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INVERSE PROBLEMS FOR HIGHER ORDER DIFFERENTIAL SYSTEMS WITH REGULAR SINGULARITIES ON STAR-TYPE GRAPHS

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Abstract. We study an inverse spectral problem for arbitrary order ordinary differential equations on compact star-type graphs when differential equations have regular singularities at boundary vertices. As the main spectral characteristics we introduce and study the so-called Weyl-type matrices which are generalizations of the Weyl function (m-function) for the classical Sturm-Liouville operator. We provide a procedure for constructing the solution of the inverse problem and prove its uniqueness.

1. Introduction

We study an inverse spectral problem for arbitrary order ordinary differential equations on compact star-type graphs when differential equations have regular singularities at boundary vertices. Boundary value problems on graphs (spatial networks, trees) often appear in natural sciences and engineering (see [1]-[6]). Inverse spectral problems consist in recovering operators from their spectral characteristics. We pay attention to the most important nonlinear inverse problems of recovering coefficients of differential equations (potentials) provided that the structure of the graph is known a priori.

For *second-order* differential operators on compact graphs inverse spectral problems have been studied fairly completely in [7-13] and other works. Inverse problems for *higher-order* differential operators on graphs were investigated in [14]-[15]. We note that inverse spectral problems for second-order and for higher-order ordinary differential operators on *an interval* have been studied by many authors (see the monographs [16]-[22] and the references therein). Arbitrary order differential operators on an interval with regular singularities were considered in [23]-[26].

In this paper we study the inverse spectral problem for arbitrary order differential operators with regular singularities on compact star-type graphs. As the main spectral characteristics in this paper we introduce and study the so-called Weyl-type matrices which are

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generalizations of the Weyl function (m-function) for the classical Sturm-Liouville operator (see [27]), of the Weyl matrix for higher-order differential operators on an interval introduced in [21]-[22], and generalizations of the Weyl-type matrices for higher-order differential operators on graphs (see [14]-[15]). We show that the specification of the Weyl-type matrices uniquely determines the coefficients of the differential equation on the graph, and we provide a constructive procedure for the solution of the inverse problem from the given Weyl-type matrices. For studying this inverse problem we develope the method of spectral mappings [21]-[22]. We also essentially use ideas from [23] on differential equations with regular singularities. The obtained results are natural generalizations of the well-known results on inverse problems for differential operators on an interval and on graphs.

2. Formulation of the inverse problem

Consider a compact star-type graph T in $\mathbf{R}^\omega$ with the set of vertices $V = \{v_0,\dots,v_p\}$ and the set of edges $\mathcal{E} = \{e_1,\dots,e_p\}$, where $v_1,\dots,v_p$ are the boundary vertices, v_0 is the internal vertex, and $e_j = [v_j, v_0]$, $j = \overline{1,p}$, $\bigcap_{j=1}^{p} e_j = \{v_0\}$. Let l_j be the length of the edge e_j. Each edge $e_j \in \mathcal{E}$ is parameterized by the parameter $x_j \in [0,l_j]$ such that $x_j = 0$ corresponds to the boundary vertices $v_1,\dots,v_p$, and $x_j = l_j$ corresponds to the internal vertex v_0. An integrable function Y on T may be represented as $Y = \{y_j\}_{j=\overline{1,p}}$, where the function $y_j(x_j)$ is defined on the edge e_j.

Consider the differential equations on T:

$$y_j^{(n)}(x_j) + \sum_{\mu=0}^{n-2}\Big(\frac{\nu_{\mu j}}{x_j^{n-\mu}} + q_{\mu j}(x_j)\Big)y_j^{(\mu)}(x_j) = \lambda y_j(x_j), \quad x_j \in (0,l_j), \quad j=\overline{1,p}, \tag{1}$$

where λ is the spectral parameter, $q_{\mu j}(x_j)$ are complex-valued integrable functions. We call $q_j = \{q_{\mu j}\}_{\mu=\overline{0,n-2}}$ the potential on the edge e_j, and we call $q = \{q_j\}_{j=\overline{1,p}}$ the potential on the graph T. Let $\{\xi_{kj}\}_{k=\overline{1,n}}$ be the roots of the characteristic polynomial

$$\delta_j(\xi) = \sum_{\mu=0}^{n}\nu_{\mu j}\prod_{k=0}^{\mu-1}(\xi-k), \quad \nu_{nj} := 1,\ \nu_{n-1,j} := 0.$$

