TAMKANG JOURNAL OF MATHEMATICS
Volume 46, Number 4, 389-399, December 2015
doi:10.5556/j.tkjm.46.2015.1768
This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

ON THE TRICYCLIC GRAPHS WITH THREE DISJOINT 6-CYCLES AND MAXIMUM MATCHING ENERGY

YUN-XIA ZHOU AND HONG-HAI LI

Abstract. The matching energy of a graph was introduced recently by Gutman and Wagner and defined as the sum of the absolute values of zeros of its matching polynomial. In this paper, we characterize graphs that attain the maximum matching energy among all connected tricyclic graphs of order n with three vertex-disjoint C_6's.

1. Introduction

A *matching* in a graph is a set of pairwise nonadjacent edges, and by $m_k(G)$ we denote the number of k-matchings of a graph G. It is both consistent and convenient to define $m_0(G) = 1$. In 2012, Gutman and Wagner [5] introduced the matching energy of a graph G, denoted by $ME(G)$, as

$$ME(G) = \frac{2}{\pi}\int_0^{\infty} \frac{1}{x^2} \ln\left[\sum_{k\geq 0} m_k(G)x^{2k}\right] dx, \tag{1}$$

which extends the formula for energy for forests to general graphs (for more on graph energy see the monograph [12] and [6, 7, 4] and the references therein). The integral on the right hand side of Eq.(1) is increasing in all of the coefficients $m_k(G)$. This means that if two graphs G and G' satisfy $m_k(G) \leq m_k(G')$ for all $k \geq 1$, then $ME(G) \leq ME(G')$. If, in addition, $m_k(G) < m_k(G')$ for at least one k, then $ME(G) < ME(G')$. It then motivates the introduction of a *quasi-order* $\succeq$, defined by

$$G \succeq H \ (or\, H \preceq G) \Longleftrightarrow m_k(G) \geq m_k(H), \qquad \text{for all nonnegative integers } k.$$

If $G \succeq H$ and there exists some k such that $m_k(G) > m_k(H)$, then we write $G \succ H$ (or $H \prec G$). We have $G \succeq H \Longrightarrow ME(G) \geq ME(H)$ and $G \succ H \Longrightarrow ME(G) > ME(H)$. From the definition it

Received October 31, 2014, accepted April 24, 2015.
2010 *Mathematics Subject Classification.* 05C50.
Key words and phrases. Tricyclic graph, matching energy, k-matching, matching polynomial.
Corresponding author: Hong-Hai Li.
Supported by National Natural Science Foundation of China (11561032, 11201198), Natural Science Foundation of Jiangxi Province (20142BAB211013), the Sponsored Program for Cultivating Youths of Outstanding Ability in Jiangxi Normal University.

is clear that if H is a subgraph of G such that the edge set of H is a proper subset of the edge set of G, then $G \succ H$.

Recall that the matching polynomial of a graph G of order n is defined as

$$\alpha(G,x) = \sum_{k\geq 0} (-1)^k m_k(G) x^{n-2k}, \tag{2}$$

where the convention that $m_k(G) = 0$ for $k < 0$ or $k > n/2$ is adopted. For any graph G, all the zeros of $\alpha(G,x)$ are real-valued.

Theorem 1.1 ([5])**.** *Let G be a simple graph, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the zeros of its matching polynomial. Then*

$$ME(G) = \sum_{i=1}^{n} |\mu_i|.$$

Given a graph G and an edge uv of G, we denote by $G-uv$ (resp. $G-v$) the graph obtained from G by deleting the edge uv (resp. the vertex v and the edges incident to it).

Lemma 1.2 ([3])**.** *If u, v are adjacent vertices of G, then for all nonnegative integers k, we have*

$$m_k(G) = m_k(G-uv) + m_{k-1}(G-u-v).$$

Denote by $\mathscr{U}_{g,n}$ the set of unicyclic graphs with n vertices and a cycle of length g. The sun graph, denoted by $C_g(P_{r_1+1},\ldots,P_{r_g+1})$, is the graph obtained from the cycle $C_g = v_1v_2\cdots v_gv_1$ by identifying one pendant vertex of the path P_{r_i+1} with vertex v_i for $i = 1,\ldots,g$. Note that $C_g(P_{n-g+1},P_1,\ldots,P_1)$ is also called *lollipop graph* and is denoted by $E_{g,n}$. For convenience, we adopt the convention that $E_{g,n} = C_n$ when $g = n$.

The proof for Lemma 3.8 in [9] actually establishes the following slightly stronger result.

Lemma 1.3. *Let n, g be any positive integers, $n > g \geq 3$. For any $G \in \mathscr{U}_{g,n} \setminus \{E_{g,n}\}$, we have $E_{g,n} \succ G$.*

We provide a simple, different proof for the following result which slightly strengthens Lemma 5 of [3].

