{g})$. For any $X \in T_pM$, JX can be decomposed into tangential and normal parts given by

$$JX = TX + FX, \;\; PX \in T_pM, \; FX \in T_p^{\perp}M, \tag{1.1}$$

where TX is the tangential component and FX is the normal component of JX. The manifold M is called an *invariant submanifold* if $F = 0$ and *anti-invariant submanifold* if $T = 0$.

In 1990, B.-Y. Chen [7] introduced slant submanifolds as a generalization of invariant submanifold and anti-invariant submanifolds as follows:

For a vector $0 \neq X_p \in T_pM$, if the angle $\theta(X_p)$ between JX_p and X_p is independent of the choice of point $p \in M$, then M is called a *slant submanifold*. Invariant submanifolds and anti-invariant submanifolds are slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively.

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Furthermore, slant distribution was introduced in [1] by J. L. Cabrerizo *et al.*, as follows:

A differentiable distribution D on M is called a *slant distribution*, if for each $p \in M$ and each non-zero vector $X \in D_p$, the angle $\theta_D(X)$ between JX and X is constant and is independent of the choice of $p \in M$ and $X \in D_p$.

In 2002, the notion of bi-slant submanifolds of an almost Hermitian manifold was introduced as a natural generalization of semi-slant submanifolds by A. Carriazzo [4] as follows:

There exist two orthogonal distributions D_1 and D_2 on M, $dimD_1 = 2d_1$ and $dimD_2 = 2d_2$ such that

(i) $TM = D_1 \oplus D_2$,

(ii) D_1 and D_2 are slant distributions with θ_1 and θ_2 angles, respectively.

Semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds, slant submanifolds are particular cases of bi-slant submanifolds. In fact, M is a semi-slant submanifold if D_1 is an invariant distribution and D_2 is a slant distribution, M is a hemi-slant submanifold if D_1 is a slant distribution and D_2 is an anti-invariant distribution, M is a CR-submanifold if $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, M is a slant submanifold if D_1 or D_2 is equal to zero.

In [15] and [16], the authors consider an orthonormal basis $\{e_1, \dots, e_n\}$ of an n-dimensional bi-slant submanifold on generalized complex space forms to compute Chen inequalities on complex space forms and Sasakian space forms such that this basis satisfies

$$e_1, e_2 = \frac{1}{\cos\theta_1} Te_1, \dots, e_{2m-1}, e_{2m} = \frac{1}{\cos\theta_1} Te_{2m-1},$$
$$e_{2m+1}, e_{2m+2} = \frac{1}{\cos\theta_2} Te_{2m+1}, \dots, e_{2n-2m} = \frac{1}{\cos\theta_2} Te_{2n-2m-1}, \tag{1.2}$$

where $dimD_1 = 2m$ and $dimD_2 = 2n-2m$. Here, Te_i is perpendicular to D_2 and Te_j is perpendicular to D_1 for $i \in \{1, \dots, 2m\}$ and $j \in \{2m+1, \dots, 2n\}$. But one can not know the angle between JD_1 and D_2 or JD_2 and D_1 for bi-slant submanifold of almost Hermitian manifolds. Therefore, this basis isn't true for bi-slant submanifolds. For this reason, we introduce a useful basis on bi-slant submanifolds in this study. Using this basis, we compute Chen inequalities and give some corollaries on bi-slant submanifolds of an almost Hermitian manifold.

The paper has been organized as follows: Section 2 is devoted to preliminaries. In section 3, we give some examples for different cases on bi-slant submanifolds of an almost Hermitian manifold. We introduce an orthonormal basis for bi-slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds, slant submanifolds. In section 4, we establish a sharp inequality involving the mean curvature vector and the Ricci curvature of bi-slant submanifolds. We investigate this inequality for semi-slant submanifolds, hemi-slant submanifolds and slant submanifolds of a generalized complex space form. In section 5, we establish an optimal inequality involving the Chen-invariant for bi-slant submanifolds of a generalized complex space form. We study this inequality for some special submanifolds of a generalized complex space form.

2. Preliminaries

Let $(\widetilde{M},\widetilde{g})$ be a k-dimensional Riemannian manifold with a Riemannian metric $\widetilde{g}$ and (M,g) be an n-dimensional submanifold of $(\widetilde{M},\widetilde{g})$ with the induced metric tensor g. We denote the inner product of both the metrics by $\langle,\rangle$. Let σ be the second fundamental form related to the shape operator A by $\langle\sigma(X,Y),N\rangle = \langle A_N X, Y\rangle$. The Gauss equation is given by

$$R(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W) + \langle\sigma(X,W),\sigma(Y,Z)\rangle - \langle\sigma(X,Z),\sigma(Y,W)\rangle \tag{2.1}$$

for all $X,Y,Z,W \in \Gamma(TM)$, where $\widetilde{R}$ and R are the Riemann curvature tensors of $\widetilde{M}$ and M, respectively.

The mean curvature vector H of the submanifold M is given by $H = \frac{1}{n}\text{trace}(\sigma)$. If $\sigma = 0$, then the submanifold is called *totally geodesic* in $\widetilde{M}$, if $H = 0$, then the submanifold is called *minimal*, if $\sigma(X,Y) = g(X,Y)H$ for all $X,Y \in \Gamma(TM)$, then the submanifold is called *totally umbilical* [19].

Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1},\dots,e_m\}$ of the normal space $T_p^\perp M$. We put

$$\sigma^r_{ij} = \langle\sigma(e_i,e_j),e_r\rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n}\langle\sigma(e_i,e_j),\sigma(e_i,e_j)\rangle. \tag{2.2}$$

We denote by K_{ij} and $\widetilde{K}_{ij}$ the sectional curvature of the plane section spanned by e_i and e_j at point p in the submanifold M and in the ambient manifold $\widetilde{M}$, respectively. In this case, using the Gauss equation, we get

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^{m}\left(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right). \tag{2.3}$$

From (2.3), it follows that

$$2\tau(p) = 2\widetilde{\tau}\left(T_pM\right) + n^2\|H\|^2 - \|\sigma\|^2. \tag{2.4}$$

Also, the squared second fundamental form and the squared mean curvature satisfy that

$$\|\sigma\|^2 = \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}(\sigma^r_{11} - \sigma^r_{22} - \cdots - \sigma^r_{nn})^2$$

$$+2\sum_{r=n+1}^{m}\sum_{j=2}^{n}(\sigma_{1j}^{r})^{2}-2\sum_{r=n+1}^{m}\sum_{2\le i<j\le n}(\sigma_{ii}^{r}\sigma_{jj}^{r}-(\sigma_{ij}^{r})^{2}). \tag{2.5}$$

Let $(\widetilde{M},J,\widetilde{g})$ be an almost Hermitian manifold and $\widetilde{\nabla}$ be the Riemannian connection of the Riemannian metric $\widetilde{g}$. The manifold is called

1. a *nearly Kaehler manifold* [9] if
$$(\widetilde{\nabla}_X J)X=0$$
for any vector field $X\in T\widetilde{M}$,
2. a *Kaehler manifold* [19] if
$$\widetilde{\nabla}J=0.$$

An almost Hermitian manifold with the J-invariant Riemannian curvature tensor $\widetilde{R}$, that is,

$$\widetilde{R}(JX,JY,JZ,JW)=\widetilde{R}(X,Y,Z,W),\qquad X,Y,Z,W\in\Gamma(T\widetilde{M}),$$

is called an *RK*-manifold [18].

An almost Hermitian manifold $\widetilde{M}$ is said to have *(pointwise) constant type* if for each $p\in\widetilde{M}$ and for all $X,Y,Z\in T_p\widetilde{M}$ such that

$$\begin{gathered}\langle X,Y\rangle=\langle X,Z\rangle=\langle X,JY\rangle=\langle X,JZ\rangle=0\ \text{ and}\\ \langle Y,Y\rangle=1=\langle Z,Z\rangle.\end{gathered} \tag{2.6}$$

And consequently, we have

$$\widetilde{R}(X,Y,X,Y)-\widetilde{R}(X,Y,JX,JY)=\widetilde{R}(X,Z,X,Z)-\widetilde{R}(X,Z,JX,JZ). \tag{2.7}$$

It is known that if $\widetilde{M}$ is an *RK*-manifold, then it has (pointwise) constant type if and only if there is a differentiable function α on $\widetilde{M}$ satisfying

$$\widetilde{R}(X,Y,X,Y)-\widetilde{R}(X,Y,JX,JY)=\alpha(\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^{2}-\langle X,JY\rangle^{2}) \tag{2.8}$$

for all $X,Y,Z\in T\widetilde{M}$. Furthermore, $\widetilde{M}$ has global constant type if α is constant. The function α is called the *constant type* of $\widetilde{M}$ [17].

A *RK*-manifold of constant holomorphic sectional curvature c and constant type α is denoted by $\widetilde{M}(c,\alpha)$. The Riemann curvature of $\widetilde{M}(c,\alpha)$ is given by

$$\begin{aligned}4\widetilde{R}(X,Y)Z=&(c+3\alpha)\{\langle Y,Z\rangle X-\langle X,Z\rangle Y\}\\ &+(c-\alpha)\{\langle X,JZ\rangle JY-\langle Y,JZ\rangle JX\end{aligned} \tag{2.9}$$

$$+2\langle X, JY\rangle JZ\}$$

for all $X, Y, Z \in T\widetilde{M}$. If $c = \alpha$, then $\widetilde{M}(c, \alpha)$ is a space of constant curvature. If $\alpha = 0$, then $\widetilde{M}(c)$ is a complex space form.

3. Bi-slant submanifolds

Let M be a $2n$-dimensional bi-slant submanifold of an almost Hermitian manifold $\widetilde{M}$ such that

$$TM = D_1 \oplus D_2, \tag{3.1}$$

where D_1 is a θ_1-slant distribution, and D_2 is a θ_2-slant distribution. Then there exist the following four cases [5]:

Case 1: M is bi-slant with $\theta_1 = \theta_2 = \theta$ and it is also θ-slant.

Case 2: M is bi-slant with $\theta_1 = \theta_2$ but it is not slant.

Case 3: M is bi-slant with $\theta_1 = \theta_2 = \theta$ and it is also α-slant with $\alpha \neq 0$.

Case 4: M is bi-slant with $\theta_1 \neq \theta_2$ and it is not slant.

Now, we are going to give an example of bi-slant submanifold for **Case 1** as follows:

Example 3.1. Let J be an almost complex structure on R^6 such that

$$J(x_1, x_2, x_3, x_4, x_5, x_6) = (x_2, -x_1, x_4, -x_3, x_6, -x_5).$$

Let M be a submanifold of R^6 given by

$$\varphi(u, v, w, t) = (u\sqrt{2}, v\sqrt{2}, u+v, u-v, w-t, t-w).$$

Then we have an orthonormal frame of M as follows:

$$X_1 = \frac{1}{2}(\sqrt{2}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}), \qquad X_2 = \frac{1}{2}(\sqrt{2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}),$$
$$X_3 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}), \qquad X_4 = \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}).$$

Put $D_1 = Span\{X_1, X_2\}$ and $D_2 = Span\{X_3, X_4\}$, then M is a bi-slant submanifold with D_1 and D_2 are anti-invariant distributions. Furthermore, M is an anti-invariant submanifold.

Now, we are going to give an example of bi-slant submanifold for **Case 2** as follows:

Example 3.2. We consider the Euclidian space R^6 with coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$. Let J be an almost complex structure on R^6 such that

$$J(x_1, x_2, x_3, x_4) = (-x_4, -x_5, -x_6, x_1, x_2, x_3).$$

Let M be a submanifold of R^6 with

$$\begin{aligned}\varphi(u_1, u_2, v_1, v_2) = (&u_1 \cos\theta_1 - u_2 \sin\theta_1, u_1 \sin\theta_1 + u_2 \cos\theta_1, 0,\\ &v_1 \cos\theta_2 - v_2 \sin\theta_2, v_1 \sin\theta_2 + v_2 \cos\theta_2, 0),\end{aligned}$$

for any $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$. Then we have an orthonormal frame of M as follows:

$$\begin{aligned} X_1 &= \cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial x_2}, \quad X_2 = \cos\theta_2 \frac{\partial}{\partial x_4} + \sin\theta_2 \frac{\partial}{\partial x_5}, \quad X_3 = \frac{\partial}{\partial x_3},\\ X_4 &= \frac{\partial}{\partial x_6}, \quad X_5 = -\sin\theta_1 \frac{\partial}{\partial x_1} + \cos\theta_1 \frac{\partial}{\partial x_2}, \quad X_6 = -\sin\theta_2 \frac{\partial}{\partial x_4} + \cos\theta_2 \frac{\partial}{\partial x_5}.\end{aligned}$$

If we put $D_1 = Span\{X_1, X_2\}$, $D_2 = Span\{X_5, X_6\}$, then M is bi-slant submanifold with D_1 and D_2 are $\theta = (\theta_1 - \theta_2)-$slant distributions. But M is not a slant submanifold.