For definiteness, we assume that $\xi_{kj}-\xi_{mj} \neq sn, s\in\mathbf{Z}$, $Re\,\xi_{1j} < \ldots < Re\,\xi_{nj}$, $\xi_{kj} \neq \overline{0,n-3}$ (other cases require minor modifications). We set $\theta_j := n-1-Re(\xi_{nj}-\xi_{1j})$, and assume that the functions $q_{\mu j}^{(\nu)}(x_j)$, $\nu=\overline{0,\mu-1}$, are absolutely continuous, and $q_{\mu j}^{(\mu)}(x_j)x_j^{\theta_j} \in L(0,l_j)$.

Let $\lambda = \rho^n$, $\varepsilon_k = \exp(2\pi i k/n)$, $k=\overline{0,n-1}$. It is known that the ρ – plane can be partitioned into sectors S of angle $\frac{\pi}{n}$ $\left(\arg\rho \in \left(\frac{k_0\pi}{n}, \frac{(k_0+1)\pi}{n}\right),\ k_0 = \overline{-n,n-1}\right)$ in which the roots $R_1,R_2,\dots,R_n$ of the equation $R^n-1=0$ can be numbered in such a way that

$$Re(\rho R_1) < Re(\rho R_2) < \ldots < Re(\rho R_n), \quad \rho\in S. \tag{2}$$

Clearly, $R_k = \varepsilon_{\eta_k}$, where $\eta_1,\dots,\eta_n$ is a permutation of the numbers $0,1,\dots,n-1$, depending on the sector. Let us agree that

$$\rho^\mu = \exp(\mu(\ln|\rho| + i\arg\rho)),\ \arg\rho \in (-\pi,\pi], \quad R_k^\mu = \exp(2\pi i\mu\eta_k/n).$$

Let the numbers c_{kj0}, $k=\overline{1,n}$, be such that

$$\prod_{k=1}^n c_{kj0} = \Big(\det[\xi_{kj}^{\nu-1}]_{k,\nu=\overline{1,n}}\Big)^{-1}.$$

Then the functions

$$C_{kj}(x_j,\lambda) = x_j^{\xi_{kj}}\sum_{\mu=0}^\infty c_{kj\mu}(\rho x_j)^{n\mu}, \quad c_{kj\mu} = c_{kj0}\Big(\prod_{s=1}^\mu \delta_j(\xi_{kj}+sn)\Big)^{-1},$$

are solutions of the differential equation in the case when $q_{\mu j}(x_j)\equiv 0$, $\mu=\overline{0,n-2}$. Moreover, $\det[C_{kj}^{(\nu-1)}(x_j,\lambda)]_{k,\nu=\overline{1,n}}\equiv 1$. Denote $\rho^* = 2n\max\limits_{\mu,j}\|q_{\mu j}\|_{L(0,l_j)}$, $\mu=\overline{0,n-2}$, $j=\overline{1,p}$. In [23] we constructed special fundamental systems of solutions $\{S_{kj}(x_j,\lambda)\}_{k=\overline{1,n}}$ and $\{E_{kj}(x_j,\rho)\}_{k=\overline{1,n}}$ of equation (1) on the edge e_j, possessing the following properties.

(1) For each $x_j\in(0,l_j]$, the functions $S_{kj}^{(\nu)}(x_j,\lambda)$, $\nu=\overline{0,n-1}$, are entire in λ. For each fixed λ, and $x_j\to 0$,

$$S_{kj}(x_j,\lambda)\sim c_{kj0}x_j^{\xi_{kj}}, \quad (S_{kj}(x_j,\lambda)-C_{kj}(x_j,\lambda))x_j^{-\xi_{kj}} = o(x_j^{\xi_{nj}-\xi_{1j}}).$$

Moreover, $\det[S_{kj}^{(\nu-1)}(x_j,\lambda)]_{k,\nu=\overline{1,n}}\equiv 1$, and $|S_{kj}^{(\nu)}(x_j,\lambda)|\le C|x^{\xi_{kj}-\nu}|$, $|\rho|x_j\le 1$. Here and below, we shall denote by the same symbol C various positive constants in the estimates independent of λ and x_j.

