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ON $Hom(-,-)$ AS DIFFERENCE ALGEBRAS

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Abstract. In this paper, we investigate some properties of $Hom(-,-)$ as difference algebras. Also we define the orthogonal subsets of difference algebras and study their properties.

1. Introduction

In [1], Hausdorff introduced the ordered group, which is a general algebraic system combing a partial ordered set and a group. In [2], Imai and Iséki introduced a new notion, called a BCK-*algebra*. The inverse of the group operation and the set difference operation have many common properties. In [4], Meng combined these properties with a partial ordered set, and constructed the notion of the difference algebra. We know that the class of BCI-algebras is a proper subclass of difference algebras; the class of ordered groups is also a proper subclass of difference algebras. It is worth to study difference algebras. Recently, Roh et al. ([6]) introduced the concept of the quotient difference via ideal and investigated some interesting properties. In [7], they also introduced the concept of nil subsets by using nilpotent elements, and investigated some related properties. In this paper, we investigate some properties of $Hom(-,-)$ as difference algebras. Also we define the orthogonal subsets of difference algebras and study their properties.

2. Preliminaries

Definition 2.1.([4]) An algebra $(X;*,\leq,0)$ with a binary operation " $*$ ", a nullary operation "0" and a partial binary relation " $\leq$ " on X is called a *difference algebra* if it satisfies, for all $x,y,z\in X$:
(D1) $(X;\leq)$ is a poset,
(D2) $x\leq y$ implies $x*z\leq y*z$,
(D3) $(x*y)*z\leq(x*z)*y$,
(D4) $0\leq x*x$,
(D5) $x\leq y$ if and only if $x*y\leq 0$.

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Roh et al. ([6]) gave examples which the conditions (D1), (D2), (D3), (D4) and (D5) are independent each other.

Example 2.2.([4]) Let $\mathbb{R}$ denote the set of all real numbers. If we specify ordinary substraction as the operation $*$, natural order as $\leq$, zero as the nullary operation 0, then $(\mathbb{R}; *, \leq, 0)$ is a difference algebra.

Example 2.3.([4]) Suppose that $(X; *, 0)$ is a BCI-algebra and $\leq$ is the BCI-order. Then $(X; *, \leq, 0)$ is a difference algebra.

Remark. In above examples, we know that the class of BCI-algebras is a proper subclass of the difference algebras; the class of ordered groups is also a proper subclass of difference algebras.

Example 2.4. ([6]) Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. Then $(X; *, \leq, 0)$ is a difference algebra.

Example 2.5.([6]) Let $(X; *, \leq, 0)$ be a difference algebra with the following table:

*	0	1	2	3	4
0	0	0	0	3	3
1	1	0	0	3	3
2	2	2	0	3	3
3	3	3	3	0	0
4	4	3	3	1	0

Define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. Then $(X; *, \leq, 0)$ is a difference algebra.

Now we give some elementary and fundamental properties of difference algebras.

Theorem 2.6.([4]) *In any difference algebra X, the following hold: for any $x, y, z \in X$,*

(1) $(x * y) * z = (x * z) * y$,
(2) $x * x = 0$,
(3) $x * y \leq z$ *implies* $x * z \leq y$,
(4) $(x * (x * y)) * y = 0$,
(5) $x \leq y$ *implies* $z * y \leq z * x$,
(6) $x * (x * (x * y)) = x * y$,
(7) $x * 0 = x$,
(8) $0 * (x * y) = (0 * x) * (0 * y)$,
(9) $(x * y) * (x * z) \leq z * y$.

In what follows, the letter X denotes a difference algebra unless otherwise specified.

Definition 2.7.([6]) A non-empty subset S of X is called a *subalgebra* of X if $x*y \in S$ whenever $x, y \in S$.

Definition 2.8.([6]) A non-empty subset A of X is called an *ideal* of X if
(I1) $0 \in A$,
(I2) $x * y \in A$ and $y \in A$ imply $x \in A$,
(I3) $x \leq y$ and $y \in A$ imply $x \in A$.