Now, we are going to give an example of bi-slant submanifold for **Case 3** as follows:

Example 3.3. Let J be an almost complex structure on R^8 such that

$$J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_5, -x_6, -x_7, -x_8, x_1, x_2, x_3, x_4).$$

Let M be a submanifold of R^8 with

$$\varphi(u, v, w, t) = (\frac{1}{2}(u - v), \frac{1}{2}(u + v), \frac{\sqrt{2}}{2} u, \frac{\sqrt{2}}{2} v, w, t, 0, 0).$$

Then we have an orthonormal frame of M as follows:

$$\begin{aligned} X_1 &= \frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3}), \quad X_2 = \frac{1}{2}(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_4}),\\ X_3 &= \frac{\partial}{\partial x_5}, \ X_4 = \frac{\partial}{\partial x_6}.\end{aligned}$$

Put $D_1 = Span\{X_1, X_2\}$ and $D_2 = Span\{X_3, X_4\}$ then M is bi-slant submanifold with D_1 and D_2 are anti-invariant distributions. Futhermore, M is a slant submanifold with slant angle $\theta = \frac{\pi}{3}$.

Let $P_i : TM \to D_i$, $i \in \{1, 2\}$, be orthogonal projections. It is well known that

$$\langle P_i X_i, Y_i \rangle = \langle X_i, P_i Y_i \rangle \tag{3.2}$$

for $X_i, Y_i \in \Gamma(D_i)$ [5, 10]. Furthermore, it can be proved that

$$(P_i T)^2 X_i = -\cos^2\theta_i X_i \tag{3.3}$$

for any $X_i \in \Gamma(D_i)$. From (3.2) and (3.3), we have

$$\begin{aligned}
\langle P_i T X_i, P_i T Y_i \rangle &= \langle T X_i, P_i T Y_i \rangle \\
&= -\langle X_i, T P_i T Y_i \rangle \\
&= -\langle X_i, P_1 T P_i T Y_i + P_2 T P_i T Y_i \rangle \\
&= -\langle X_i, -\cos^2\theta_i Y_i \rangle \\
&= \cos^2\theta_i \langle X_i, Y_i \rangle.
\end{aligned} \tag{3.4}$$

Since $\|TX\|^2 + \|FX\|^2 = \|JX\|^2$, we get

$$\|P_1 T X\|^2 + \|P_2 T X\|^2 + \|FX\|^2 = \|JX\|^2 \tag{3.5}$$

for any $X \in TM$. Now, we choose orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM such that $D_1 = Span\{e_1, \ldots, e_{2m}\}$ and $D_2 = Span\{e_{2n+1}, \ldots, e_{2n}\}$. Using (3.4) and (3.5), we have

$$\|P_2 Te_i\|^2 + \|Fe_i\|^2 = \sin^2\theta_1 \tag{3.6}$$

and

$$\|P_1 Te_j\|^2 + \|Fe_j\|^2 = \sin^2\theta_2 \tag{3.7}$$

for $i \in \{1, \ldots, 2m\}$ and $j \in \{2m+1, \ldots, 2n\}$. Therefore, we can choose a bi-slant orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of T_pM satisfying that

$$\begin{aligned}
Te_1 &= \cos\theta_1 e_2 + P_2 Te_1,\ Te_2 = -\cos\theta_1 e_1 + P_2 Te_2, \ldots, \\
Te_{2m-1} &= \cos\theta_1 e_{2m} + P_2 Te_{2m-1},\ Te_{2m} = -\cos\theta_1 e_{2m-1} + P_2 Te_{2m}, \\
Te_{2m+1} &= \cos\theta_2 e_{2m+2} + P_1 Te_{2m+1},\ Te_{2m+2} = -\cos\theta_2 e_{2m+1} + P_1 Te_{2m+2} \\
&\quad , \ldots, Te_{2n} = -\cos\theta_{2n-1} e_{2n} + P_1 Te_{2n-1}.
\end{aligned} \tag{3.8}$$

Since both D_1 and D_2 are slant distributions, we note that $P_i Te_{\ell+2}$ is orthogonal to both e_ℓ and $e_{\ell+1}$ vectors, where e_ℓ, $e_{\ell+1}$ and $e_{\ell+2}$ are any mutually orthogonal vectors in D_i, $i \in \{1,2\}$.

If M is a semi-slant submanifold of $\widetilde{M}$, then $\theta_1 = 0$. Thus, we have

$$\langle JX_1, Y_2 \rangle = 0 \tag{3.9}$$

for all $X_1 \in \Gamma(D_1)$ and $Y_2 \in \Gamma(D_2)$. Taking into consideration (3.8) and (3.9), we can choose an orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of semi-slant submanifolds satisfies that

$$\begin{aligned}
Te_1 &= e_2, \ldots, Te_{2m-1} = e_{2m}, \\
Te_{2m+1} &= \cos\theta_2 e_{2m+2}, \ldots, Te_{2n-1} = \cos\theta_2 e_{2n}.
\end{aligned} \tag{3.10}$$

If M is a hemi-slant submanifold of $\widetilde{M}$, then $\theta_2 = \frac{\pi}{2}$. Thus, we have

$$\langle JX_2, Y_2 \rangle = 0 \tag{3.11}$$

for all $X_2, Y_2 \in \Gamma(D_2)$. From (3.8) and (3.11), we can choose an orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of hemi-slant submanifolds satisfying that

$$\begin{aligned} \|P_1 Te_j\|^2 &= \|Te_j\|^2, \quad i \in \{1, \ldots, 2m\}, \\ \|P_2 Te_i\|^2 &= 0, \quad j \in \{2m+1, \ldots, 2n\}. \end{aligned} \tag{3.12}$$