Lemma 1.4 ([3])**.** *Let u, v be adjacent vertices of a graph G. If G_1 (resp., G_2) is the graph obtained from G by inserting t vertices into the edge uv (resp., by joining the vertex u to an end vertex of a path P_t) then $G_1 \succeq G_2$. If in addition $d_G(u) \geq 2$, then $G_1 \succ G_2$.*

Proof. Let v_p (resp., v_q) denote the vertex on the path in G_1 joining u and v, adjacent to u (resp., v). It is easy to see that the graphs $G_1 - v_qv$ and $G_2 - uv$ are isomorphic. Moreover, $G_2 - u - v$ is isomorphic with a graph, say H, obtained from $G_1 - v_q - v$ by deleting all edges in

G_1 incident with u, except for the edge uv_p. If $d_G(u) \geq 2$, the edge set of H is proper subset of that of $G_1 - v_q - v$. By Lemma 1.2, for any positive integer k, we have $m_k(G_2) = m_k(G_2 - uv) + m_{k-1}(G_2 - u - v) = m_k(G_1 - v_q v) + m_{k-1}(H) \leq m_k(G_1 - v_q v) + m_{k-1}(G_1 - v_q - v) = m_k(G_1)$, where the inequality strictly holds for at least one k if $d_G(u) \geq 2$, because then the edge set of H is a proper subset of that of $G_1 - v_q - v$. Hence our result follows. □

We adopt the convention that P_0 stands for the null graph. So $P_0 \cup G = G$ for any graph G.

Lemma 1.5 ([12]). *Let n be a given positive integer, and let l, l' be nonnegative integers less than or equal to $\lfloor n/2 \rfloor$. We have*

(i) *If l, l' are even and $l < l'$ then $P_l \cup P_{n-l} \succ P_{l'} \cup P_{n-l'}$.*
(ii) *If l, l' are odd and $l < l'$ then $P_l \cup P_{n-l} \prec P_{l'} \cup P_{n-l'}$.*
(iii) *If l is even and l' is odd, then $P_l \cup P_{n-l} \succ P_{l'} \cup P_{n-l'}$.*

In[5], Gutman and Wagner first introduced the notion of matching energy and characterized the extremal (maximal or minimal with respect to matching energy) graphs among some special graph classes. Li and Yan [11] characterized the connected graph with the given connectivity (resp. chromatic number) which has maximum matching energy. Ji, Li and Shi [8] characterized the graphs with the extremal matching energy among all bicyclic graphs. Recently, Chen et.al [1, 2] further investigated unicyclic graphs, bicyclic graphs and tricyclic graphs for extremal matching energy. For more on matching energy see[10, 14] and the references therein.

In [13], Li, Shi and Wei characterized graphs that attain the maximal energy among all connected tricyclic graphs on n vertices with three disjoint cycles. In this paper, we treat a similar problem for the matching energy, i.e, we characterize graphs that attain the maximum matching energy among all connected tricyclic graphs of order n with three vertex-disjoint cycles of length 6.

2. Main results

We borrow part of the definitions and notations from [13]. We say H is the central structure of G if G can be obtained from H by planting some trees on it. Let $G_{6,n}$ denote the set of all connected tricyclic graphs on n vertices with three disjoint C_6's. Given positive integers l_1, l_2, l_3 such that $l_1, l_2 \geq 2$ and $1 \leq l_3 \leq 4$, let $\Phi_6^I(l_1, l_2; l_3)$ denote the graph in $G_{6,n}$, as shown in Figure 1, where the first C_6 and the second C_6 are joined by a path $P_1 = v \cdots u$ on l_1 vertices, the second C_6 and the third C_6 are joined by a path $P_2 = u' \cdots w$ on l_2 vertices, the smaller part $u \cdots u'$ of the second C_6 has l_3 vertices. Note that when $u = u'$, we have $l_3 = 1$. Similarly, given positive integers $l_1, l_2, l_3 \geq 2$ and $n = l_1 + l_2 + l_3 + 13$, let $\Phi_6^{II}(l_1, l_2, l_3)$ denote the graph as shown in Figure 1: it has a center vertex v such that the three cycles C_6 are joined to v by paths $P_1 = u_1 \cdots v$, $P_2 = u_2 \cdots v$, $P_3 = u_3 \cdots v$ with l_1, l_2 and l_3 vertices, respectively. Note that $\Phi_6^I(l_1, l_2; l_3) = \Phi_6^I(l_2, l_1; l_3)$ and $\Phi_6^{\Pi}(l_1, l_2, l_3) = \Phi_6^{\Pi}(l_{\sigma(1)}, l_{\sigma(2)}, l_{\sigma(3)})$ for any permutation $\sigma \in S_3$. Denote by $\Phi_{6,n}^I$ (resp., $\Phi_{6,n}^{\Pi}$) the set of all graphs of the form $\Phi_6^I(l_1, l_2; l_3)$ (resp., $\Phi_6^{\Pi}(l_1, l_2, l_3)$) for some l_1, l_2, l_3. Denote by $G_{6,n}^I$(resp., $G_{6,n}^{\Pi}$) the set of all graphs in $G_{6,n}$ that