(2) For each $x_j>0$ and for each sector S with property (2), the functions $E_{kj}^{(\nu)}(x_j,\rho)$, $\nu=\overline{0,n-1}$, are regular with respect to $\rho\in S$, $|\rho|>\rho^*$, and continuous for $\rho\in\overline{S}$, $|\rho|\ge\rho^*$. Moreover,

$$|E_{kj}^{(\nu)}(x_j,\rho)(\rho R_k)^{-\nu}\exp(-\rho R_k x_j)-1|\le C(|\rho|x_j), \quad \rho\in\overline{S}, \quad |\rho|x_j\ge 1.$$

(3) The relation

$$E_{kj}(x_j,\rho) = \sum_{\mu=1}^n b_{kj\mu}(\rho)S_{\mu j}(x_j,\lambda), \tag{3}$$

holds, where

$$b_{kj\mu}(\rho) = b_{\mu j}^0 R_k^{\xi_{\mu j}}\rho^{\xi_{\mu j}}[1], \quad b_{\mu j}^0\ne 0, \quad \rho\in\overline{S}, \quad \rho\to\infty,$$
$$\prod_{\mu=1}^n b_{\mu j}^0 = \det[R_k^{\nu-1}]_{k,\nu=\overline{1,n}}\Big(\det[R_k^{\xi_{\mu j}}]_{k,\mu=\overline{1,n}}\Big)^{-1}, \tag{4}$$

where $[1] = 1+O(\rho^{-1})$.

Note that the asymptotical formula (4) is the most important and nontrivial property of these solutions. This property allows one to study both direct and inverse problems for arbitrary order differential operators with regular singularities (see [24]-[26]).

Consider the linear forms

$$U_{j\nu}(y_j) = \sum_{\mu=0}^{\nu} \gamma_{j\nu\mu} y_j^{(\mu)}(l_j),\ j=\overline{1,p},\ \nu=\overline{0,n-1},$$

where $\gamma_{j\nu\mu}$ are complex numbers, $\gamma_{j\nu} := \gamma_{j\nu\nu} \neq 0$. The linear forms $U_{j\nu}$ will be used in matching conditions at the internal vertex v_0 for boundary value problems and for the correspondung special solutions of equation (1).

Fix $s=\overline{1,p},\ k=\overline{1,n-1}$. Let $\Psi_{sk} = \{\psi_{skj}\}_{j=\overline{1,p}}$ be solutions of equation (1) on the graph T under the boundary conditions

$$\psi_{sks}(x_s,\lambda) \sim c_{ks0} x_s^{\xi_{ks}}, \quad x_s \to 0, \tag{5}$$

$$\psi_{skj}(x_j,\lambda) = O(x_j^{\xi_{n-k+1,j}}), \quad x_j \to 0, \quad j=\overline{1,p},\ j\neq s, \tag{6}$$

and the matching conditions at the vertex v_0:

$$U_{1\nu}(\psi_{sk1}) = U_{j\nu}(\psi_{skj}),\ j=\overline{2,p},\ \nu=\overline{0,k-1}, \tag{7}$$

$$\sum_{j=1}^{p} U_{j\nu}(\psi_{skj}) = 0, \quad \nu=\overline{k,n-1}. \tag{8}$$

The function Ψ_{sk} is called the Weyl-type solution of order k with respect to the boundary vertex v_s. Define additionally $\psi_{sns}(x_s,\lambda) := S_{ns}(x_s,\lambda)$.

Using the fundamental system of solutions $\{S_{\mu j}(x_j,\lambda)\}$ on the edge e_j, one can write

$$\psi_{skj}(x_j,\lambda) = \sum_{\mu=1}^{n} M_{skj\mu}(\lambda) S_{\mu j}(x_j,\lambda), \quad j=\overline{1,p}, \quad k=\overline{1,n-1}, \tag{9}$$

where the coefficients $M_{skj\mu}(\lambda)$ do not depend on x_j.

It follows from (9) and the boundary condition (6) for the Weyl-type solutions that

$$\psi_{skj}(x_j,\lambda) = \sum_{\mu=n-k+1}^{n} M_{skj\mu}(\lambda) S_{\mu j}(x_j,\lambda), \quad j=\overline{1,p}\setminus s. \tag{10}$$

Similarly, using (5) one gets

$$\psi_{sks}(x_s,\lambda) = S_{ks}(x_s,\lambda) + \sum_{\mu=k+1}^{n} M_{sk\mu}(\lambda) S_{\mu s}(x_s,\lambda), \quad M_{sk\mu}(\lambda) := M_{sks\mu}(\lambda). \tag{11}$$

We introduce the matrices $M_s(\lambda),\ s=\overline{1,p}$, as follows:

$$M_s(\lambda) = [M_{sk\mu}(\lambda)]_{k,\mu=\overline{1,n}}, \quad M_{sk\mu}(\lambda) := \delta_{k\mu} \text{ for } k \geq \nu.$$

The matrix $M_s(\lambda)$ is called the Weyl-type matrix with respect to the boundary vertex v_s. The inverse problem is formulated as follows. Fix $N = \overline{1,p}$.