Now, we shall need the following lemma:

Lemma 3.4 ([5])**.** *Let M be a (θ_1, θ_2) bi-slant submanifold of an almost Hermitian manifold $\widetilde{M}$. Given $\theta \in [0, \frac{\pi}{2}]$, M is θ-slant if and only if the following equations hold:*

$$P_2 TP_1 TP_1 + P_2 TP_2 TP_1 = 0, \tag{3.13}$$

$$P_1 TP_1 TP_2 + P_1 TP_2 TP_2 = 0, \tag{3.14}$$

$$P_1 TP_2 TP_1 = (\cos^2\theta_1 - \cos^2\theta) P_1, \tag{3.15}$$

$$P_2 TP_1 TP_2 = (\cos^2\theta_2 - \cos^2\theta) P_2. \tag{3.16}$$

Let M be a (θ_1, θ_2) bi-slant submanifold. Also, if M is a θ-slant submanifold, then taking into consideration (3.15), we have

$$\begin{aligned} \|P_2 Te_i\|^2 = \langle P_2 Te_i, P_2 Te_i \rangle &= -\langle e_i, TP_2 Te_i \rangle \\ &= -\langle e_i, P_1 TP_2 Te_i \rangle \\ &= \cos^2\theta - \cos^2\theta_1 \end{aligned} \tag{3.17}$$

for $i \in \{1, \ldots, 2m\}$. Using the similar way, we have

$$\|P_1 Te_j\|^2 = \cos^2\theta - \cos^2\theta_2 \tag{3.18}$$

for $j \in \{2m+1, \ldots, 2n\}$. From (3.8), (3.17) and (3.18), if M is a slant submanifold with $\theta_1 = \theta_2 = \theta$, then we can choose orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of T_pM such that

$$Te_1 = \cos\theta e_2, \ldots, Te_{2n-1} = \cos\theta\ e_{2n}. \tag{3.19}$$

We note that the orthonormal basis given in (3.19) was firstly given by B.-Y. Chen in [7].

4. Ricci curvature

In this section, we are going to give an inequality involving Ricci curvature of bi-slant submanifold on generalized complex space form and study this inequality for semi-slant submanifold, hemi-slant submanifold and slant submanifold of generalized complex space forms.

Theorem 4.1. *Let M be a $2n$-dimensional (θ_1,θ_2) bi-slant submanifold of a $2k$-dimensional generalized complex space form. Then*

(***a***) *For $X \in T_p^1M = \{X \in T_pM \mid \langle X,X\rangle = 1\}$, it follows that*

$$\operatorname{Ric}(X) \le n^2\|H\|^2 + \frac{c+3\alpha}{4}(2n-1) + \frac{3(c-\alpha)}{8}(\cos^2\theta_1 + \cos^2\theta_2) + \|P_1TX_{D_2}\|^2 + \|P_2TX_{D_1}\|^2), \tag{4.1}$$

where θ_1 is slant angle of D_1, θ_2 is slant angle of D_2 and $Ric(X)$ is the Ricci curvature of M.

(***b***) *The equality case of (4.1) is satisfied by $X \in T_p^1M$ if and only if*

$$\begin{cases} \sigma(X,Y) = 0, & \text{for all } Y \in T_pM \text{ orthogonal to } X, \\ \sigma(X,X) = nH(p). \end{cases} \tag{4.2}$$

(***c***) *The equality case of (4.1) holds for all $X \in T_p^1M$ if and only if p is a totally geodesic point.*

Proof. From (2.4) and (2.5), we have

$$n^2\|H\|^2 = \tau(p) - \widetilde{\tau}(T_pM) + \frac{1}{4}\sum_{r=n+1}^{2k}(\sigma_{11}^r - \sigma_{22}^r - \cdots - \sigma_{nn}^r)^2 + \sum_{r=2n+1}^{2k}\sum_{j=2}^{2n}(\sigma_{1j}^r)^2 - \sum_{r=2n+1}^{2k}\sum_{1\ne i<j\le 2n}(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2). \tag{4.3}$$

Using

$$\sum_{r=2n+1}^{2k}\sum_{1\ne i<j\le 2n}\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2 = \sum_{1\ne i<j\le 2n}K_{ij} - \widetilde{K}_{ij} \tag{4.4}$$

and

$$\sum_{1\ne i<j\le 2n}K_{ij} = \tau(p) - Ric(e_1), \tag{4.5}$$

we obtain

$$Ric(e_1) = n^2\|H\|^2 + \widetilde{Ric}_{T_pM}(e_1) - \sum_{r=2n+1}^{2k}\sum_{j=2}^{2n}(\sigma_{1j}^r)^2 - \frac{1}{4}\sum_{r=2n+1}^{2k}(\sigma_{11}^r - \sigma_{22}^r - \cdots - \sigma_{nn}^r)^2. \tag{4.6}$$

Since we can choose $e_1 = X$ as any unit vector in T_p^1M, we get the following inequality:

$$\operatorname{Ric}(X) \le n^2\|H\|^2 + \widetilde{\operatorname{Ric}}_{(T_pM)}(X). \tag{4.7}$$