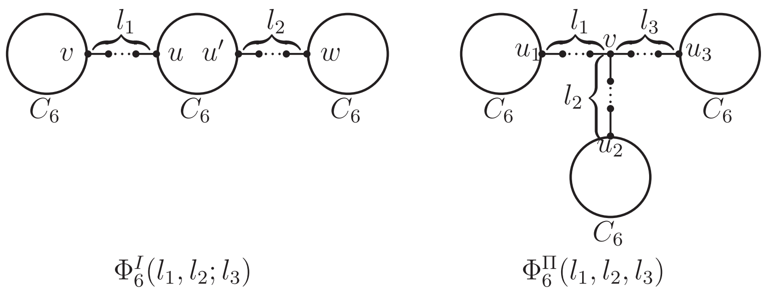

Figure 1: $\Phi_6^I(l_1, l_2; l_3)$ and $\Phi_6^{\Pi}(l_1, l_2, l_3)$.

have $\Phi_6^I(l_1, l_2; l_3)$(resp., $\Phi_6^{\Pi}(l_1, l_2, l_3)$) for some appropriate l_1, l_2, l_3 as their central structures. Now note that $G_{6,n} = G_{6,n}^I \cup G_{6,n}^{\Pi}$.

Lemma 2.1. *Let uv be a bridge of a connected graph G such that G_1, H are the connected components of $G - uv$ with $u \in V(G_1)$ and $v \in V(H)$. Let G_2 be a graph, vertex-disjoint from H, that satisfies $G_2 \succeq G_1$. Let G' be the graph obtained from G_2, H by adding an edge joining v to some vertex u' of G_2. Suppose that $G_2 - u' \succeq G_1 - u$. Then $G' \succeq G$. If, in addition, one of the quasi-inequalities $G_2 \succeq G_1$ and $G_2 - u' \succeq G_1 - u$ is strict, then $G' \succ G$.*

Proof. By Lemma 1.2, for every positive integer k, we have $m_k(G) = m_k(G - uv) + m_{k-1}(G - u - v) = m_k(G_1 \cup H) + m_{k-1}((G_1 - u) \cup (H - v))$ and $m_k(G') = m_k(G_2 \cup H) + m_(k-1)((G_2 - u') \cup (H - v))$. Since $G_2 \succeq G_1$ and $G_2 - u' \succeq G_1 - u$, we have $m_k(G_2 \cup H) \geq m_k(G_1 \cup H)$ and $m_{k-1}((G_2 - u') \cup (H - v)) \geq m_{k-1}((G_1 - u) \cup (H - v))$; hence $m_k(G') \geq m_k(G)$. This establishes the quasi-inequality $G' \succeq G$. It is clear that the latter quasi-inequality is strict if one of the quasi-inequalities $G_2 \succeq G_1$ and $G_2 - u' \succeq G_1 - u$ is strict. □

Lemma 2.2. *For any $G \in G_{6,n}^I \setminus \Phi_{6,n}^I$, $G \prec \Phi_6^I(l_1, l_2; 2)$ for some positive integers $l_1, l_2 \geq 2$ that satisfy $l_1 + l_2 = n - 14$.*

Proof. By Lemma 1.4 we may assume that G has no nontrivial rooted tree, where the root is an internal vertex of one of the two paths joining the C_6's. So we may assume that G is of the form as shown in Figure 2, which can be obtained by connecting sun graphs $C_6(P_{r_1+1}, \ldots, P_{r_5+1}, P_1)$ and $\mathscr{S}$ by a path P_{l_1}, $\mathscr{S}$ and $C_6(P_1, P_{t_1+1}, P_{t_2+1}, \ldots, P_{t_5+1})$ by a path P_{l_2}, where the middle sun

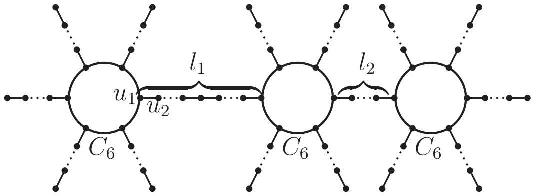

Figure 2: G when $l_3 = 3$.

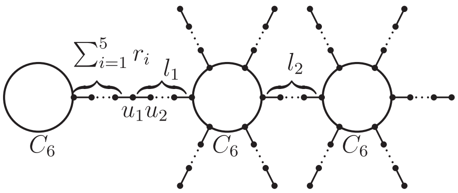

Figure 3: G'.

graph $\mathscr{S}$ has either five pendent paths of length s_1, s_2, s_3, s_4, s_5 or four pendent paths of length s_1, s_2, s_3, s_4 according to $l_3 = 1$ or not.

