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ON SOME PARAMETRIC CLASSIFICATIONS OF QUASI-SYMMETRIC 2-DESIGNS

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Abstract. Quasi-symmetric 2-designs with block intersection numbers x and y, where $y = x+4$ and $x > 0$ are considered. If $D(v, b, r, k, \lambda; x, y)$ is a quasi-symmetric 2-design with above condition, then it is shown that the number of such designs is finite, whenever $3 \le x \le 68$. Moreover, the non-existence of triangle free quasi-symmetric 2-designs under these parameters is obtained.

1. Introduction

During the last several decades quasi-symmetric 2-designs and their classification play an important role to the study of design theory see for example [6, 10, 12, 13, 15, 16, 18]. Many results have been developed in the theory of binary codes, basically on self-complementary codes, self-dual codes using such designs. The main aim of this article is to classify these designs with certain parametrical restrictions.

Before going further, we recall some of the necessary definitions and results. Let X be a finite set with v elements, called points and β be a finite family of b distinct k-subset of X, called blocks. Then the pair $D = (X, \beta)$ is called a balanced incomplete block design (or 2-design) with parameters (v, b, r, k, λ), $(v > k \ge 3)$, if each element of X is contained in exactly r blocks, each block is of size k and any 2-subset of X is contained in exactly λ blocks. This r is also known as replication parameter.

For $0 \le x < k$, x is known to be a block intersection number, if there exist B, $B' \in \beta$ such that $|B \cap B'| = x$. A symmetric design is a 2-(v, k, λ) design such that $b = \lambda_0 = v$, $r = \lambda_1 = k$ and any 2 distinct blocks intersect in λ points. Now a slight generalization in the above definition will sufficiently broaden to include all symmetric designs. That generalization has been done in Fisher's inequality by $b \ge v$. Under this consideration we get 2-designs having more than one intersection number. A 2-design with exactly two intersection numbers is said to be

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a quasi-symmetric 2-design. We denote these intersection numbers by x and y and assume to be $0 \le x < y < k$. In this article we consider only the proper quasi-symmetric 2-designs, i.e. both the intersection numbers are positive and not equal. Examples of quasi-symmetric 2-designs which are not related to symmetric designs or affine designs are rare, therefore construction methods for such quasi-symmetric 2-designs are of great interest. However, some examples of quasi-symmetric 2-designs can be introduced through symmetric designs by adjoining some new blocks to the symmetric designs, so that it has exactly two intersection numbers λ and 0. Again if D is a multiple of a symmetric 2-(v, k, λ) design, then D is clearly a quasi-symmetric 2-design with $x = \lambda$ and $y = k$.

Let Γ be the usual block graph associated with a quasi-symmetric 2-design D, whose vertices are the blocks of D and two distinct vertices are adjacent iff the corresponding blocks intersect in y points. In [5], it has been shown that Γ is a strongly regular graph. A reasonable number of investigations on quasi-symmetric 2-designs are simplified by such block graphs Γ or sometimes by the complement $\bar{\Gamma}$ of Γ. Let $\bar{c}$ denote the number of triangles on any edge of $\bar{\Gamma}$. Then for any fixed values of $x, y \ge 2$ and $\bar{c} \ge 0$, there exist only finitely many such designs. For further reading, the reader may consult with [2] regarding the basic terminology of design theory and [17] related to the results on quasi-symmetric 2-designs and strongly regular graph.

Some results have been established on quasi-symmetric 2-design with $x = 0$ as one intersection number in [1]. In [8, 9], it has been shown that the number of such designs is finite provided k or $\lambda(\ge 2)$ is fixed. Again the quasi-symmetric 2-designs with $y = \lambda$ has been studied in [7]. In [5], several necessary conditions are obtained for the existence of a quasi-symmetric 2-design with parameter set $D(v, b, r, k, \lambda; x, y)$ by imposing the divisibility restrictions on $y - x$. It was also shown in [16] that there are finitely many such designs for $y \ge 2$ and fixed block size k. Most of the recent works on quasi-symmetric 2-designs have been concentrated on the difference of the intersection numbers x and y, where $y = x + 2$ in [14] and $y = x + 3$ in [10] i.e. the difference of the intersection numbers is 2 and 3.