Now, we calculate $\widetilde{\operatorname{Ric}}_{(T_pM)}(X)$. Let $p \in M$ and $\{e_1,\ldots,e_{2m},e_{2m+1},\ldots,e_{2n}\}$ be orthonormal basis of M such that $\{e_1,\ldots,e_{2m}\}$ is an orthonormal basis of D_1 and $\{e_{2m+1},\ldots,e_{2n}\}$ is an orthonormal basis of D_2. From (2.9) and (3.8), we have the following equalities:

$$\widetilde{Ric}_{D_1}(e_1) = \sum_{j=1}^{2m}\widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1)$$

$$= \frac{c+3\alpha}{4}(2m-1) + \frac{3(c-\alpha)}{4}\cos^2\theta_1, \tag{4.8}$$

$$\widetilde{Ric}_{D_2}(e_{2m+1}) = \sum_{j=2m+2}^{2n} \widetilde{g}(\widetilde{R}(e_{2m+1}, e_j)e_j, e_{2m+1})$$

$$= \frac{c+3\alpha}{4}(2n-2m-1) + \frac{3(c-\alpha)}{4}\cos^2\theta_2, \tag{4.9}$$

$$\sum_{j=2m+1}^{2n} \widetilde{g}(\widetilde{R}(e_1, e_j)e_j, e_1) = \frac{c+3\alpha}{4}(2n-2m) + \frac{3(c-\alpha)}{4}\sum_{j=2m+1}^{2n}\langle P_2 Te_1, e_j\rangle^2, \tag{4.10}$$

$$\sum_{j=1}^{2m} \widetilde{g}(\widetilde{R}(e_{2m+1}, e_j)e_j, e_{2m+1}) = \frac{c+3\alpha}{4}(2m) + \frac{3(c-\alpha)}{4}\sum_{j=1}^{2m}\langle P_1 Te_{2m+1}, e_j\rangle^2, \tag{4.11}$$

$$\sum_{j=1}^{2m} \widetilde{g}(\widetilde{R}(e_1, e_j)e_j, e_{2m+1}) = \frac{3(c-\alpha)}{4}\sum_{j=1}^{2m}\langle Te_1, e_j\rangle\langle Te_{2m+1}, e_j\rangle$$

$$= \frac{3(c-\alpha)}{4}\langle Je_1, Je_{2m+1}\rangle$$

$$= 0. \tag{4.12}$$

Now, we choose $X = \frac{1}{\sqrt{2}}(e_1 + e_{2m+1})$. Then $\|X\| = 1$ and $X \in T_p^1 M$. Thus, we get

$$\begin{aligned}\widetilde{Ric}_{T_pM}(X) = \frac{1}{2}\{&\widetilde{Ric}_{D_1}(e_1) + \widetilde{Ric}_{D_2}(e_{2m+1}) \\ &+ \sum_{j=2m+1}^{2n}\widetilde{g}(\widetilde{R}(e_1, e_j)e_j, e_1) + \sum_{j=1}^{2m}\widetilde{g}(\widetilde{R}(e_{2m+1}, e_j)e_j, e_{2m+1})\} \\ &+ \sum_{j=1}^{2m}\widetilde{g}(\widetilde{R}(e_1, e_j)e_j, e_{2m+1}).\end{aligned} \tag{4.13}$$

If we put (4.8), (4.9), (4.10), (4.11) and (4.12) in (4.13), we have

$$\begin{aligned}\widetilde{Ric}_{T_pM}(X) = &\frac{c+3\alpha}{4}(2n-1) + \frac{3(c-\alpha)}{8}(\cos^2\theta_1 + \cos^2\theta_2) \\ &+ \sum_{j=2m+1}^{2n}\langle P_2 Te_1, e_j\rangle^2 + \sum_{j=1}^{2m}\langle P_1 Te_{2m+1}, e_j\rangle^2\}.\end{aligned} \tag{4.14}$$

From (4.14) and (4.7), we get (4.1). The equality in (4.1) is valid if and only if

$$\sigma_{12}^r = \cdots = \sigma_{12n}^r = 0 \text{ and } \sigma_{11}^r = \sigma_{22}^r + \cdots + \sigma_{2n2n}^r, \quad r \in \{2n+1, \ldots, 2k\}, \tag{4.15}$$

which is equivalent to (4.2).

Now, we prove the statement **(c)**. Assuming the equality case of (4.1) for all $X \in T_p^1 M$, in view of (4.15), for each $r \in \{2n+1, \ldots, 2k\}$, we have

$$\sigma_{ij}^r = 0, \qquad i \neq j, \tag{4.16}$$

$$2\sigma^r_{ii} = \sigma^r_{11} + \sigma^r_{22} + \cdots + \sigma^r_{2n2n}, \qquad i \in \{1, \ldots, 2n\}. \tag{4.17}$$

From (4.17), we have $2\sigma^r_{11} = 2\sigma^r_{22} = \cdots = 2\sigma^r_{2n2n} = \sigma^r_{11} + \sigma^r_{22} + \cdots + \sigma^r_{2n2n}$, which implies that

$$2(n-1)\left(\sigma^r_{11} + \sigma^r_{22} + \cdots + \sigma^r_{2n2n}\right) = 0.$$

Since $n \neq 1$, $\sigma^r_{11} + \sigma^r_{22} + \cdots + \sigma^r_{nn} = 0$ is valid. Then in view of (4.17), we get $\sigma^r_{ii} = 0$ for all $i \in \{1, \ldots, n\}$. This together with (4.16) gives $\sigma^r_{ij} = 0$ for all $i, j \in \{1, \ldots, 2n\}$ and $r \in \{2n+1, \ldots, 2k\}$, that is, p is a totally geodesic point. The proof of the converse part is straightforward.

From Theorem 4.1, we get the following corollaries:

Corollary 4.2.

(***a***) *Let M be a $2n$-dimensional submanifold of a generalized complex space form $\widetilde{M}(c, \alpha)$. We have the following table:*

	$\widetilde{M}$	M	Inequality
(1)	$\widetilde{M}(c,\alpha)$	bi – slant	$\mathrm{Ric}(X) \leq n^2\|H\|^2 + \frac{c+3\alpha}{4}(2n-1) + \frac{3(c-\alpha)}{8}(\cos^2\theta_1 + \cos^2\theta_2 + \|P_1TX_{D_2}\|^2 + \|P_2TX_{D_1}\|^2)$
(2)	$\widetilde{M}(c,\alpha)$	semi – slant	$\mathrm{Ric}(X) \leq n^2\|H\|^2 + \frac{c+3\alpha}{4}(2n-1) + \frac{3(c-\alpha)}{8}(1 + \cos^2\theta_2)$
(3)	$\widetilde{M}(c,\alpha)$	hemi – slant	$\mathrm{Ric}(X) \leq n^2\|H\|^2 + \frac{c+3\alpha}{4}(2n-1) + \frac{3(c-\alpha)}{8}(\cos^2\theta_1 + \|TX_{D_2}\|^2 + \|P_2TX_{D_1}\|^2)$
(4)	$\widetilde{M}(c,\alpha)$	CR	$\mathrm{Ric}(X) \leq n^2\|H\|^2 + \frac{c+3\alpha}{4}(2n-1) + \frac{3(c-\alpha)}{8}$
(5)	$\widetilde{M}(c,\alpha)$	θ – slant *with* $\theta_1 = \theta_2 = \theta$ *or* $\theta_1 = \theta_2 \neq \theta$	$\mathrm{Ric}(X) \leq n^2\|H\|^2 + \frac{c+3\alpha}{4}(2n-1) + \frac{3(c-\alpha)}{4}\cos^2\theta$
(6)	$\widetilde{M}(c,\alpha)$	invariant	$\mathrm{Ric}(X) \leq n^2\|H\|^2 + \frac{c+3\alpha}{4}(2n-1) + \frac{3(c-\alpha)}{4}$
(7)	$\widetilde{M}(c,\alpha)$	anti – invariant	$\mathrm{Ric}(X) \leq n^2\|H\|^2 + \frac{c+3\alpha}{4}(2n-1)$