If $\sum_{i=1}^{5} r_i \geq 1$, then by applying Lemma 2.1 with G_1 equal to $C_6(P_{r_1+1}, \ldots, P_{r_5+1}, P_1)$, H equal to the graph obtained from G by deleting $V(G_1)$ and all incident edges, G' equal to the graph as shown in Figure 3, G_2 equal to $E_{6,r_1+r_2+\cdots+r_5+6}$ and $u = u' = u_1$, $v = u_2$, and noting that $C_6(P_{r_1+1}, \ldots, P_{r_5+1}, P_1) \prec E_{6,r_1+r_2+\cdots+r_5+6}$ by Lemma 1.3 and $C_6(P_{r_1+1}, \ldots, P_{r_5+1}, P_1) - u_1 \prec P_{r_1+r_2+\cdots+r_5+5} \prec E_{6,r_1+r_2+\cdots+r_5+5}$, we obtain $G \prec G'$.

Similarly, if $\sum_{i=1}^{5} t_i \geq 1$, then by applying Lemma 2.1, we can also show that $G \prec G'$ for some $G' \in \Phi^I_{6,n}$. So we can assume that G is obtained from some $\Phi^I_6(l'_1, l'_2; l_3)$ by attaching a pendant path at each vertex of degree two in the middle cycle; say, the lengths of the pendant paths are s_1, s_2, s_3, s_4 or s_1, s_2, s_3, s_4, s_5 in case $l_3 = 1$. Figure 4 shows one possibility for G, namely, when the middle cycle is $C_6(P_1, P_{s_1+1}, P_{s_2+1}, P_1, P_{s_3+1}, P_{s_4+1})$, i.e., $l_3 = 4$.

Let $G'' = \Phi^I_6(l'_1, l'_2 + s_1 + s_2 + s_3 + s_4; 2)$ or $\Phi^I_6(l'_1, l'_2 + s_1 + s_2 + s_3 + s_4 + s_5; 2)$ when $l_3 = 1$. It suffices to show that if $l_3 \neq 2$ or $l_3 = 2$ and $s_1 + s_2 + s_3 + s_4 \geq 1$, then $G \prec G''$.

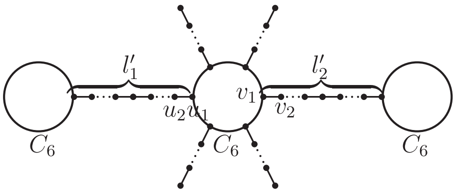

Figure 4: G ($l_3 = 4$).

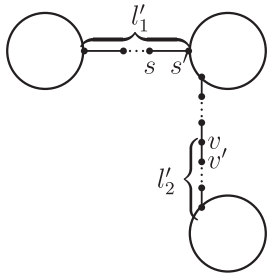

Figure 5: G''.

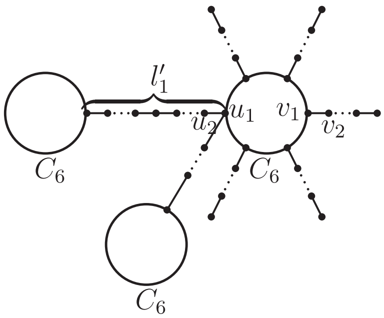

Figure 6: G ($l_3 = 1$).

Now take G_1 to the graph obtained by connecting a C_6 and the middle sun graph of G by a path $P_{l'_2}$ and take G_2 to the graph obtained by connecting two $C'_6 s$ by a path $P_{l'_2+s_1+\cdots+s_4}$(or $P_{l'_2+s_1+\cdots+s_5}$ in case $l_3 = 1$). By Lemma 2.1 one readily shows that $G_1 \preceq G_2$, and with strict inequality if $\sum s_i \geq 1$. Refer to Figure 4 or Figure 6 for graph G and to Figure 5 for graph G''. Now take H to be E_{6,l'_1+4} and apply Lemma 2.1 with G, G'', u_1, s', u_2 (and s) playing the role of G, G', u, u', v respectively. when $l_3 \neq 1$, $G_1 - u_1$ is a unicyclic graph and $G_2 - s'$ is a lollipop graph of the same order. So $G_1 - u_1 \preceq G_2 - s'$ and with strict inequality, if $G_1 - u_1$ is not a lollipop, i.e., provided that $l_3 \neq 2$ or $l_3 = 2$ and $s_1 + \cdots + s_4 \geq 1$. As we have noted, $G_1 \preceq G_2$, and with strict inequality when $\sum s_i \geq 1$. So by Lemma 2.1 we obtain $G \prec G''$. When $l_3 = 1$, $G_2 - s'$ equals the lollipop graph $E_{6,l'_2+s_1+\cdots+s_5+9}$ and $G_1 - u_1$ equals the union of E_{6,l'_2+4} and a tree, say T, of order $s_1 + \cdots + s_5 + 5$. By adding a suitable edge joining a vertex of E_{6,l'_2+4} to a vertex of T one obtains a unicyclic graph F with the same order as $G_2 - s'$. Hence $G_2 - s' \succ G_1 - u_1$. Then by applying Lemma 2.1 we also obtain $G \prec G''$ for this case. □

Lemma 2.3. *For any positive integers $a, b \geq 6$, if* $\min\{a, b\} > 6$ *then* $E_{6,a} \cup E_{6,b} \prec C_6 \cup E_{6,a+b-6}$.