In this article we study proper quasi-symmetric 2-designs with intersection pair (x, y), where lower intersection number x is not necessarily zero. However, our aim is to develop the classification of proper quasi-symmetric 2-designs with the difference of intersection numbers four. The assumptions under which our work has been done are $v \ge 2k$, $\lambda > 1$ and $1 \le x < y$. In particular, we show that the number of such designs is 18, whenever $3 \le x \le 68$. These are listed in Table 3. Finally in the last section, we prove that there does not exist any triangle-free quasi-symmetric 2-designs having non-zero intersection numbers with their difference four. In this paper we use the software Mathematica 8.0.1 and Maple 14 for carrying out the major mathematical calculations.

2. Preliminaries

We recall here some of the results which are useful for the development of this paper. In addition to these readers can see [5, 17]. *For the rest of the paper* by quasi-symmetric designs we mean quasi-symmetric 2-designs.

Lemma 1. *In a t-(v,k,λ) design, let λ_i denote the number of blocks containing any given i-tuple, $i=0,1,\dots,t$ with $\lambda_t=\lambda$, $\lambda_0=b$ and $\lambda_1=r$. Then*

$$\lambda_i = \frac{(v-i)}{(k-i)}\lambda_{i+1},\ \ i=0,1,\dots,t-1.$$

Lemma 2 ([3])**.** *Let B be a 2-(v,k,λ) design with intersection numbers $s_1,s_2,\dots,s_n$; where $s_1\equiv s_2\equiv\cdots\equiv s_n\equiv s\,(mod\ 2)$. Then either*

(1) $r\equiv\lambda\,(mod\ 4)$ *or*
(2) $s\equiv 0\,(mod\ 2),\ k\equiv 0\,(mod\ 4),\ v\equiv\pm1\,(mod\ 8)$ *or*
(3) $s\equiv 1\,(mod\ 2),\ k\equiv v\,(mod\ 4),\ v\equiv\pm1\,(mod\ 8)$.

Lemma 3 ([17])**.** *Let D be a quasi-symmetric design with standard parameter set $(v,b,r,k,\lambda;x,y)$. Then the following relations hold:*

(1) $vr=bk$ *and* $\lambda(v-1)=r(k-1)$.
(2) $k(r-1)(x+y-1)-xy(b-1)=k(k-1)(\lambda-1)$.
(3) $y-x$ *divides* $k-x$ *and* $r-\lambda$.
(4) $r(-r+kr+\lambda)=bk\lambda$.
(5) $\lambda\le\frac{k(k-1)}{2}$.

Lemma 4 ([16])**.** *Let D be a quasi-symmetric design with standard parameter set $(v,b,r,k,\lambda;x,y)$ and $\lambda>1$. Then,*

(1) $b>v$.
(2) $x\le\frac{k^2}{v}<\lambda$.
(3) *If* $v\ge 2k$ *then* $2x<k$.
(4) *For* $z=y-x$ *and if* $x\ge 1+z+z^3$ *then* $x<\lambda<x+1+z+z^3$.

Lemma 5 ([10])**.** *Let D be a quasi-symmetric design with standard parameter set $(v,b,r,k,\lambda;x,y)$ and a is the number of blocks intersecting a given block in y points. Then,*

$$a = -\frac{(\lambda-1)k^2+(r-rx+x-\lambda)k}{(x-y)y}, \tag{2.1}$$

$$b = -\frac{(\lambda-1)k^2+(r-rx+x-\lambda-ry+y)k-xy}{xy}. \tag{2.2}$$

Lemma 6 ([13])**.** *Let D be a quasi-symmetric design with standard parameter set $(v,b,r,k,\lambda;x,y)$. Then the following relation holds:*

$$a_1r^2+b_1r+c_1=0 \tag{2.3}$$

where

$$\begin{aligned}
a_1 &= (-1+k)xy,\\
b_1 &= (xy-k^2(-1+x+y))\lambda,\\
c_1 &= k(-(xy)+k(x+y-\lambda)+k^2(-1+\lambda))\lambda.
\end{aligned}$$