(***b***) *The equality case of inequalities in the previous table is satisfied for* $X \in T_p^1 M$ *if and only if*

$$\begin{cases} \sigma(X,Y) = 0, & \text{for all } Y \in T_p M \text{ orthogonal to } X, \\ 2\sigma(X,X) = nH(p). \end{cases}$$

If $H(p) = 0$*, then* $X \in T_p^1 M$ *satisfies the equality case of inequalities in the previous table if and only if* $X \in \mathcal{N}_p = \{X \in T_p M : \sigma(X,Y) = 0,\ \forall Y \in T_p M\}$.

(***c***) *The equality case of inequalities* **(1)**–**(4)** *is satisfied for all* $X \in T_p^1 M$ *if and only if* p *is a totally geodesic point.*

(***d***) *The equality case of inequalities* **(5)**–**(7)** *is satisfied for all* $X \in T_p^1 M$ *if and only if either* p *is a totally geodesic point or* $n = 1$ *and* p *is a totally umbilical point.*

Proof. From Theorem 4.1, we have the inequality (1) immediately. Taking (3.10) into consideration, we have

$$P_1 T X_{D_2} = P_2 T X_{D_1} = 0, \tag{4.18}$$

where $X = X_{D_1} + X_{D_2}$ for $X_{D_1} \in D_1$ and $X_{D_2} \in D_2$. If we write (4.18) and (3.12) in the inequality (1), we get the inequalities (2) and (3), respectively. Next, putting $\theta_2 = \frac{\pi}{2}$ in the inequality (2), we have the inequality (4) or putting $\theta_1 = 0$ in the inequality (3), then

$$T X_{D_2} = P_2 T X_{D_1} = 0. \tag{4.19}$$

Thus, we have the inequality (4) again. If we put (3.17) and (3.18) in the inequality (1), we get the inequality (5). Writing $\theta = 0$ and $\theta = \frac{\pi}{2}$ in the inequality (5), we obtain the inequalities (6) and (7), respectively. Hence proof of the **(a)** is complete. The statements **(b-d)** are straightforward from Theorem 4.1.

Remark 4.3. The inequalities (5-7) on the above table were proved by S. Hong and M. M. Tripathi in [11]. The authors also studied generic submanifolds of generalized complex space forms. Since generic submanifolds don't contain slant distributions, we note that bi-slant submanifolds are not a particular case of generic submanifold.

Corollary 4.4.

(***a***) *Let* M *be a* $2n$*-dimensional submanifold of a complex space form* $\widetilde{M}(4c)$*. We have the following table:*

	$\widetilde{M}$	M	**Inequality**
(1)	$\widetilde{M}(4c)$	bi – slant	$\mathrm{Ric}(X) \le n^2\|H\|^2 + (2n-1)c + \frac{3c}{2}(\cos^2\theta_1 + \cos^2\theta_2 + \|P_1TX_{D_2}\|^2 + \|P_2TX_{D_1}\|^2)$
(2)	$\widetilde{M}(4c)$	semi – slant	$\mathrm{Ric}(X) \le n^2\|H\|^2 + (2n-1)c + \frac{3c}{2}(1 + \cos^2\theta_2)$
(3)	$\widetilde{M}(4c)$	hemi – slant	$\mathrm{Ric}(X) \le n^2\|H\|^2 + (2n-1)c + \frac{3c}{2}(\cos^2\theta_1 + \|TX_{D_2}\|^2 + \|P_2TX_{D_1}\|^2)$
(4)	$\widetilde{M}(4c)$	CR	$\mathrm{Ric}(X) \le n^2\|H\|^2 + 2(n+1)c + \frac{3c}{2}$
(5)	$\widetilde{M}(4c)$	θ – slant ***with*** $\theta_1 = \theta_2 = \theta$ ***or*** $\theta_1 = \theta_2 \neq \theta$	$\mathrm{Ric}(X) \le n^2\|H\|^2 + (2n-1)c + 3c\cos^2\theta$
(6)	$\widetilde{M}(4c)$	anti – invariant	$\mathrm{Ric}(X) \le n^2\|H\|^2 + (2n-1)c$

(***b***) *The equality case of inequalities in the previous table is satisfied for $X \in T_p^1M$ if and only if*

$$\begin{cases} \sigma(X,Y) = 0, & \text{for all } Y \in T_pM \text{ orthogonal to } X, \\ 2\sigma(X,X) = nH(p). \end{cases}$$

If $H(p) = 0$, then $X \in T_p^1M$ satisfies the equality case of inequalities in the previous table if and only if $X \in \mathcal{N}_p = \{X \in T_pM : \sigma(X,Y) = 0,\ \forall Y \in T_pM\}$.

(***c***) *The equality case of inequalities* **(1)–(4)** *is satisfied for all $X \in T_p^1M$ if and only if p is a totally geodesic point.*

(***d***) *The equality case of inequalities* **(5)–(6)** *is satisfied for all $X \in T_p^1M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Remark 4.5. In [13], K. Matsumoto, I. Mihai and A. Oiaga compute the inequalities (5-6). In addition to this, using the fact that every invariant submanifold of a (nearly) Kaehler manifold is minimal, they gave the following corollary.