Inverse problem 1. Given $\{M_s(\lambda)\}$, $s = \overline{1,p} \setminus N$, construct q on T.

We note that the notion of the Weyl-type matrices M_s is a generalization of the notion of the Weyl function (m-function) for the classical Sturm-Liouville operator ([19, 27]) and is a generalization of the notion of Weyl matrices introduced in [14, 15, 21, 22, 24] for higher-order differential operators on an interval and on graphs. Thus, Inverse Problem 1 is a generalization of the well-known inverse problems for differential operators on an interval and on graphs.

We also note that in Inverse problem 1 we do not need to specify all matrices $M_s(\lambda)$, $s = \overline{1,p}$; one of them can be omitted. This last fact was first noticed in [8], where the inverse problem was solved for the Sturm-Liouville operators on an arbitrary tree.

In Section 3 properties of the Weyl-type solutions and the Weyl-type matrices are studied. Section 4 is devoted to the solution of auxiliary inverse problems of recovering the potential on a fixed edge. In section 5 we study Inverse Problem 1. For this inverse problem we provide a constructive procedure for the solution and prove its uniqueness.

3. Properties of spectral characteristics

Fix $s = \overline{1,p}$, $k = \overline{1,n-1}$. Substituting (10)-(11) into matching conditions (7)-(8), we obtain a linear algebraic system with respect to $M_{skj\mu}(\lambda)$. Solving this system by Cramer's rule one gets $M_{skj\mu}(\lambda) = \Delta_{skj\mu}(\lambda)/\Delta_{sk}(\lambda)$, where the functions $\Delta_{skj\mu}(\lambda)$ and $\Delta_{sk}(\lambda)$ are entire in λ. Thus, the functions $M_{skj\mu}(\lambda)$ are meromorphic in λ, and consequently, the Weyl-type solutions and the Weyl-type matrices are meromorphic in λ. In particular,

$$M_{sk\mu}(\lambda) = \frac{\Delta_{sk\mu}(\lambda)}{\Delta_{sk}(\lambda)}, \quad k \le \mu, \tag{12}$$

where $\Delta_{sk\mu}(\lambda) := \Delta_{sks\mu}(\lambda)$. We note that the function $\Delta_{sk}(\lambda)$ in (12) is the characteristic function for the boundary value problem L_{sk} for equation (1) under the conditions

$$y_s(x_s) = O(x_s^{\xi_{k+1,s}}),\ x_s \to 0, \qquad y_j(x_j) = O(x_j^{\xi_{n-k+1,j}}),\ x_j \to 0,\ j = \overline{1,p},\ j \ne s,$$

$$U_{1\nu}(y_1) = U_{j\nu}(y_j),\ j = \overline{2,p},\ \nu = \overline{0,k-1}, \qquad \sum_{j=1}^{p} U_{j\nu}(y_j) = 0,\ \nu = \overline{k,n-1}.$$

Zeros of $\Delta_{sk}(\lambda)$ coincide with the eigenvalues of L_{sk}. Denote

$$\Omega_{kj} = \det[R_l^{\xi_{\mu j}}]_{l,\mu=\overline{1,k}},\ \Omega_{0j} := 1, \quad \omega_{kj} := \frac{\Omega_{k-1,j}}{\Omega_{kj}},\ k = \overline{1,n}.$$

Lemma 1. *Fix $j=\overline{1,p}$, and fix a sector S with property* (2).

1) *Let $k=\overline{1,n-1}$, and let $y_j(x_j,\lambda)$ be a solution of equation* (1) *on the edge e_j under the condition*

$$y_j(x_j,\lambda)=O(x_j^{\xi_{k+1,j}}),\quad x_j\to 0. \tag{13}$$

Then for $x_j\in(0,l_j]$, $\nu=\overline{0,n-1}$, $\rho\in S$, $|\rho|\to\infty$,

$$y_j^{(\nu)}(x_j,\lambda)=\sum_{\mu=k+1}^{n}A_{\mu j}(\rho)(\rho R_\mu)^\nu\exp(\rho R_\mu x_j)[1], \tag{14}$$

where the coefficients $A_{\mu j}(\rho)$ do not depend on x_j. Here and below we assume that $\arg\rho=$ *const, when* $|\rho|\to\infty$.