Lemma 7 ([4])**.** *Let D be a quasi-symmetric design with standard parameter set $(v,b,r,k,\lambda;x,y)$. Then the following inequalities hold:*

$$\begin{aligned}
0 \le\ & k(v-6)(v-3)(v-k)(2k-x-y)^2\\
&-2k(v-3)(v-k)(2k(v-k)-3v)(2k-x-y)\\
&+(6-v)(v-3)(v-1)(k-x)(k-y)(2k-x-y)\\
&+k(v-k)(5v+3k(v-k)(k(v-k)-2(v-1))-3)\\
&+(v-3)(k(v-k)(3v+2)-6(v-1)v)(k-x)(k-y).
\end{aligned}$$

$$\begin{aligned}
0 \le\ & k(v-k)(k(v-k)-1)+(v-2)(v-1)(k-x)(k-y)\\
&-k(v-2)(v-k)(2k-x-y).
\end{aligned}$$

3. Parametric characterization

In this section, we give some characterization of parameters for the proper quasi-symmetric designs with difference of intersection numbers four. We will use D to denote a proper quasi-symmetric designs with intersection numbers x and $y=x+4$, $x>0$.

Theorem 1. *Let D be a proper quasi-symmetric design with standard parameters and $y=x+4$, $x>0$. Then $\lambda \le x+26$, provided $x \ge 69$.*

Proof. From the quadratic equation (2.3) we get

$$\Delta = b_1^2-4a_1c_1=\lambda(\lambda F(k,x)+G(k,x)),$$

where $F(k,x)=4(-1+k)k^2x(4+x)-4(-1+k)k^3x(4+x)+(x(4+x)-k^2(3+2x))^2$ and $G(k,x)=4kx(k-1)(x+4)(k-x)(k-x-4)$.

Next suppose $x\ge 69$, we find out the range of λ for which $\Delta<0$. Let $F^i(k,x)=\frac{\partial^i F(k,x)}{\partial k^i}$. As a result $F^4(k,x)=216-96x<0$ for $x\ge 69$. We see that if we increase the value of k then the value of $F^3(k,x)$ decreases. For the larger values of x, we consider $x=69+p$, $p\ge 0$, we get

$$F^3(2x+1,x)=-24(27039+806p+6p^2)<0.$$

So, $F^3(k,x) < 0$ for $k \geq 2x+1$. Also,

$$\begin{aligned}
F^2(2x+1,x) &= -4(7794696 + 349844p + 5227p^2 + 26p^3)\\
F^1(2x+1,x) &= -933045840 - 56372784p - 1274588p^2 - 12784p^3 - 48p^4\\
F(2x+1,x) &= -19259788176 - 1480946112p - 45372944p^2 - 692672p^3\\
&\quad -5271p^4 - 16p^5
\end{aligned}$$

and as above argument, we conclude $F(k,x) < 0$ for $k \geq 2x+1$. We will show that $\lambda F(k,x) + G(k,x) < 0$ when $\lambda \geq x+26$. Here we calculate the case for minimum value of $\lambda = x+26$ i.e. $E(k,x) = (x+26)F(k,x) + G(k,x)$. Denoting partial derivatives of $E(k,x)$ with respect to k as before, we find $E^4(k,x) = 24(234-79x) < 0$ for all $x \geq 3$. Thus,

$$\begin{aligned}
E^3(2x+1,x) &= -24(488424 + 14741p + 111p^2).\\
E^2(2x+1,x) &= -4(121207044 + 5558948p + 84707p^2 + 429p^3).\\
E^1(2x+1,x) &= -4(2573504496 + 164364396p + 3899548p^2 + 40797p^3 + 159p^4).\\
E(2x+1,x) &= -44147192400 - 5483493360p - 224491768p^2 - 4193944p^3\\
&\quad -37137p^4 - 127p^5.
\end{aligned}$$