Corollary 4.6. *Let M be an $2n$-dimensional invariant submanifold of a complex space form $\widetilde{M}(4c)$. Then*

(a) *Any* $X \in T_p^1 M = \{X \in T_p M \mid \langle X, X \rangle = 1\}$ *satisfies*

$$\mathrm{Ric}\,(X) \leq 2(n+1)c. \tag{4.20}$$

(b) *An* $X \in T_p^1 M$ *satisfies the equality case of* (4.20) *if and only if* $X \in \mathcal{N}_p$.

(c) *The equality case of* (4.20) *is satisfied for all* $X \in T_p^1 M$ *if and only if* p *is a totally geodesic point.*

5. Scalar curvature and δ-invariant

Now, we recall the following definition of B.-Y. Chen and the following Lemma for future uses [8].

Definition 5.1. Let M be a submanifold of a Riemannian manifold $\widetilde{M}$. The Chen invariant at a point $p \in M$, denoted by $\delta_M(p)$, is defined by

$$\delta_M(p) = \tau(p) - \inf(K)(p), \tag{5.1}$$

where $\tau(p)$ is the scalar curvature and

$$\inf(K)(p) = \inf\{K(\Pi) : K(\Pi) \text{ is a plane section of } T_p M\}.$$

Lemma 5.2. *If* $n > k \geq 2$ *and* $a_1, \ldots, a_n, a$ *are real numbers such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 \;=\; (n-1)(\sum_{i=1}^{n} a_i^2 + a), \tag{5.2}$$

then

$$2a_1 a_2 \geq a,$$

with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

Let $\Pi = Span\{X, Y\}$ be any 2-dimensional plane section of $T_p M$. We set

$$T(\Pi) = \langle TX, Y \rangle^2. \tag{5.3}$$

Now, we are going to give an optimal inequality involving the Chen invariant as follows:

Theorem 5.3. *Let M be a $2n$-dimensional (θ_1,θ_2) bi-slant submanifold of a $2k$-dimensional generalized complex space form. For any 2-dimensional plane section $\Pi = Span\{X,Y\}$ in T_pM, we have*

$$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c+3\alpha}{4}(2n+1)(n-1) + \frac{3(c-\alpha)}{8}[n\cos^2\theta_1 + n\cos^2\theta_2 + \frac{1}{2}\|P_1T\|^2 + \frac{1}{2}\|P_2T\|^2 - 2T(\Pi)]. \tag{5.4}$$

The equality case of the inequality (5.4) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1,\dots,e_{2n}\}$ of T_pM and an orthonormal basis $\{e_{2n+1},\dots,e_{2k}\}$ of $T_p^\perp M$ such that the shape operators of M have the following forms:

$$A_{2n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}, \quad a+b=\mu, \tag{5.5}$$

$$A_r = \begin{pmatrix} c & d & 0 & \dots & 0 \\ d & -c & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad r \in \{2n+2,\dots,2k\}. \tag{5.6}$$

Proof. From the Gauss equation and (4.14), we have

$$2\tau(p) = n^2\|H\|^2 - \|\sigma\|^2 + \frac{c+3\alpha}{2}(2n-1)n + \frac{3(c-\alpha)}{8}[2n\cos^2\theta_1 + 2n\cos^2\theta_2 + \|P_1T\|^2 + \|P_2T\|^2]. \tag{5.7}$$

In equation (5.7), if we put

$$w = 2\tau(p) - \frac{n^2(n-1)}{n-2}\|H\|^2 - \frac{c+3\alpha}{2}(2n-1)n - \frac{3(c-\alpha)}{8}[2n\cos^2\theta_1 + 2n\cos^2\theta_2 + \|P_1T\|^2 + \|P_2T\|^2], \tag{5.8}$$

we get

$$n^2\|H\|^2 = (n-1)(w + \|\sigma\|^2). \tag{5.9}$$

If we choose e_{2n+1} in the direction of the mean curvature vector $H(p)$, then (5.9) gives

$$(\sum_{i=1}^{2n}\sigma_{ii}^{2n+1})^2 = (n-1)[\sum_{i=1}^{2n}(\sigma_{ii}^{2n+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{2n+1})^2 + \sum_{r=2n+2}^{2k}\sum_{i,j=1}^{2n}(\sigma_{ij}^r)^2 + w]. \tag{5.10}$$

Applying Lemma 5.2 and equation (5.10), we obtain

$$2\sigma_{11}^{2n+1}\sigma_{22}^{2n+1} \geq \sum_{i\neq j}(\sigma_{ij}^{2n+1})^2 + \sum_{r=2n+2}^{2k}\sum_{i,j=1}^{2n}(\sigma_{ij}^{r})^2 + w. \tag{5.11}$$

Therefore, we have

$$\begin{aligned} K(\Pi) &= \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4}T(\Pi) + \sum_{r=2n+1}^{2k}[\sigma_{11}^r\sigma_{22}^r - (\sigma_{12})^2] \\ &\geq \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4}T(\Pi) + \frac{1}{2}[\sum_{i\neq j}(\sigma_{ij}^{2n+1})^2 \\ &\quad + \sum_{r=2n+2}^{2k}\sum_{i,j=1}^{2n}(\sigma_{ij}^r)^2 + w] + \sum_{r=2n+2}^{2k}\sigma_{11}^r\sigma_{22}^r - \sum_{r=2n+2}^{2k}(\sigma_{12})^2 \\ &= \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4}T(\Pi) + \frac{1}{2}\sum_{i\neq j}(\sigma_{ij}^{2n+1})^2 \\ &\quad + \frac{1}{2}\sum_{r=2n+2}^{2k}\sum_{i,j=1}^{2n}(\sigma_{ij}^r)^2 + \frac{1}{2}\sum_{r=2n+2}^{2k}(\sigma_{11}^r + \sigma_{22}^r)^2 \\ &\quad + \sum_{j>2}[(\sigma_{1j}^{2n+1})^2 + (\sigma_{2j}^{2n+1})^2] + \frac{w}{2} \\ &\geq \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4}T(\Pi) + \frac{w}{2}. \end{aligned} \tag{5.12}$$

From (5.8) and (5.12), we have

$$\begin{aligned} \inf K(\Pi) \geq{}& \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4}T(\Pi) + \tau(p) - \frac{n^2(n-1)}{n-2}\|H\|^2 \\ &-\frac{3(c-\alpha)}{8}(2n\cos^2\theta_1 + 2n\cos^2\theta_2 + \|P_1T\|^2 \\ &+\|P_2T\|^2). \end{aligned} \tag{5.13}$$

From (5.1) and (5.13), we get (5.4).