2) *Let $k=\overline{1,n}$, and let $y_j(x_j,\lambda)$ be a solution of equation* (1) *on the edge e_j under the condition*

$$y_j(x_j,\lambda)\sim c_{kj0}x_j^{\xi_{kj}},\ x_j\to 0. \tag{15}$$

Then for $x_j\in(0,l_j]$, $\nu=\overline{0,n-1}$, $\rho\in S$, $|\rho|\to\infty$,

$$y_j^{(\nu)}(x_j,\lambda)=\frac{\omega_{kj}}{\rho^{\xi_{kj}}}(\rho R_k)^\nu\exp(\rho R_k x_j)[1]+\sum_{\mu=k+1}^{n}B_{\mu j}(\rho)(\rho R_\mu)^\nu\exp(\rho R_\mu x_j)[1], \tag{16}$$

where the coefficients $B_{\mu j}(\rho)$ do not depend on x_j.

Proof. It follows from (13) that

$$y_j(x_j,\lambda)=\sum_{\mu=k+1}^{n}a_{\mu j}(\lambda)S_{\mu j}(x_j,\lambda). \tag{17}$$

Using the fundamental system of solutions $\{E_{kj}(x_j,\rho)\}_{k=\overline{1,n}}$, one can write

$$y_j(x_j,\lambda)=\sum_{m=1}^{n}A_{mj}(\rho)E_{mj}(x_j,\rho). \tag{18}$$

By virtue of (3), we calculate

$$y_j(x_j,\lambda)=\sum_{m=1}^{n}A_{mj}(\rho)\sum_{\mu=1}^{n}b_{mj\mu}(\rho)S_{\mu j}(x_j,\lambda)=\sum_{\mu=1}^{n}S_{\mu j}(x_j,\lambda)\sum_{m=1}^{n}A_{mj}(\rho)b_{mj\mu}(\rho).$$

Comparing this relation with (17), we obtain

$$\sum_{m=1}^{n}A_{mj}(\rho)b_{mj\mu}(\rho)=0,\quad \mu=\overline{1,k}. \tag{19}$$

We consider (19) as a linear algebraic system with respect to $A_j(\rho),A_{2j}(\rho),\dots,A_{kj}(\rho)$. Solving this system by Cramer's rule and taking (4) into account we get

$$A_{mj}(\rho)=\sum_{\mu=k+1}^{n}(\alpha_{m\mu j}+O(\rho^{-1}))A_{\mu j}(\rho),\quad m=\overline{1,k}, \tag{20}$$

where $\alpha_{m\mu j}$ are constants. Substituting (20) into (18) and using (2) we arrive at (14). Relations (16) are proved analogously by using (15) instead of (13). ☐

Now we are going to study the asymptotic behavior of the Weyl-type solutions.

Lemma 2. *Fix $s=\overline{1,p}$, $k=\overline{1,n}$, and fix a sector S with property* (2). *For $x_s\in(0,l_s)$, $\nu=\overline{0,n-1}$, the following asymptotic formula holds*

$$\psi^{(\nu)}_{sks}(x_s,\lambda)=\frac{\omega_{ks}}{\rho^{\xi_{ks}}}(\rho R_k)^\nu\exp(\rho R_k x_s)[1],\quad \rho\in S,\ |\rho|\to\infty. \tag{21}$$

Proof. For $k=n$, (21) follows from Lemma 1. Fix $s=\overline{1,p}$, $k=\overline{1,n-1}$. Using Lemma 1 and boundary conditions for Ψ_{sk} we get the following asymptotic formulae for $\rho\in S$, $|\rho|\to\infty$:

$$\psi^{(\nu)}_{sks}(x_s,\lambda)=\frac{\omega_{ks}}{\rho^{\xi_{ks}}}(\rho R_k)^\nu\exp(\rho R_k x_s)[1]+\sum_{\mu=k+1}^{n}A^{sk}_{\mu s}(\rho)(\rho R_\mu)^\nu\exp(\rho R_\mu x_s)[1],\ x_s\in(0,l_s], \tag{22}$$

$$\psi^{(\nu)}_{skj}(x_j,\lambda)=\sum_{\mu=n-k+1}^{n}A^{sk}_{\mu j}(\rho)(\rho R_\mu)^\nu\exp(\rho R_\mu x_j)[1],\ j=\overline{1,p}\setminus s,\ x_j\in(0,l_j]. \tag{23}$$