So, for $p \geq 0$, $E^i(2x+1,x) < 0$ for all $i = 1,2,3$ and $E(2x+1,x) < 0$ also. Therefore by previous argument $E(2x+1,x) \leq E(k,x) < 0$ for all $k \geq 2x+1$ and $x \geq 69$. But, again for $x \geq 69$ and $\lambda \geq x+26$, $\Delta < 0$ - which is not considerable. So, λ is restricted to $x+26$ when x is greater than or equal to 69. Hence we are done. ☐

Theorem 2. *If D be a quasi-symmetric design with $x \geq 61$ and $\lambda \geq x+8$, then*

$$k+1 \leq r \leq 8 + \frac{\lambda(k^2(2x+3) - x(x+4))}{2(k-1)x(x+4)}. \tag{3.1}$$

Proof. On substitution $y = x+4$ and $\lambda = x+p$, $p \geq 8$ in the quadratic equation (2.3), we carry out the following calculation. Let Δ denote the discriminant of the quadratic equation (2.3). To calculate the larger root, we concentrate on $\Delta - (16a_1)^2 < 0$ for certain range of x and λ.

Now we calculate

$$f_1(k,x,p) = \frac{\partial f(k,x,p)}{\partial k}, \quad f_2(k,x,p) = \frac{\partial f_1(k,x,p)}{\partial k},$$

$$f_3(k,x,p) = \frac{\partial f_2(k,x,p)}{\partial k}, \quad f_4(k,x,p) = \frac{\partial f_3(k,x,p)}{\partial k} = -24\lambda\{(\lambda - x - 8)(4x-9) + 7x - 72\}$$

which is negative for $x \geq 11$ and $\lambda \geq x+8$ and hence by earlier argument $f_3(k,x,p)$ decreases as k increases. Further computation on $f_i(k,x,p), i = 0,\ldots,3$, we observe that all these expressions are negative when $k \geq 2x+1$, $x \geq 61$ and $\lambda \geq x+8$. Substituting x by $61+q$ and λ by $69+p+q$ and $k = 2x+1$ in $f_i(k,x,p)$ for $i = 0,\ldots,3$, we get,

$$f_3(q,p) = -24(69+p+q)(6pq^2 + 3q^2 + 710pq + 185q + 20975p + 50).$$

$$\begin{aligned}
f_2(q,p) = &-4248244880 - 292088512q - 7475732q^2 - 84488q^3 - 356q^4\\
&-1412448880p - 93738272qp - 2321472q^2p - 25432q^3p - 104q^4p\\
&-21268640p^2 - 1084816qp^2 - 18412q^2p^2 - 104q^3p^2.\\
f_1(q,p) = &-700525319200 - 57035211440q - 1856075592q^2 - 30178172q^3\\
&-245152q^4 - 796q^5 - 34373402800p - 2950320080qp\\
&-100557908q^2p - 1702688q^3p - 14332q^4p - 48q^5p - 557288400p^2\\
&-38335600qp^2 - 986204q^2p^2 - 11248q^3p^2 - 48q^4p^2.\\
f_0(q,p) = f(q,p) = &-48377043441520 - 4695317242208q - 189826216296q^2\\
&-4091868976q^3 - 49600255q^4 - 320566q^5 - 863q^6\\
&-521755144320p - 57358651968qp - 2580191144q^2p\\
&-61024040q^3p - 802478q^4p - 5574q^5p - 16q^6p\\
&-9982505360p^2 - 877504704qp^2 - 30690960q^2p^2\\
&-534240q^3p^2 - 4631q^4p^2 - 16q^5p^2.
\end{aligned}$$