Example 2.9.([6]) Let X be the set of all real numbers and $F(X)$ be the set of all real-valued functions on X. Suppose that $\leq$ is the pointwise order:

$$f \leq g \text{ if and if for all } x \in X, f(x) \leq g(x).$$

The operation $*$ is defined by the formula

$$(f * g)(x) := f(x) - g(x) \text{ for all } x \in X.$$

The nullary operation 0 is the constant function 0. Then $(F(X); *, \leq, 0)$ is a difference algebra. Let $P(X) := \{f \in F(X) | f(x) \geq 0\}$ for all $x \in R$. Then $P(X)$ is an ideal of $F(X)$, but it is not a subalgebra of $F(X)$. Thus we know that an ideal of a difference algebra may not be a subalgebra.

Definition 2.10.([7]) An ideal A of X is said to be *closed* if
(I4) $0 * x \in A$ for all $x \in A$.

We know that every closed ideal in a difference algebra is a subalgebra but converse is not true (See [6]).

3. $Hom(-,-)$ as Difference Algebras

Let $X := (X; *_X, \leq, 0_X)$ and $Y := (Y; *_Y, \leq', 0_Y)$ be difference algebras. A mapping f from a poset $(X; \leq)$ into a poset $(Y; \leq')$ is called a *Harris map* ([5]) if for any incomparable elements $x, y \in X$, either $f(x) = f(y)$ or $f(x)$ and $f(y)$ are incomparable. We denote the fact that x and y are incomparable by $x||y$. A mapping $f : X \to Y$ is called a (*difference*) *homomorphism* if

(Hi) $f(x *_X y) = f(x) *_Y f(y)$ for any $x, y \in X$,

(Hii) f is a Harris map.

Note that $f(0_X) = 0_Y$, since $x * x = 0$. Let $Ker(f) := \{x \in X | f(x) = 0_Y\}$ be the kernel of f. If a mapping $f : X \to Y$ satisfies (Hi), then it is order preserving, i.e., $x \leq y$ implies $0_Y = f(x * y) = f(x) * f(y)$, i.e, $f(x) \leq f(y)$. Define the trivial homomorphism 0 as $0(x) = 0$ for all $x \in X$.

Example 3.1. In Example 2.5, if we define a map $\varphi : X \to X$ by $\varphi(0) = \varphi(1) = \varphi(2) = 0$, $\varphi(3) = \varphi(4) = 3$, then it is easy to verify that φ is a homomorphism.

Let $hom(X, Y)$ denotes the set of all homomorphisms between difference algebras X and Y. For any $f, g \in hom(X, Y)$, define a mapping $f * g : X \to Y$ by $(f * g)(x) := f(x) * g(x)$ for any $x \in X$. A subset A of $hom(X, Y)$ is said to have a *kernel intersection property* if $Ker(f * g) = Ker(f) \cap Ker(g)$ for any $f, g \in A$. Denote by $Hom(X, Y)$ the set of all homomorphisms having the kernel intersection property.

Example 3.2. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X; *, \leq, 0)$ is a difference algebra. By routine calculations, there are 16 homomorphisms from X to itself. Among them we can find that $Hom(X, X) = \{f_1, f_2, f_3, f_4\}$, where $f_1(0) = f_1(1) = f_1(2) = f_1(3) = 0$; $f_2(0) = f_2(1) = 0, f_2(2) = f_2(3) = 1$; $f_3(0) = f_3(1) = 0, f_3(2) = f_3(3) = 2$; $f_4(0) = f_4(1) = 0$, $f_4(2) = f_4(3) = 3$.

Definition 3.3. A difference algebra X is said to be *medial* if for any $x, y, z, w \in X$,

$$(x * y) * (z * w) = (x * z) * (y * w).$$

Example 3.4. Every lattice ordered group (or po-group) G is a difference algebra. Since for any $x, y, z, w \in G$, $(x - y) - (z - w) = (x - z) - (y - w)$, $(G; -, \leq, 0)$ is a medial difference algebra.

Lemma 3.5. *Let X be a difference algebra and let Y be a medial difference algebra. Then $f * g \in Hom(X, Y)$ for any $f, g \in Hom(X, Y)$.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}(f*g)(x*y) &= f(x*y)*g(x*y)\\ &= (f(x)*f(y))*(g(x)*g(y))\\ &= (f(x)*g(x))*(f(y)*g(y)) \quad [\because Y : \text{medial}]\\ &= (f*g)(x)*(f*g)(y).\end{aligned}$$

This means that $f*g$ satisfies (Hi).