Substituting (22)-(23) into matching conditions (7)-(8) for Ψ_{sk}, we obtain the linear algebraic system with respect to $A^{sk}_{\mu j}(\rho)$. Solving this system by Cramer's rule, we obtain in particular,

$$A^{sk}_{\mu j}(\rho)=O(\rho^{-\xi_{ks}}\exp(\rho(R_k-R_\mu)l_j)). \tag{24}$$

Substituting (24) into (22) we arrive at (21). ☐

It follows from the proof of Lemma 2 that one can also get the asymptotics for $\psi^{(\nu)}_{skj}(x_j,\lambda)$, $j\neq s$; but for our purposes only (21) is needed.

4. Auxiliary inverse problems

In this section we consider auxiliary inverse problems of recovering differential operator on each fixed edge. Fix $s=\overline{1,p}$, and consider the following inverse problem on the edge e_s.

IP(s). Given the Weyl-type matrix M_s, construct the potential q_s on the edge e_s.

In this inverse problem we construct the potential only on the edge e_s, but the Weyl-type matrix M_s brings a global information from the whole graph. In other words, this problem is not a local inverse problem related only to the edge e_s.

Let us formulate the uniqueness theorem for the solution of the inverse problem $IP(s)$. For this purpose together with q we consider a potential $\tilde q$. Everywhere below if a symbol α denotes an object related to q, then $\tilde\alpha$ will denote the analogous object related to $\tilde q$.

Theorem 1. *Fix $s = \overline{1,p}$. If $M_s = \tilde{M}_s$, then $q_s = \tilde{q}_s$. Thus, the specification of the Weyl-type matrix M_s uniquely determines the potential q_s on the edge e_s.*

We omit the proof since it is similar to that in [22, Ch.2]. Moreover, using the method of spectral mappings and Lemma 2, one can get a constructive procedure for the solution of the inverse problem $IP(s)$. It can be obtained by the same arguments as for n-th order differential operators on a finite interval (see [22, Ch.2] for details). Note that like in [22], the nonlinear inverse problem $IP(s)$ is reduced to the solution of a linear equation in the corresponding Banach space of sequences. The unique solvability of this linear equation is proved by the same arguments as in [22].

Fix $j = \overline{1,p}$. Now we define an auxiliary Weyl-type matrix with respect to the internal vertex v_0 and the edge e_j. Let $\varphi_{kj}(x_j,\lambda)$, $k = \overline{1,n}$, be solutions of equation (1) on the edge e_j under the conditions

$$\varphi_{kj}^{(\nu-1)}(l_j,\lambda) = \delta_{k\nu},\ \nu = \overline{1,k}, \qquad \varphi_{kj}(x_j,\lambda) = O(x_j^{\xi_{n-k+1,j}}),\ x_j \to 0.$$

We introduce the matrix $m_j(\lambda) = [m_{jk\nu}(\lambda)]_{k,\nu=\overline{1,n}}$, where $m_{jk\nu}(\lambda) := \varphi_{kj}^{(\nu-1)}(l_j,\lambda)$. Clearly, $m_{jk\nu}(\lambda) = \delta_{k\nu}$ for $k \ge \nu$, and $\det m_j(\lambda) \equiv 1$. The matrix $m_j(\lambda)$ is called the Weyl-type matrix with respect to the internal vertex v_0 and the edge e_j. Consider the following inverse problem on the edge e_j.

IP[j]. Given the Weyl-type matrix m_j, construct the potential q_j on the edge e_j.

This inverse problem is the classical one, since it is the inverse problem of recovering a higher-order differential equation on a finite interval from its Weyl-type matrix. This inverse problem has been solved in [22], where the uniqueness theorem is proved. Moreover, in [22] an algorithm for the solution of the inverse problem $IP[j]$ is given, and necessary and sufficient conditions for the solvability of this inverse problem are provided.

5. Solution of Inverse Problem 1

In this section we obtain a constructive procedure for the solution of Inverse problem 1 and prove its uniqueness. First we prove an auxiliary assertion.