Thus the right hand side of the required relation of r can be established, as $\frac{\sqrt{\Delta}}{2a_1} < 8$, when $x \geq 61$ and $\lambda \geq x+8$. Hence $r \leq \frac{-b_1}{2a_1} + 8$. Now from Lemma 6, we have $a_1 = (k-1)x(x+4)$ and $b_1 = \lambda(x(x+4) - k^2(2x+3))$. Therefore

$$r < 8 + \frac{\{(2x+3)k^2 - x^2 - 4x\}\lambda}{2(k-1)x(x+4)}.$$

But we already know that $k < r$ for quasi-symmetric designs. These two conclude the equation (3.1). $\square$

Theorem 3. *Let D be a quasi-symmetric design with standard parameters and $y = x+4$, then either $x \leq 68$ or $\lambda < x+8$ or parameters of D will be assigned by any one of the six cases listed in Table 1.*

Proof. Assume that $x \geq 69$ and $\lambda \geq x+9$. Again from Theorem 1, we have $\lambda \leq x+26$. Rewriting $\Delta = \lambda L(k,x,\lambda)$, where $L(k,x,\lambda) = \lambda F(k,x) + G(k,x)$ and $F(k,x)$ and $G(k,x)$ are as in Theorem 1. Then $L(k,x,\lambda) = -4kx^2(4+x)^2 + 2k^2x(4+x)(8+2x^2-2x(-6+\lambda)-5\lambda) - 4k^3x(4+x)(5+2x-2\lambda) + x^2(4+x)^2\lambda + k^4(4x^2-4x(-4+\lambda)+9\lambda)$. Hence $L^4(k,x,\lambda) = 24(4x(x-\lambda)+(16x+9\lambda)) < 0$, for $\lambda \geq x+9$ and $x \geq 69$.

Now, for $x = 69+p$ and $\lambda = 78+p+q$, with $p \geq 0$ and $0 \leq q \leq 17$. We get the following derivations:

$$\begin{aligned}
L^3(6x-11,x,\lambda) &= -4986072 - 159720p - 1272p^2 - 2340648q - 70320pq - 528p^2q.\\
L^2(6x-11,x,\lambda) &= -704065200 - 34173664p - 549120p^2 - 2924p^3
\end{aligned}$$

Table 1:

x	v	b	r	k	λ	q	t	m
91	435	465	217	203	101	1	14	6
93	497	568	248	217	108	6	31	11
96	1616	1717	425	400	105	0	25	5
124	657	730	320	288	140	7	32	12
136	568	639	315	280	155	10	35	15
441	1891	1953	945	915	457	7	30	14

$$\begin{aligned}
&-425803872q-19214192pq-288940p^2q-1448p^3q.\\
L^1(6x-11,x,\lambda) &= -56450920176-3783789120p-93878932p^2\\
&-1024420p^3-4156p^4-51429293136q-3097519152pq\\
&-69943996p^2q-701792p^3q-2640p^4q.\\
L(6x-11,x,\lambda) &= -2081510713152-202441177584p-7436993544p^2\\
&-131624648p^3-1134842p^4-3839p^5-4639106021904q\\
&-349596892800pq-10535705648p^2q-158721200p^3q\\
&-1195319p^4q-3600p^5q.
\end{aligned}$$

Thus $L(k,x,\lambda)<0$ for all $k \geq 6x-11$. So, there are no such designs for $x \geq 69$, $\lambda \geq x+9$ and $k \geq 6x-11$.

Now we calculate for $2x+1 \leq k \leq 6x-12$. Let $R(k,x,q) = (x+9+q)(k^2(2x+3)-x(x+4)) - 2(k-1)x(x+4)(k+41+13q)$. Hence $R^2(k,x,q) = 2(27+3q+13x+2qx) > 0$, where $R^2(k,x,q) = \frac{\partial^2 R(k,x,q)}{\partial k^2}$. Then for $x = 69+p$, we get $R(2x+1,x,q) = -38138688-15353532q-1624512p-654524pq-23054p^2-9297p^2q-109p^3-44p^3q$ and $R(6x-12,x,q) = -(12647676+29734635q)-(430721+1263014q)p-(4482+17855q)p^2-(13+84q)p^3$. Hence $R(k,x,q)<0$ for $2x+1 \leq k \leq 6x-12$. Thus $\frac{(x+9+q)(k^2(2x+3)-x(x+4))}{2(k-1)x(x+4)} < k+41+13q$. So, from Theorem 2, we get $k+1 \leq r \leq k+41+13q$.