Assume that $f*g$ is not a Harris map, where f, g are Harris maps. Then there are incomparable elements $x, y \in X$ such that $(f*g)(x)$ and $(f*g)(y)$ are not equal and not incomparable. This means that either $(f*g)(x) < (f*g)(y)$ or $(f*g)(y) < (f*g)(x)$. Without loss of generality, we let $(f*g)(x) < (f*g)(y)$. Since Y is medial, $0 = (f*g)(x)*(f*g)(y) = (f(x)*g(x))*(f(y)*g(y)) = (f(x)*f(y))*(g(x)*g(y)) = f(x*y)*g(x*y) = (f*g)(x*y)$ and hence $x*y \in Ker(f*g) = Ker(f) \cap Ker(g)$. Hence $0 = f(x*y) = f(x)*f(y)$, $0 = g(x*y) = g(x)*g(y)$, i.e., $f(x) \leq f(y), g(x) \leq g(y)$. Since f, g are Harris maps and x and y are incomparable, we obtain $f(x) = f(y)$ and $g(x) = g(y)$. Thus, $(f*g)(x) = f(x)*g(x) = f(y)*g(y) = (f*g)(y)$, a contradiction. This proves that $f*g$ satisfies (Hii). Hence $f*g \in Hom(X, Y)$.

Theorem 3.6. *Let X be a difference algebra and let Y be a medial difference algebra. Then $(Hom(X, Y); *, 0)$ is also a medial difference algebra.*

Proof. Straightforward.

A difference algebra X is called a Γ-*difference algebra* if whenever $x*y = y*x$, $x = y$ for every $x, y \in X$.

Theorem 3.7. *If X is a difference algebra and Y is a medial Γ-difference algebra, then $Hom(X, Y)$ is a medial Γ-difference algebra.*

Proof. Assume that $f*g = g*f$ for $f, g \in Hom(X, Y)$. Then $f(x)*g(x) = (f*g)(x) = (g*f)(x) = g(x)*f(x)$ for any $x \in X$. Since Y is a Γ-difference algebra, we have $f(x) = g(x)$ for any $x \in X$. Hence $f = g$. Thus $Hom(X, Y)$ is a medial Γ-difference algebra.

For any elements x, y in a difference algebra X, let us write $x*y^n$ for

$$(\cdots((x*y)*y)*\cdots)*y$$

where y occurs n times. We say that an element x in a difference algebra X is a nilpotent element if $0*x^n = 0$ for some positive integer n. If every element x of X is nilpotent, then X is called a *nil difference algebra* ([7]).

Example 3.8. The difference algebra in Example 3.2 is nil.

Theorem 3.9. *Let X be a difference algebra and let Y be a medial difference algebra. If Y is nil, then $Hom(X, Y)$ is nil.*

Proof. Let $f \in Hom(X,Y)$ and let $x \in X$. Since Y is nil, there exists $n \in \mathbb{Z}^+$ such that $0 * f(x)^n = 0$. Hence

$$\begin{aligned} 0(x) &= 0 * f(x)^n \\ &= (\cdots((0(x) * f(x)) * f(x)) * \cdots) * f(x) \qquad [f(x) \text{ occurs } \ n \ \text{ times}] \\ &= (\cdots((0 * f) * f) * \cdots) * f)(x) \qquad [f \text{ occurs } \ n \ \text{ times}] \\ &= (0 * f^n)(x). \end{aligned}$$

Thus $0 * f^n = 0$. The proof is complete.

Let X be a difference algebra and let Y be a medial difference algebra. Let M and Θ be subsets of X and $Hom(X,Y)$, respectively. We define orthogonal subsets $M^\perp$ and $\Theta^\perp$ of M and Θ respectively by

$$M^\perp := \{f \in Hom(X,Y) | f(x) = 0 \quad \text{for} \quad \text{all } x \in M\}$$

and

$$\Theta^\perp := \{x \in X | f(x) = 0 \quad \text{for} \quad \text{all } f \in \Theta\}.$$