Proof. Without loss of generality, assume that $\min\{a, b\} > 6$. By Lemma 1.2 we have

$$m_k(E_{6,a} \cup E_{6,b}) = m_k(E_{6,b} \cup C_6 \cup P_{a-6}) + m_{k-1}(E_{6,b} \cup P_{a-7} \cup P_5)$$

$$m_k(C_6 \cup E_{6,a+b-6}) = m_k(E_{6,b} \cup C_6 \cup P_{a-6}) + m_{k-1}(E_{6,b-1} \cup P_{a-7} \cup C_6).$$

Now we only need to show that $E_{6,b} \cup P_5 \prec E_{6,b-1} \cup C_6$. We proceed by induction on b. When $b = 7$, by Lemma 1.2, we have

$$\begin{aligned} m_k(E_{6,7} \cup P_5) &= m_k(C_6 \cup P_1 \cup P_5) + m_{k-1}(P_5 \cup P_5) \\ &= m_k(C_6 \cup P_1 \cup P_5) + m_{k-1}(P_5 \cup P_1 \cup P_4) + m_{k-2}(P_5 \cup P_3) \end{aligned}$$

and

$$\begin{aligned} m_k(C_6 \cup C_6) &= m_k(C_6 \cup P_6) + m_{k-1}(C_6 \cup P_4) \\ &= m_k(C_6 \cup P_6) + m_{k-1}(P_6 \cup P_4) + m_{k-2}(P_4 \cup P_4). \end{aligned}$$

By Lemma 1.5 we have $P_1 \cup P_5 \prec P_6$, $P_5 \cup P_3 \prec P_4 \cup P_4$. So $E_{6,7} \cup P_5 \prec C_6 \cup C_6$. For the case when $b = 8$, by calculation, we have Table 1. So we have $E_{6,8} \cup P_5 \prec E_{6,7} \cup C_6$.

Table 1: The k-matching numbers of $E_{6,8} \cup P_5$ and $E_{6,7} \cup C_6$ for $k = 1, \ldots, 6$.

$k =$	1	2	3	4	5	6
$m_k(E_{6,8} \cup P_5)$	12	54	114	115	50	6
$m_k(E_{6,7} \cup C_6)$	13	64	148	161	71	10

For $b \geq 9$, we have

$$m_k(E_{6,b} \cup P_5) = m_k(E_{6,b-1} \cup P_5) + m_{k-1}(E_{6,b-2} \cup P_5)$$
$$m_k(E_{6,b-1} \cup C_6) = m_k(E_{6,b-2} \cup C_6) + m_{k-1}(E_{6,b-3} \cup C_6).$$

By induction, we have $E_{6,b-1} \cup P_5 \prec E_{6,b-2} \cup C_6$ and $E_{6,b-2} \cup P_5 \prec E_{6,b-3} \cup C_6$; hence $E_{6,b} \cup P_5 \prec E_{6,b-1} \cup C_6$. □

Lemma 2.4. *For any positive integers $l_1, l_2 \geq 2$ with $l_1 + l_2 = n - 14$, if $\min\{l_1, l_2\} > 2$ then $\Phi_6^I(l_1, l_2; 2) \prec \Phi_6^I(n-16, 2; 2)$.*

Proof. For brevity, we denote $\Phi_6^I(l_1, l_2; 2)$ and $\Phi_6^I(n-16, 2; 2)$ respectively by G and H. Referring to Figure 7, by Lemma 1.2 we have

$$m_k(G) = m_k(G - u_1 v_1) + m_{k-1}(E_{6,l_1+4} \cup E_{6,l_2+4} \cup P_4),$$
$$m_k(H) = m_k(H - u_0 v_0) + m_{k-1}(E_{6,n-12} \cup C_6 \cup P_4).$$

Because $G - u_1 v_1 = H - u_0 v_0$ and $E_{6,l_1+4} \cup E_{6,l_2+4} \prec E_{6,n-12} \cup C_6$ by Lemma 2.3, we obtain $G \prec H$. □

Lemma 2.5. *For any graph $G \in G_{6,n}^{\Pi} \setminus \Phi_6^{\Pi}(n-17, 2, 2)$ with $n \geq 19$, $G \prec \Phi_6^{\Pi}(n-17, 2, 2)$.*

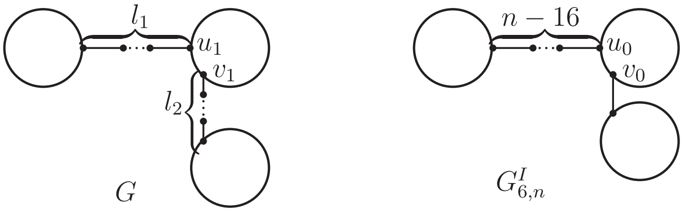

Figure 7: The graphs G and H.