Lemma 3. *Fix $j = \overline{1,p}$. Then for each fixed $s = \overline{1,p} \setminus j$,*

$$m_{j1\nu}(\lambda) = \frac{\psi_{s1j}^{(\nu-1)}(l_j,\lambda)}{\psi_{s1j}(l_j,\lambda)}, \quad \nu = \overline{2,n}, \tag{25}$$

$$m_{jk\nu}(\lambda) = \frac{\det[\psi_{s\mu j}(l_j,\lambda),\dots,\psi_{s\mu j}^{(k-2)}(l_j,\lambda),\psi_{s\mu j}^{(\nu-1)}(l_j,\lambda)]_{\mu=\overline{1,k}}}{\det[\psi_{s\mu j}^{(\xi-1)}(l_j,\lambda)]_{\xi,\mu=\overline{1,k}}},\ 2 \le k < \nu \le n. \tag{26}$$

Proof. Denote
$$w_{js}(x_j,\lambda) := \frac{\psi_{s1j}(x_j,\lambda)}{\psi_{s1j}(l_j,\lambda)}.$$
The function $w_{js}(x_j,\lambda)$ is a solution of equation (1) on the edge e_j, and $w_{js}(l_j,\lambda)=1$. Moreover, by virtue of the boundary conditions on Ψ_{s1}, one has $w_{js}(x_j,\lambda)=O(x_j^{\xi_{nj}})$, $x_j\to 0$. Hence, $w_{js}(x_j,\lambda)\equiv\varphi_{1j}(x_j,\lambda)$, i.e.
$$\varphi_{1j}(x_j,\lambda) = \frac{\psi_{s1j}(x_j,\lambda)}{\psi_{s1j}(l_j,\lambda)}. \tag{27}$$
Similarly, we calculate
$$\varphi_{kj}(x_j,\lambda) = \frac{\det[\psi_{s\mu j}(l_j,\lambda),\ldots,\psi^{(k-2)}_{s\mu j}(l_j,\lambda),\psi_{s\mu j}(x_j,\lambda)]_{\mu=\overline{1,k}}}{\det[\psi^{(\xi-1)}_{s\mu j}(l_j,\lambda)]_{\xi,\mu=\overline{1,k}}}, \quad k=\overline{2,n-1}. \tag{28}$$
Since $m_{jk\nu}(\lambda)=\varphi^{(\nu-1)}_{kj}(l_j,\lambda)$, it follows from (27) that (25) holds. Similarly, (26) follows from (28). $\square$

Now we are going to obtain a constructive procedure for the solution of Inverse problem 1. Our plan is the following.

Step 1. Let the Weyl-type matrices $\{M_s(\lambda)\}$, $s=\overline{1,p}\setminus N$, be given. Solving the inverse problem $IP(s)$ for each fixed $s=\overline{1,p}\setminus N$, we find the potentials q_s on the edges e_s, $s=\overline{1,p}\setminus N$.

Step 2. Using the knowledge of the potential on the edges e_s, $s=\overline{1,p}\setminus N$, we construct the Weyl-type matrix $m_N(\lambda)$.

Step 3. Solving the inverse problem $IP[N]$, we find the potential q_N on e_N.

Steps 1 and 3 have been already studied in Section 4. It remains to fulfil Step 2.

Suppose that Step 1 is already made, and we found the potentials q_s, $s=\overline{1,p}\setminus N$, on the edges e_s, $s=\overline{1,p}\setminus N$. Then we calculate the functions $S_{kj}(x_j,\lambda)$, $j=\overline{1,p}\setminus N$, $k=\overline{1,n}$.

Fix $s=\overline{1,p}\setminus N$. All calculations below will be made for this fixed s.

Our goal now is to construct the Weyl-type matrix $m_N(\lambda)$. For this purpose we will use Lemma 3. According to (25)-(26), in order to construct $m_N(\lambda)$ we have to calculate the functions
$$\psi^{(\nu)}_{skN}(l_N,\lambda), \quad k=\overline{1,n-1},\ \nu=\overline{0,n-1}. \tag{29}$$
We will find the functions (29) by the following steps.