Proposition 3.10. *Let X be a difference algebra and let Y be a medial difference algebra. Then we have the following:*
(1) $\{0\}^\perp = Hom(X,Y)$, *where* 0 *is the zero element of* X.
(2) $X^\perp = \{0\}$, *where* 0 *is the zero homomorphism.*
(3) *If* $M_1 \subseteq M_2 \subseteq X$, *then* $M_2^\perp \subseteq M_1^\perp$.
(4) $M \subseteq (M^\perp)^\perp$, *where* $M \subseteq X$.
(5) $M^\perp = ((M^\perp)^\perp)^\perp$, *where* $M \subseteq X$.
(6) $\{0\}^\perp = X$, *where* 0 *is the zero homomorphism.*
(7) $Hom(X,Y)^\perp = \{0\}$, *where* 0 *is the zero element of* X.
(8) *If* $N_1 \subseteq N_2 \subseteq Hom(X,Y)$, *then* $N_2^\perp \subseteq N_1^\perp$.
(9) $N \subseteq (N^\perp)^\perp$, *where* $N \subseteq Hom(X,Y)$.
(10) $N^\perp = ((N^\perp)^\perp)^\perp$, *where* $N \subseteq Hom(X,Y)$.

Proof. (1), (2), (6) and (7) follows easily from definitions of $M^\perp$ and $\Theta^\perp$. (4) and (9) are easy.
(3) Assume that $M_1 \subseteq M_2 \subseteq X$. Let $f \in M_2^\perp$. Then $f(x) = 0$ for all $x \in M_2$. This implies $f(x) = 0$ for all $x \in M_1$, because $M_1 \subseteq M_2$. Hence $f \in M_1^\perp$ and $M_2^\perp \subseteq M_1^\perp$.
For (5) apply (9) to $M^\perp$ for $M^\perp \subseteq ((M^\perp)^\perp)^\perp$ and apply (3) to (4) for $((M^\perp)^\perp)^\perp \subseteq M^\perp$.
(8) and (10) are similar to the cases of (3) and (5) respectively.

Theorem 3.11. *Let X be a difference algebra and let Y be a medial difference algebra. Let M and Θ be subsets of X and $Hom(X,Y)$ respectively. Then $M^\perp$ and $\Theta^\perp$ are subalgebras of $Hom(X,Y)$ and X respectively.*

Proof. Straightforward.

A non-empty subset N of X is said to be *normal* of X if $(x*a)*(y*b) \in N$ for any $x*y, a*b \in N$. Since $x*0 = x$, every normal subset of X is a subalgebra of X, but the converse need not be true in general.

Example 3.12. In Example 2.9, let $S := \{f \in F(X) | f(0) = 0\}$. Then S is a normal subalgebra of a difference algebra $(F(X); *, \leq, 0)$.

Example 3.13. In Example 2.4, let $N := \{0, 3\}$. Then N is a closed ideal of X. But N is not normal, since $2*1 = 3,\ 3*2 = 0 \in N$ but $(2*3)*(1*2) = 1*3 = 2 \notin N$.

Example 3.14. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Define a binary relation " $\leq$ " by $x \leq y$ if and only if $x*y = 0$. Then $(X; *, \leq, 0)$ is a difference algebra and $N := \{0, 3\}$ is a normal subalgebra of X.

Example 3.15. In Example 2.5, if we let $I := \{0, 3\}$, then I is a subalgebra, but not normal, since $3*2 = 3$, $4*2 = 3 \in I$, but $(4*3)*(2*2) = 1*0 = 1 \notin I$.

Theorem 3.16. *Let X be a difference algebra, let Y be a medial difference algebra, $M \subseteq X$ and $\Theta \subseteq Hom(X, Y)$. Then $M^{\perp}$ and $\Theta^{\perp}$ are normal subalgebras of $Hom(X, Y)$ and X, respectively.*

Proof. Let $f*g, h*k \in M^{\perp}$. Then for any $x \in M$, we have $(f*g)(x) = f(x)*g(x) = 0$ and $(h*k)(x) = h(x)*k(x) = 0$. Since Y is medial, $(f*h)(x)*(g*k)(x) = (f(x)*h(x))*(g(x)*k(x)) = (f(x)*g(x))*(h(x)*k(x)) = 0$. Therefore $(f*h)*(g*k) \in M^{\perp}$. Thus $M^{\perp}$ is a normal subalgebra of $Hom(X, Y)$.