Proof. First, using Lemma 2.1, we can show that for any graph $G \in G^{\Pi}_{6,n} \setminus \Phi^{\Pi}_{6,n}$, $G \prec \Phi^{\Pi}_6(l_1, l_2, l_3)$ for some positive integers l_1, l_2, l_3 with $\min\{l_1, l_2, l_3\} \geq 2$. Thus hereafter we assume that $G = \Phi^{\Pi}_6(l_1, l_2, l_3)$. If $\min\{l_1, l_2, l_3\} > 2$, then we show that $\Phi^{\Pi}_6(l_1, l_2, l_3) \prec \Phi^{\Pi}_6(l'_1, l'_2, 2)$ for some positive integers $l'_1, l'_2 \geq 2$ with $l'_1 + l'_2 = n-15$ as follows: Let G and $H = \Phi^{\Pi}_6(l_1, l'_2, 2)$ with $l_1 + l'_2 = n-15$ be as given in Figure 8. Without loss of generality, assume that $l_3 = \min\{l_1, l_2, l_3\} > 2$. By

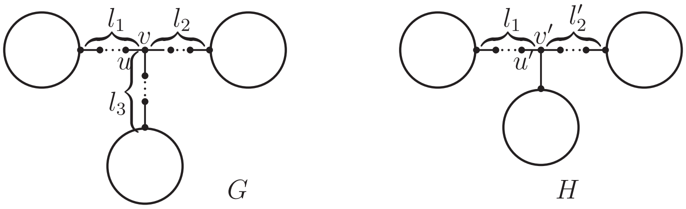

Figure 8: The graphs G and H.

Lemma 1.2 we have

$$m_k(G) = m_k(G - uv) + m_{k-1}(E_{6,l_1+3} \cup E_{6,l_2+4} \cup E_{6,l_3+4}).$$
$$m_k(H) = m_k(H - u'v') + m_{k-1}(E_{6,l_1+3} \cup E_{6,l'_2+4} \cup C_6).$$

Note that $G - uv = H - u'v'$ and by Lemma 2.3, $E_{6,l_2+4} \cup E_{6,l_3+4} \prec E_{6,l'_2+4} \cup C_6$. Hence $\Phi^{\Pi}_6(l_1, l_2, l_3) \prec \Phi^{\Pi}_6(l_1, l'_2, 2)$ and we may assume that $G = \Phi^{\Pi}_6(l_1, l_2, 2)$ for some positive integers $l_1, l_2 \geq 2$ with $l_1 + l_2 = n-15$.

If $\min\{l_1, l_2\} > 2$, then we prove that $\Phi^{\Pi}_6(l_1, l_2, 2) \prec \Phi^{\Pi}_6(n-17, 2, 2)$. Let G and $H = \Phi^{\Pi}_6(n-17, 2, 2)$ be as shown in Figure 9.

By Lemma 1.2 we have

$$m_k(G) = m_k(G - uv) + m_{k-1}(E_{6,l_1+4} \cup E_{6,l_2+4} \cup P_5),$$
$$m_k(H) = m_k(H - u'v') + m_{k-1}(E_{6,l_1+l_2+2} \cup C_6 \cup P_5).$$

Since we have $G - uv = H - u'v'$ and $E_{6,l_1+4} \cup E_{6,l_2+4} \prec E_{6,l_1+l_2+2} \cup C_6$ by Lemma 2.3, we are done. □

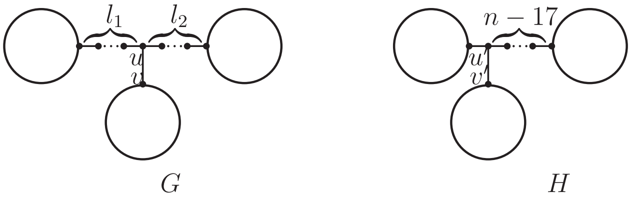

Figure 9: The graphs G and H.

Lemma 2.6. *For $n \geq 19$, we have $ME(\Phi_6^I(n-16,2;2)) < ME(\Phi_6^{II}(n-17,2,2))$ and, moreover, $\Phi_6^I(n-16,2;2) \prec \Phi_6^{II}(n-17,2,2)$ except when $n = 20, 22$.*

Proof. For convenience, let $G = \Phi_6^I(n-16,2;2)$ and $H = \Phi_6^{II}(n-17,2,2)$, as shown in Figure 10.

