TAMKANG JOURNAL OF MATHEMATICS
Volume 37, Number 1, 11-14, Spring 2006

NOTE ON ALZER'S INEQUALITY

CHAO-PING CHEN AND FENG QI

Abstract. If the sequence $\{a_i\}_{i=1}^{\infty}$ satisfies $\triangle a_i = a_{i+1} - a_i > 0$, $\triangle^2 a_i = \triangle(\triangle a_i) = a_{i+2} - 2a_{i+1} + a_i \geqslant 0$, $i = 0, 1, 2, \dots$, $a_0 = 0$. Then

$$\frac{a_n}{a_{n+1}} < \left(\frac{1}{a_n} \sum_{i=1}^{n} a_i^r \Big/ \frac{1}{a_{n+1}} \sum_{i=1}^{n+1} a_i^r \right)^{1/r}$$

for all natural numbers n, and all real $r > 0$.

1. Introduction

The Alzer's inequality to which title refers is

$$\frac{n}{n+1} < \left(\frac{1}{n} \sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{1/r} \tag{1}$$

for all natural numbers n, and all real $r > 0$ (see [1, 2, 4, 5]).

Elezović and Pečarić [3] genealized Alzer's inequality as follows: If the positive sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$1 \leqslant \left(\frac{a_{n+2}}{a_{n+1}} \right)^r \left[\frac{a_{n+2}}{a_{n+1}} - 1 + \left(\frac{a_n}{a_{n+1}} \right)^{r+1} \right], \ n \geqslant 0, \ a_0 = 0, \tag{2}$$

then, for $r > 0$,

$$\frac{a_n}{a_{n+1}} \leq \left(\frac{1}{a_n} \sum_{i=1}^{n} a_i^r \Big/ \frac{1}{a_{n+1}} \sum_{i=1}^{n+1} a_i^r \right)^{1/r}. \tag{3}$$

In this paper, we consider the sufficient conditions relating to the sequence $\{a_n\}_{n=1}^{\infty}$ only, so that (3) strictly holds for all natural numbers n, and all real $r > 0$.

Received and revised June 17, 2004.
2000 *Mathematics Subject Classification*. Primary 26D15.
Key words and phrases. Alzer's inequality, convex sequence, mathematical induction, Lagrange's mean value theorem, Cauchy's mean value theorem.
The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, Doctor Fund of Jiaozuo Institute of Technology, China.

Theorem . *If the sequence* $\{a_i\}_{i=1}^{\infty}$ *satisfies*

$$\triangle a_i = a_{i+1} - a_i > 0 \tag{4}$$

and

$$\triangle^2 a_i = \triangle(\triangle a_i) = a_{i+2} - 2a_{i+1} + a_i \geq 0 \tag{5}$$

for $i = 0, 1, 2, \ldots, a_0 = 0$*, then*

$$\frac{a_n}{a_{n+1}} < \left(\frac{1}{a_n} \sum_{i=1}^{n} a_i^r \bigg/ \frac{1}{a_{n+1}} \sum_{i=1}^{n+1} a_i^r \right)^{1/r} \tag{6}$$

for all natural numbers n*, and all real* $r > 0$*.*

2. Lemma

Lemma. *For any fixed real* $r > 0$*, the function*

$$f(x, y) = \frac{(x+y)^{r+1} - x^{r+1}}{(x+y)^r} \quad (x > 0,\ y > 0)$$

is strictly increasing with both x *and* y*.*

Proof. Easy calculation yields

$$f'_x(x, y) = \frac{[(x+y)^{r+1} - x^{r+1}] - (r+1)x^r y}{(x+y)^{r+1}}.$$

By Lagrange's mean value theorem, there exists at least one point $\xi \in (x, x+y)$ such that

$$(x+y)^{r+1} - x^{r+1} = (r+1)y\xi^r, \quad x < \xi < x+y$$

and therefore

$$[(x+y)^{r+1} - x^{r+1}] - (r+1)x^r y = (r+1)y(\xi^r - x^r) > 0,$$

which implies that $f'_x(x, y) > 0$. Clearly,

$$f'_y(x, y) = 1 + \frac{rx^{r+1}}{(x+y)^{r+1}} > 0.$$

The proof is complete.

3. Proofs of Theorem.

The first proof. (6) is equivalent to

$$\sum_{i=1}^{n} a_i^r > \frac{a_n^{r+1} a_{n+1}^r}{a_{n+1}^{r+1} - a_n^{r+1}}. \tag{7}$$

We now prove (7) by mathematical induction. When $n = 1$, (7) is $a_1^r > \frac{a_1^{r+1} a_2^r}{a_2^{r+1} - a_1^{r+1}}$, i.e. $\frac{a_2}{a_1} - \left(\frac{a_2}{a_1}\right)^r > 1$, so (7) is ture.