The equality case of (5.4) is satisfied at $p \in M$ if and only if

$$\begin{aligned} &\sigma_{ij}^{2n+1} = 0,\ \forall i \neq j,\ i,j > 2, \\ &\sigma_{ij}^r = 0,\ \forall i \neq j,\ i,j > 2,\ r \in \{2n+1,\dots,2k\}, \\ &\sigma_{11}^r + \sigma_{22}^r = 0,\ \forall r \in \{2n+2,\dots,2k\}, \\ &\sigma_{1j}^{2n+1} = \sigma_{2j}^{2n+1} = 0,\ \forall j > 2, \\ &\sigma_{11}^{2n+1} = \sigma_{22}^{2n+1} = \sigma_{33}^{2n+1} = \dots = \sigma_{2n2n}^{2n+1}, \end{aligned} \tag{5.14}$$

which shows that the shape operators of M at $p \in M$ take the form of (5.5) and (5.6).

From Theorem 5.3, we get the following corollaries:

Corollary 5.4. *Let M be a submanifold of a generalized complex space form $\widetilde{M}(c,\alpha)$. For any 2-dimensional plane section Π in T_pM, we have the following table:*

	$\widetilde{M}$	M	Inequality
(1)	$\widetilde{M}(c,\alpha)$	bi – slant	$\delta_M(p) \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c+3\alpha}{4}(2n+1)(n-1) + \frac{3(c-\alpha)}{8}[n\cos^2\theta_1 + n\cos^2\theta_2 + \frac{1}{2}\|P_1T\|^2 + \frac{1}{2}\|P_2T\|^2 - 2T(\Pi)]$
(2)	$\widetilde{M}(c,\alpha)$	semi – slant	$\delta_M(p) \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c+3\alpha}{4}(2n+1)(n-1) + \frac{3(c-\alpha)}{8}[n(1+\cos^2\theta_2) - 2T(\Pi)]$
(3)	$\widetilde{M}(c,\alpha)$	hemi – slant	$\delta_M(p) \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c+3\alpha}{4}(2n+1)(n-1) + \frac{3(c-\alpha)}{8}[n\cos^2\theta_1 + \frac{1}{2}\|T\|^2 + \frac{1}{2}\|P_2T\|^2 - 2T(\Pi)]$
(4)	$\widetilde{M}(c,\alpha)$	CR	$\delta_M(p) \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c+3\alpha}{4}(2n+1)(n-1) + \frac{3(c-\alpha)}{8}[n - 2T(\Pi)]$
(5)	$\widetilde{M}(c,\alpha)$	θ – slant ***with*** $\theta_1 = \theta_2 = \theta$ ***or*** $\theta_1 = \theta_2 \neq \theta$	$\delta_M(p) \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c+3\alpha}{4}(2n+1)(n-1) + \frac{3(c-\alpha)}{4}[n\cos^2\theta - T(\Pi)]$
(6)	$\widetilde{M}(c,\alpha)$	invariant	$\delta_M(p) \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c+3\alpha}{4}(2n+1)(n-1) + \frac{3(c-\alpha)}{4}[n - T(\Pi)]$
(7)	$\widetilde{M}(c,\alpha)$	anti – invariant	$\delta_M(p) \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c+3\alpha}{4}(2n+1)(n-1)$

The equality case of the inequalities holds at a point $p \in M$ if and only if the shape operators of M take the form of (5.5) *and* (5.6).

Remark 5.5. The inequalities (5-7) were proved by J-S. Kim, Y-M. Song and M. M. Tripathi in [12] and by A. Mihai in [14].

Corollary 5.6. *Let M be a submanifold of a complex space form $\widetilde{M}(4c)$. For any 2-dimensional* *plane section* Π *in* T_pM*, we have the following table:*

	$\widetilde{M}$	M	**Inequality**
(1)	$\widetilde{M}(c,\alpha)$	bi – slant	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + (2n+1)(n-1)c + \frac{3c}{2}[n\cos^2\theta_1 + n\cos^2\theta_2 + \frac{1}{2}\|P_1T\|^2 + \frac{1}{2}\|P_2T\|^2 - 2T(\Pi)]$
(2)	$\widetilde{M}(c,\alpha)$	semi – slant	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + (2n+1)(n-1)c + \frac{3c}{2}[n(1+\cos^2\theta_2) - 2T(\Pi)]$
(3)	$\widetilde{M}(c,\alpha)$	hemi – slant	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + (2n+1)(n-1)c + \frac{3c}{2}[n\cos^2\theta_1 + \frac{1}{2}\|T\|^2 + \frac{1}{2}\|P_2T\|^2 - 2T(\Pi)]$
(4)	$\widetilde{M}(c,\alpha)$	CR	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + (2n+1)(n-1)c + \frac{3c}{2}[n - 2T(\Pi)]$
(5)	$\widetilde{M}(c,\alpha)$	θ – slant ***with*** $\theta_1 = \theta_2 = \theta$ ***or*** $\theta_1 = \theta_2 \neq \theta$	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + (2n+1)(n-1)c + 3c[n\cos^2\theta - T(\Pi)]$
(6)	$\widetilde{M}(c,\alpha)$	anti – invariant	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + (2n+1)(n-1)c$

The equality case of the inequalities holds at a point $p \in M$ *if and only if the shape operators of* M *take the form of* (5.5) *and* (5.6).

Corollary 5.7. *Let* M *be a* $2n$*-dimensional invariant submanifold of a complex space form* $\widetilde{M}(4c)$*. For any 2-dimensional plane section* Π *in* T_pM*, we have*

$$\delta_M(p) \leq (2n+1)(n-1)c + 3c[n - T(\Pi)]. \tag{5.15}$$

The equality case of (5.15) *satisfied at* $p \in M$ *if and only if the shape operators of* M *take form as* (5.5) *and* (5.6).

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