Consider $r = k+t$, then $1 \leq t \leq 41+13q$. From Lemma 4, k is a factor of $r(\lambda-r)$ i.e. k is a factor of $t(x+9+q-t)$. Let m be the resultant number when k divides $t(x+9+q-t)$. Then clearly $0<m<t$.

If we substitute the values of $k = \frac{t(x+9+q-t)}{m}$, $r = k+t$, $\lambda = x+9+q$ and $y = x+4$ in equation (2.3), we get a quadratic polynomial in x and examine for the each values of $q = 0,1,\ldots,17$; $t = 1,2,\ldots,41+13q$ and $m = 1,2,\ldots,t-1$. These set of values provide a large number of parameters of D. But we are interested only those values which satisfy the Calderbank criteria. Accordingly a list of feasible values for the parameters is given in Table 1. □

Theorem 4. *Let D be a quasi-symmetric design with standard parameter set having $y = x+4$. If $3 \le x \le 68$, then D will be one of among 18 set of values given in Table 3.*

Proof. We have the expression for Δ as $\Delta = \lambda^2 F(k,x)+\lambda G(k,x)$, where the expression of $F(k,x)$ and $G(k,x)$ are given in Theorem 1. It is clear that $G(k,x) > 0$, when $k > x+4$. Since the highest power of k in $F(k,x)$ is negative, so, there is some integral values of k depending on x, say $k(x)$ such that $F(k,x) < 0$ when $k \ge k(x)$. Again from Theorem 1, $\lambda F(k,x)+G(k,x) < 0$ for some integral values of λ. Hence there exist $k(x)$ and $\lambda(x)$ for which $\Delta < 0$ when $k \ge k(x)$ and $\lambda \ge \lambda(x)$. Let $x = 3+p$ and $k = 6x+37+q$, we get

$$\begin{aligned} F(q,p) = &-897984-27348000p-11651960p^2-1873016p^3-134039p^4-3600p^5 \\ &-522720q-2295744pq-703804p^2q-74336p^3q-2640p^4q-27192q^2-68560pq^2 \\ &-13790p^2q^2-724p^3q^2-492q^3-872pq^3-88p^2q^3-3q^4-4pq^4 \end{aligned}$$

and

$G(q,p) = 4(3+p)(7+p)(48+5p+q)(52+5p+q)(54+6p+q)(55+6p+q)$.

So, the integer k depending on x can take its value as $k(x) = 6x+37$.

Again for $\lambda = x+691+s$, we have

$$\begin{aligned} \Delta = &(694+p+s)(498816+18422024928p+7918207496p^2+1276525888p^3+91482058p^4 \\ &+2459039p^5+314966304q+1555924128pq+479692724p^2q+50764740p^3q \\ &+1804636p^4q+17496504q^2+46663896pq^2+9420392p^2q^2+495230p^3q^2+323892q^3 \\ &+595452pq^3+60228p^2q^3+1998q^4+2739pq^4+897984s+27348000ps+61651960p^2s \\ &+1873016p^3s+134039p^4s+3600p^5s+522720qs+2295744pqs+703804p^2qs \\ &+74336p^3qs+2640p^4qs+27192q^2s+68560pq^2s+13790p^2q^2s+724p^3q^2s+492q^3s \\ &+872pq^3s+88p^2q^3s+3q^4s+4pq^4s). \end{aligned}$$

Similarly the values of λ depending on x can be taken as $\lambda(x) = x+691$. Using these values as upper bounds for k and λ depending upon x, we consider some smaller values than obtained by the formula for $k(x)$ and $\lambda(x)$, which satisfy the required criteria. Those values are listed in Table 2.