1) Using (11) we construct the functions
$$\psi^{(\nu)}_{sks}(l_s,\lambda),\ k=\overline{1,n-1},\ \nu=\overline{0,n-1}, \tag{30}$$

by the formula

$$\psi^{(\nu)}_{sks}(l_s,\lambda)=S^{(\nu)}_{ks}(l_s,\lambda)+\sum_{\mu=k+1}^{n}M_{sk\mu}(\lambda)S^{(\nu)}_{\mu s}(l_s,\lambda). \tag{31}$$

2) Using the matching conditions (7) on Ψ_{sk}, we get, in particular,

$$U_{j\nu}(\psi_{skj})=U_{s\nu}(\psi_{sks}),\quad 0\le\nu<k\le n-1,\quad j=\overline{1,p}\setminus s. \tag{32}$$

Since the functions (30) were already calculated, it follows that the right-hand sides in (32) are known. For each fixed $k=\overline{1,n-1}$, we successively use (32) for $\nu=0,1,\dots,k-1$, and calculate recurrently the functions

$$\psi^{(\nu)}_{skj}(l_j,\lambda),\ k=\overline{1,n-1},\ \nu=\overline{0,k-1},\ j=\overline{1,p}\setminus s. \tag{33}$$

In particular we found the functions (29) for $\nu=\overline{0,k-1}$.

3) It follows from (10) that

$$\sum_{\mu=n-k+1}^{n}M_{skj\mu}(\lambda)S^{(\nu)}_{\mu j}(l_j,\lambda)=\psi^{(\nu)}_{skj}(l_j,\lambda),\ k=\overline{1,n-1},\ j=\overline{1,p}\setminus s,\ \nu=\overline{0,n-1}. \tag{34}$$

Fix $k=\overline{1,n-1}$, $j=\overline{1,p}$, $j\ne s$, $j\ne N$, and consider a part of relations (34), namely, for $\nu=\overline{0,k-1}$. For this choice of the parameters, the right-hand sides in (34) are known, since the functions (33) are known. Relations (34) for $\nu=\overline{0,k-1}$, form a linear algebraic system σ_{skj} with respect to the coefficients $M_{skj\mu}(\lambda)$, $\mu=\overline{n-k+1,n}$. Solving this system by Cramer's rule, we find this functions. Substituting them into (34), we calculate the functions

$$\psi^{(\nu)}_{skj}(l_j,\lambda),\quad k=\overline{1,n-1},\ j=\overline{1,p}\setminus N,\ \nu=\overline{0,n-1}. \tag{35}$$

Note that for $j=s$ these functions were found earlier (see (31)).

4) Let us now use the generalized Kirchhoff's conditions (8) for Ψ_{sk}. Since the functions (35) are known, one can construct by (8) the functions (29) for $k=\overline{1,n-1}$, $\nu=\overline{k,n-1}$. Thus, the functions (29) are known for $k=\overline{1,n-1}$, $\nu=\overline{0,n-1}$.

Since the functions (29) are known, we construct the Weyl-type matrix $m_N(\lambda)$ via (25)-(26) for $j=N$. Thus, we have obtained the solution of Inverse problem 1 and proved its uniqueness, i.e. the following assertion holds.

Theorem 2. *The specification of the Weyl-type matrices $M_s(\lambda)$, $s=\overline{1,p}\setminus N$, uniquely determines the potential q on T. The solution of Inverse problem 1 can be obtained by the following algorithm.*

Algorithm 1. *Given the Weyl-type matrices $M_s(\lambda)$, $s=\overline{1,p}\setminus N$.*

1) *Find the potentials* q_s, $s = \overline{1,p} \setminus N$, *by solving the inverse problem* $IP(s)$ *for each fixed* $s = \overline{1,p} \setminus N$.
2) *Calculate* $S_{kj}^{(\nu)}(l_j, \lambda)$, $j = \overline{1,p} \setminus N$, $k = \overline{1,n}$, $\nu = \overline{0,n-1}$.
3) *Fix* $s = \overline{1,p} \setminus N$. *All calculations below will be made for this fixed* s. *Construct the functions* (30) *via* (31).
4) *Calculate the functions* (33) *using* (32).
5) *Find the functions* $M_{skj\mu}(\lambda)$, *by solving the linear algebraic systems* σ_{skj}.
6) *Construct the functions* (35) *using* (34).
7) *Find the functions* (29) *using* (33), (35) *and* (8).
8) *Calculate the Weyl-type matrix* $m_N(\lambda)$ *via* (25)-(26) *for* $j = N$.
9) *Construct the potential* q_N *on the edge* e_N *by solving the inverse problem* $IP[N]$.

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