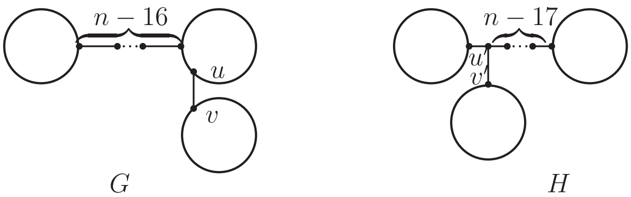

Figure 10: The graphs G and H.

First, assume that $n \geq 20$. By Lemma 1.2 we have

$$\begin{aligned} m_k(G) =& m_k(G-uv) + m_{k-1}(G-u-v) \\ =& m_k(G-uv) + m_{k-1}(E_{6,n-7} \cup P_5) \\ =& m_k(G-uv) + m_{k-1}(E_{6,n-13} \cup P_6 \cup P_5) + m_{k-2}(E_{6,n-14} \cup P_5 \cup P_5), \\ m_k(H) =& m_k(H-u'v') + m_{k-1}(H-u'-v') \\ =& m_k(H-u'v') + m_{k-1}(E_{6,n-13} \cup C_6 \cup P_5) \\ =& m_k(H-u'v') + m_{k-1}(E_{6,n-13} \cup P_6 \cup P_5) + m_{k-2}(E_{6,n-13} \cup P_4 \cup P_5). \end{aligned}$$

Since $G-uv = H-u'v'$, we only need to consider $E_{6,n-14} \cup P_5 \prec E_{6,n-13} \cup P_4$. By Lemma 1.2 we have

$$\begin{aligned} m_k(E_{6,n-14} \cup P_5) =& m_k(P_{n-14} \cup P_5) + m_{k-1}(P_{n-20} \cup P_4 \cup P_5) \\ m_k(E_{6,n-13} \cup P_4) =& m_k(P_{n-13} \cup P_4) + m_{k-1}(P_{n-19} \cup P_4 \cup P_4). \end{aligned}$$

Note that the assumption $n \geq 20$ guarantees that $E_{6,n-14}$ is defined. By Lemma 1.5 (iii) we have $P_{n-14} \cup P_5 \prec P_{n-13} \cup P_4$, provided that $5 \leq n-14$ and $4 \leq n-13$, i.e., $n \geq 19$. Since we are assuming $n \geq 20$, the said quasi-inequality holds. Similarly, we have $P_{n-20} \cup P_5 \prec P_{n-19} \cup P_4$, provided that $n \geq 25$. If $n = 24$, $P_{n-20} \cup P_5$ and $P_{n-19} \cup P_4$ are both equal to $P_4 \cup P_5$. If $n = 23$, we have $P_{n-20} \cup P_5 = P_3 \cup P_5 \prec P_{n-19} \cup P_4 = P_4 \cup P_4$ by Lemma 1.5 (iii). If $n = 22$, then $P_{n-20} \cup P_5 \succ P_3 \cup P_4 = P_{n-19} \cup P_4$. Similarly, we have $P_{n-20} \cup P_5 \prec P_{n-19} \cup P_4$ for $n = 21$ and $P_{n-20} \cup P_5 \prec P_{n-19} \cup P_4$ for $n = 20$. So for $n \geq 20$, we have $P_{n-20} \cup P_5 \preceq P_{n-19} \cup P_4$ and hence $E_{6,n-14} \cup P_5 \prec E_{6,n-13} \cup P_4$ except for $n = 20, 22$. Thus for $n \geq 20$, we always have $G \prec H$ except possibly when $n = 20, 22$.

Now we treat the case $n = 19$. We have

$$\begin{aligned} m_k(G) =& m_k(G - uv) + m_{k-1}(G - u - v) \\ =& m_k(G - uv) + m_{k-1}(E_{6,12} \cup P_5), \\ m_k(H) =& m_k(H - u'v') + m_{k-1}(H - u' - v') \\ =& m_k(H - u'v') + m_{k-1}(C_6 \cup C_6 \cup P_5). \end{aligned}$$

Note that $G - uv = H - u'v'$, so we only need to establish $E_{6,12} \prec C_6 \cup C_6$. Now for all positive integers k, since

$$\begin{aligned} m_k(E_{6,12}) =& m_k(P_{12}) + m_{k-1}(P_4 \cup P_6) \\ =& m_k(P_6 \cup P_6) + m_{k-1}(P_5 \cup P_5) + m_{k-1}(P_4 \cup P_6) \\ =& m_k(P_6 \cup P_6) + m_{k-1}(P_4 \cup P_5) + m_{k-2}(P_3 \cup P_5) + m_{k-1}(P_4 \cup P_6) \\ \leq& m_k(P_6 \cup P_6) + m_{k-1}(P_4 \cup P_6) + m_{k-2}(P_4 \cup P_4) + m_{k-1}(P_4 \cup P_6) \\ =& m_k(C_6 \cup C_6), \end{aligned}$$

where at least one inequality holds strictly, we are done. So we also obtain $G \prec H$ in this case.