Now depending upon the previous restrictions given in Lemmas 3(5) and 4(3) for k and λ along with $k \le k(x)$ and $\lambda \le -G(k,x)/F(k,x)$ provided $F(k,x) < 0$, we calculate for each case of x, the specific values for k and λ. Then depending on each of these values for x, k and λ, we search the integral values for r as a root of the quadratic equation (2.3) and hence remaining all the parameters related to a quasi-symmetric design are evaluated. Through these calculations we obtain 18 new set of values for the parameters of quasi-symmetric designs, which are tabulated in Table 3. □

Table 2:

x	$\lambda(x)$	$k(x)$	x	$\lambda(x)$	$k(x)$	x	$\lambda(x)$	$k(x)$	x	$\lambda(x)$	$k(x)$	x	$\lambda(x)$	$k(x)$
3	694	54	17	51	45	31	47	91	45	54	254	59	68	311
4	233	35	18	50	47	32	47	98	46	55	258	60	69	315
5	99	32	19	50	49	33	47	106	47	56	262	61	70	319
6	98	30	20	50	51	34	48	108	48	57	266	62	71	322
7	113	29	21	50	53	35	48	118	49	58	270	63	72	326
8	122	29	22	49	56	36	48	132	50	59	274	64	73	331
9	64	32	23	48	59	37	47	176	51	60	277	65	74	336
10	62	33	24	48	62	38	47	127	52	61	282	66	75	340
11	61	34	25	48	65	39	52	128	53	62	286	67	76	343
12	55	36	26	48	68	40	52	142	54	63	291	68	77	348
13	55	37	27	48	71	41	51	189	55	64	295			
14	55	39	28	48	74	42	52	192	56	65	299			
15	53	41	29	48	78	43	52	247	57	66	303			
16	51	43	30	47	84	44	53	251	58	67	307			

Table 3:

x	y	v	b	r	k	λ
3	7	63	651	155	15	35
4	8	55	495	144	16	40
4	8	56	231	66	16	18
6	10	91	715	220	28	66
6	10	126	525	125	30	29
6	10	100	330	99	30	29
6	10	120	952	238	30	58
6	10	106	742	210	30	58
8	12	63	651	248	24	92
8	12	64	336	126	24	46
11	15	89	267	99	33	36
11	15	106	265	90	36	30
12	16	78	273	119	34	51
12	16	70	161	69	30	29
15	19	91	273	120	40	52
16	20	76	190	90	36	42
18	22	100	225	99	44	43
22	26	118	177	78	52	34

4. Triangle free designs

In this section we confine ourselves on the triangle-free quasi-symmetric 2-designs. In particular, we concentrate only those quasi-symmetric 2-designs D, having no three mutually disjoint blocks. This is equivalent to say that for a triangle-free quasi-symmetric 2-design the corresponding block graph Γ whose complement does not contain any triangle. These type of quasi-symmetric 2-designs are of special interests. Some works have been done on triangle-free quasi-symmetric 2-designs for the intersection numbers 0 and y in [8]. Later, many works have been developed in [10], [12] and [13]. We present here some of the relevant results.

Lemma 8 ([12])**.** *Let D be a triangle free quasi-symmetric 2-design with the standard parameter set $(v,b,r,k,\lambda;x,y)$. Then*

$$f(\lambda)\equiv A\lambda^2+B\lambda+C=0;$$

where A, B and C are polynomials in k, x and y given by

$$\begin{aligned}
A &= k[k^3y-x^2(x^2-xy+y^2)+k(x^3+2x^2y-xy^2+y^3)+\\
&\quad k^2(x^3-y^2(1+y)+xy(-1+3y)-x^2(1+3y))];\\
B &= -2k^4y^2+x^3y(-2x+y)-k^2(2x^4-5xy^3-x^3(1+y)+y^3(1+y)\\
&\quad +3x^2y(1+3y))+kx(-4x^2y^2+3xy^2(1+y)-y^3(1+y)+x^3(1+4y))\\
&\quad +k^3y(-2x^3+x^2(3+6y)+x(-1+3y-6y^2)+(y+y^3));\, and\\
C &= (-1+k)(k-y)y(-2x^2+ky+xy)^2.
\end{aligned}$$