If $x*y, a*b \in \Theta^{\perp}$, then $f(x*y) = 0$ and $f(a*b) = 0$ for all $f \in \Theta$. Hence we have

$$\begin{aligned} f((x*a)*(y*b)) &= f(x*a)*f(y*b) \\ &= (f(x)*f(a))*(f(y)*f(b)) \\ &= (f(x)*f(y))*(f(a)*f(b)) \quad [\because Y : \text{medial}] \\ &= f(x*y)*f(a*b) = 0*0 = 0. \end{aligned}$$

Therefore $(x*a)*(y*b) \in \Theta^{\perp}$. Thus $\Theta^{\perp}$ is a normal subalgebra of X.

Theorem 3.17. *Let X, Y and Z be difference algebras. If Z is medial, then to each homomorphism $f : X \to Y$ there corresponds a unique homomorphism*

$$f^* : Hom(Y, Z) \to Hom(X, Z) \qquad (*)$$

that satisfies $f^*(g)(x) = (g \circ f)(x)$ for all $x \in X$ and all $g \in Hom(Y, Z)$.

Proof. For each $g \in Hom(Y, Z)$ we define a mapping $\mu : X \to Z$ by $\mu(x) := g(f(x))$ for all $x \in X$. Since g and f are homomorphisms, μ is a homomorphism and $\mu \in Hom(X, Z)$. Define $f^* : Hom(Y, Z) \to Hom(X, Z)$ by $f^*(g) := g \circ f$, for any $g \in Hom(Y, Z)$.

To prove that f^* is a homomorphism, let $g, g' \in Hom(Y, Z)$. Then for any $x \in X$,

$$\begin{aligned} f^*(g * g')(x) &= ((g * g') \circ f)(x) = (g * g')(f(x)) \\ &= g(f(x)) * g'(f(x)) = f^*(g)(x) * f^*(g')(x) \\ &= (f^*(g) * f^*(g'))(x). \end{aligned}$$

Since x is arbitrary, it follows that $f^*(g * g') = f^*(g) * f^*(g')$. We claim that f^* is a Harris map. For any g_1 and g_2 of $Hom(Y, Z)$ with $g_1(y)||g_2(y)$ for any $y \in Y$, we have

$$f^*(g_1)(x) = (g_1 \circ f)(x) = g_1(f(x))||g_2(f(x)) = (g_2 \circ f)(x) = f^*(g_2)(x)$$

for any $x \in X$. Therefore f^* is a homomorphism. The fact that $(*)$ holds for all $x \in X$ obviously determines $f^*(g)$ uniquely. This completes the proof.

Theorem 3.18. *Let X, Y and Z be difference algebras and let $f : X \to Y$ be a homomorphism. If Z is medial, then $Ker(f^*) = Im(f)^\perp$ and $Ker(f) = Im(f^*)^\perp$.*

Proof. Let $\phi \in Ker(f^*)$. Then $f^*(\phi) = 0$ and hence $f^*(\phi)(x) = (\phi \circ f)(x) = 0$ for all $x \in X$. Hence $\phi \in Im(f)^\perp$ and $Ker(f^*) \subseteq Im(f)^\perp$. Similarly $Im(f)^\perp \subseteq Ker(f^*)$. Hence $Ker(f^*) = Im(f)^\perp$. Next for any $\mu \in Im(f^*)$ there exists a homomorphism $g : Y \to Z$ such that $f^*(g) = \mu$. Then for any $x \in Ker(f)$, $\mu(x) = f^*(g)(x) = (g \circ f)(x) = g(f(x)) = g(0) = 0$, which implies that $x \in Im(f^*)^\perp$ and that $Ker(f) \subseteq Im(f^*)^\perp$.

Conversely, let $x \in Im(f^*)^\perp$. Assume that $x \notin Ker(f)$, i.e., $f(x) \neq 0$. Choose a homomorphism $g : Y \to Z$ with $g(f(x)) \neq 0$. If we set $f^*(g) := \mu$, then $\mu \in Im(f^*)$ and hence $\mu(x) = f^*(g)(x) = (g \circ f)(x) \neq 0$. This means that $x \notin Im(f^*)^\perp$ which is a contradiction. Thus $x \in Ker(f)$ and $Im(f^*)^\perp \subseteq Ker(f)$. This completes the proof.

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