For $n = 20$, and $n = 22$, by calculation we obtain Tables 2 and 3. So, for these cases, we have neither $G \preceq H$ nor $G \succeq H$. Making use of Theorem 1.1, we also obtain $ME(G) = 30.0168$ and $ME(H) = 30.0334$ when $n = 20$, and $ME(G) = 31.877$ and $ME(H) = 31.9742$ when $n = 22$. In either case, we have $ME(G) < ME(H)$.

Table 2: The k-matching numbers of G and H for $k = 1, \dots, 10$.

$k =$	1	2	3	4	5	6	7	8	9	10
$m_k(G)$	22	203	1014	4239	5683	6311	4008	1321	191	8
$m_k(H)$	22	203	1024	4238	5674	6281	3964	1296	188	8

The proof is completed. □

In conclusion, we obtain the following result:

Theorem 2.7. *For any graph $G \in G_{6,n} \setminus \{\Phi_6^{II}(n-17,2,2)\}$ with $n \geq 19$, we have $ME(G) < ME(\Phi_6^{II}(n-17,2,2))$ and, moreover, $G \prec \Phi_6^{II}(n-17,2,2)$ except when $n = 20, 22$ and $G = \Phi_6^{I}(n-16,2;2)$.*

Table 3: The k-matching numbers of G and H for $k = 1, \ldots, 11$.

$k =$	1	2	3	4	5	6	7	8	9	10	11
$m_k(G)$	24	246	1410	4760	9888	14011	13600	7217	1956	237	8
$m_k(H)$	24	246	1410	4760	9888	13594	13600	7207	1964	240	8

We would like to add that when $n = 18$, $G^{\Pi}_{6,n}$ is empty, $G^{I}_{6,n} = \Phi^{I}_{6,n}$ and $\Phi^{I}_{6,n}(2,2;2) \succ G$ for every $G \in G_{6,n} \setminus \Phi^{I}_{6,n}(2,2;2)$. When $n < 18$, $G_{6,n}$ is empty.

References

[1] L. Chen and Y. Shi, *The maximal matching energy of tricyclic graphs*, MATCH Commun. Math. Comput. Chem., **73** (2015), 105–119.

[2] L. Chen, J. Liu and Y. Shi, *Matching energy of unicyclic and bicyclic graphs with a given diameter*, Complexity, to appear.

[3] I. Gutman, *Graphs with greatest number of matchings*, Publ. Inst. Math. (Beograd), **27** (1980), 67–76.

[4] I. Gutman, X. Li, Y. Shi and J. Zhang, *Hypoenergetic trees*, MATCH Commun. Math. Comput. Chem., **60** (2008), 415–426.

[5] I. Gutman nad S. Wagner, *The matching energy of a graph*, Discr. Appl. Math., **160** (2012), 2177–2187.

[6] B. Huo, X. Li and Y. Shi, *Complete solution to a conjecture on the maximal energy of unicyclic graphs*, *European J. Combin.*, **32** (2011), 662–673.

[7] B. Huo, X. Li and Y. Shi, *Complete solution to a problem on the maximal energy of unicyclic bipartite graphs*, Lin. Algebra Appl., **434**(2011), 1370–1377.

[8] S. J. Ji, X. L. Li and Y. T. Shi, *Extremal matching energy of bicyclic graphs*, MATCH Commun. Math. Comput. Chem., **70** (2013), 697–706.

[9] H. H. Li, B. S. Tam and L. Su, *On the signless Laplacian coeffcients of unicyclic graphs*, Lin. Algebra Appl., **439** (2013), 2008–2009.

[10] H. H. Li, Y. X. Zhou, L. Su, *Graphs with extremal matching energies and prescribed parameters*, MATCH Commun. Math. Comput. Chem., **72** (2014), 239–248.

[11] S. L. Li and W. G. Yan, *The matching energy of graphs with given parameters*, Discr. Appl. Math., **162** (2014), 415–420.

[12] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.

[13] X. L. Li, Y. T. Shi and M. Q. Wei, *On a conjecture about tricyclic graphs with maximal energy*, MATCH Commun. Math. Comput. Chem., **72** (2014), 183–214.

[14] K. Xu, K. C. Das and Z. Zheng, *The minimal matching energy of* (n, m)*-graphs with a given matching number*, MATCH Commun. Math. Comput. Chem., **73** (2015), 93–104.

College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, China.

E-mail: yunxia2008@163.com

College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, China.

E-mail: lhh@jxnu.edu.cn