Lemma 9 ([12])**.** *Let D be a triangle free quasi-symmetric 2-design with the standard parameter set $(v,b,r,k,\lambda;x,y)$. Then the following relation holds:*

$$b(y-x)^2 = (k(r-1)+x(1-b))(y-x)-(r-\lambda-k+x)(x-k)+2(y-x)^2.$$

Lemma 10 ([13])**.** *Let D be a triangle free quasi-symmetric 2-design with the standard parameter set $(v,b,r,k,\lambda;x,y)$ and $v\geq 2k$. Let $z=y-x$. Then $x\leq z+z^2$.*

Theorem 5. *There does not exist any triangle-free quasi-symmetric 2-designs having non-zero intersection numbers with the difference of two intersection numbers four.*

Proof. Let D be a quasi-symmetric 2-design with standard parameter set $(v,b,r,k,\lambda;x,y=x+4)$. We may assume $v\geq 2k$. Then by Lemmas 3 and 10, we have $k>2x$ and $x\leq 20$. Let $\Delta_0=B^2-4AC=(x^2+3kx+12k)^2(-16k^5(4+x)+k^4(1344+208x+49x^2)-4k^3(320+40x+60x^2+13x^3)+2k^2(128-512x-80x^2+56x^3+11x^4)-4kx(-128-64x-8x^2+4x^3+x^4)+x^2(x^2-16)^2)$. Above expression contains one perfect square term. So, we verify for remaining terms under following parametrical conditions. Let the verified term is denoted by M, where $M=-16k^5(4+x)+k^4(1344+208x+49x^2)-4k^3(320+40x+60x^2+13x^3)+2k^2(128-512x-80x^2+56x^3+11x^4)-4kx(-128-64x-8x^2+4x^3+x^4)+x^2(x^2-16)^2$. For $k=x+19+p$, $M=7993984-6205248p-1551360p^2-130176p^3-4736p^4-64p^5-20173440x-8033392px-1047584p^2x-$

$61056p^3x - 1632p^4x - 16p^5x - 1795103x^2 - 626020px^2 - 62458p^2x^2 - 2404p^3x^2 - 31p^4x^2 - 24240x^3 - 12720px^3 - 912p^2x^3 - 16p^3x^3 + 1024x^4$ - which is clearly negative, for $1 \le x \le 19$ and $p \ge 0$. Therefore, under these parametrical considerations we are unable to find any triangle free quasi-symmetric 2-designs.

Again, we have $k > y = x + 4$. Then, for $1 \le x \le 19$ and $max\{2x+1, x+5\} \le k \le x+18$ we examine M. In all the cases, M is found to be a non-perfect square, except for a few, such as $(x, k) = (1, 6)$, then $\sqrt{M} = 45\sqrt{521}$, for $(x, k) = (7, 23)$, then $\sqrt{M} = 64i\sqrt{6819}$ etc. But the exceptional cases where D has its value perfect square are $(x, k, M) = (1, 5, 720), (1, 9, 2112), (1, 15, 3780), (2, 10, 2624), (2, 18, 2496), (3, 7, 1456), (3, 11, 3200), (4, 12, 3840), (4, 19, 2964), (5, 13, 4544), (6, 14, 5312), (7, 15, 6144), (9, 24, 2841)$. But for these values of x, k and M, we are not getting any positive integral solutions of λ from the quadratic equation $A\lambda^2 + B\lambda + C = 0$ and hence no triangle free quasi-symmetric 2-designs can exist under these parametrical restrictions. $\square$

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