Suppose (7) holds for some $n \geq 1$, then

$$\sum_{i=1}^{n+1} a_i^r = \sum_{i=1}^{n} a_i^r + a_{n+1}^r > \frac{a_n^{r+1} a_{n+1}^r}{a_{n+1}^{r+1} - a_n^{r+1}} + a_{n+1}^r = \frac{a_{n+1}^{2r+1}}{a_{n+1}^{r+1} - a_n^{r+1}}.$$

In order to prove (7) for $n + 1$, it is sufficient to show that

$$\frac{a_{n+1}^{2r+1}}{a_{n+1}^{r+1} - a_n^{r+1}} > \frac{a_{n+1}^{r+1} a_{n+2}^r}{a_{n+2}^{r+1} - a_{n+1}^{r+1}}, \tag{8}$$

which can be rearranged as

$$\frac{a_{n+1}^{r+1} - a_n^{r+1}}{a_{n+1}^r} < \frac{a_{n+2}^{r+1} - a_{n+1}^{r+1}}{a_{n+2}^r}. \tag{9}$$

From (4) and (5) we have

$$0 = a_0 < a_n < a_{n+1}, \quad n = 1, 2, \ldots,$$
$$\triangle a_n = a_{n+1} - a_n \leq a_{n+2} - a_{n+1} = \triangle a_{n+1}, \quad n = 0, 1, 2, \ldots.$$

In other words, the sequence $\{a_n\}_{n=1}^{\infty}$ is a positive, strictly increasing and convex one. By Lemma , we have

$$\begin{aligned} \frac{a_{n+1}^{r+1} - a_n^{r+1}}{a_{n+1}^r} &= \frac{(a_n + \triangle a_n)^{r+1} - a_n^{r+1}}{(a_n + \triangle a_n)^r} < \frac{(a_{n+1} + \triangle a_n)^{r+1} - a_{n+1}^{r+1}}{(a_{n+1} + \triangle a_n)^r} \\ &\leq \frac{(a_{n+1} + \triangle a_{n+1})^{r+1} - a_{n+1}^{r+1}}{(a_{n+1} + \triangle a_{n+1})^r} = \frac{a_{n+2}^{r+1} - a_{n+1}^{r+1}}{a_{n+2}^r}. \end{aligned}$$

The proof of the theorem is complete.

The second proof. (8) is also equivalant to

$$\frac{a_{n+2}^{r+1} - a_{n+1}^{r+1}}{a_{n+1}^{r+1} - a_n^{r+1}} > \left(\frac{a_{n+2}}{a_{n+1}}\right)^r. \tag{10}$$

It is easy to see that

$$\begin{aligned} \frac{a_{n+2}^{r+1} - a_{n+1}^{r+1}}{a_{n+1}^{r+1} - a_n^{r+1}} &= \frac{(a_n + \triangle a_n + \triangle a_{n+1})^{r+1} - (a_n + \triangle a_n)^{r+1}}{(a_n + \triangle a_n)^{r+1} - a_n^{r+1}} \\ &\geq \frac{(a_n + \triangle a_n + \triangle a_{n+1})^{r+1} - (a_n + \triangle a_{n+1})^{r+1}}{(a_n + \triangle a_n)^{r+1} - a_n^{r+1}}. \end{aligned} \tag{11}$$

Define

$$f(x) = (x + \triangle a_{n+1})^{r+1}, \quad x \in [a_n, a_n + \triangle a_n],$$
$$g(x) = x^{r+1}, \quad x \in [a_n, a_n + \triangle a_n].$$

By Cauchy's mean value theorem, there exists at least one point $\eta \in (a_n, a_n + \triangle a_n)$ such that

$$\frac{(a_n + \triangle a_n + \triangle a_{n+1})^{r+1} - (a_n + \triangle a_{n+1})^{r+1}}{(a_n + \triangle a_n)^{r+1} - a_n^{r+1}} = \frac{(\eta + \triangle a_{n+1})^r}{\eta^r} = \left(1 + \frac{\triangle a_{n+1}}{\eta}\right)^r. \tag{12}$$

Because of $1/\eta > 1/(a_n + \triangle a_n) = 1/a_{n+1}$, we have

$$\left(1 + \frac{\triangle a_{n+1}}{\eta}\right)^r > \left(1 + \frac{\triangle a_{n+1}}{a_{n+1}}\right)^r = \left(\frac{a_{n+2}}{a_{n+1}}\right)^r. \tag{13}$$

The combination of (11), (12) and (13) implies (10), and thus, the proof of the theorem is complete.

References

[1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396-402.
[2] Ch.-P. Chen and F. Qi, *Notes on proofs of Alzer's inequality*, Octogon Mathematical Magazine **11**(2003), 29-33.
[3] N. Elezović and J. Pečarić, *On Alzer's inequality*, J. Math. Anal. Appl. **223**(1998), 366-369.
[4] J. Sándor, *On an inequality of Alzer*, J. Math. Anal. Appl. **192** (1995), 1034-1035.
[5] J. S. Ume, *An elementary proof of H. Alzer's inequality*, Math. Japon. **44** (1996), 521-522.

Department of Applied Mathematics and Informatics, Henan Polytechnic University, #142, Mid-Jiefang Road, Jiaozuo City, Henan 454000, China.

E-mail: chenchaoping@hpu.edu.cn

Department of Applied Mathematics and Informatics, Henan Polytechnic University, #142, Mid-Jiefang Road, Jiaozuo City, Henan 454000, China.

E-mail: qifeng@jzit.edu.cn