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ON THE SLANT HELICES ACCORDING TO BISHOP FRAME OF THE TIMELIKE CURVE IN LORENTZIAN SPACE

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Abstract. T.Ikawa obtained the following differential equation

$$D_T D_T D_T T - K D_T T, K = \kappa^2 - \tau^2$$

for the cırcular helix which corresponds the case that the curvature κ and torsion τ of timelike curve α on the Lorentzian manifold M_1 are constant [5]. In this paper, we have defined a slant helix according to Bishop frame of the timelike curve. Furthermore, we have given some necessary and sufficent conditions for the slant helix and T.Ikawa's result is generalized to the case of the general slant helix.

1. Preliminaries

Let M be an 3-dimensional smooth manifold equipped with a metric $\langle,\rangle_L$, where the metric $\langle,\rangle_L$ means a symmetric non-degenerate (0; 2)-tensor field on M with constant signature. A tangent space $T_P(M)$ at a point $P \in M$ is furnished with the canonical inner product. If the signature of the metric $\langle,\rangle_L$ is i, then we call M an indefinite-Riemannian manifold of signature i and denoted by M_i. If $\langle,\rangle_L$ is positive definite, then M is a Riemannian manifold. Especially if $i = 1$, then M is called a Lorentzian manifold. A tangent vector x of M_i is said to be spacelike, if $\langle x, x\rangle_L > 0$ or $x = 0$, timelike, if $\langle x, x\rangle_L < 0$ and null or lightlike if $\langle x, x\rangle_L = 0$ and $x \neq 0$ [3, 4].

A curve in an indefinite-Riemannian manifold M_i is a smooth mapping $\alpha : I \to M_i$, where I is an open interval in the real line R^1. As an open submanifold of R^1, I has a coordinate system consisting of the identity map u of I. The velocity vector of α at $s \in I$

$$\alpha'(s) = \frac{d\alpha(u)}{du}\Big|_s \in T_{\alpha(s)}(M_i).$$

A curve $\alpha(s)$ is said to be regular if $\alpha'(s)$ is not equal to zero for any s. If $\alpha(s)$ is a spacelike or timelike curve, we can reparameterize it such that $\langle \alpha'(s), \alpha'(s)\rangle_L = 1$ and $\langle \alpha'(s), \alpha'(s)\rangle_L = -1$, respectively. In this case $\alpha(s)$ is said to be unit speed or arc length parametrization [3, 4].

Let $R^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in R\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in IR^3. The Lorentz scalar product of x and y is defined by

$$\langle x, y\rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

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$E_1^3 = (R^3, \langle x,y\rangle_L)$ is called 3-dimensional Lorentzian space, Minkowski 3-Space or 3- dimensional Semi-Euclidean space.For any $x, y \in E_1^3$, Lorentzian vectoral product of $x$and y is defined by

$$x \wedge_L y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1)[2].$$

Let $\alpha(s)$ be a timelike curve in M_1. Denote by $\{T, N, B\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and binormal vectors of the curve α respectively. If α is a timelike curve, then this set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

$$\begin{aligned} \alpha'(s) &= T \\ D_T T &= \kappa N \\ D_T N &= \kappa T + \tau B \\ D_T B &= -\tau N, \end{aligned}$$

where D denotes the covariant differentiation in M_1 and $\{T, N, B\}$ are mutually orthogonal vectors satisfying the following equations

$$\langle T,T\rangle_L = -1, \ \ \langle N,N\rangle_L = 1, \ \ \langle B,B\rangle_L = 1 \ [8].$$

In a Lorentzian manifold M_1, a curve is described by the Frenet formula. For example, if all curvatures of a curve are identically zero, then the curve is a geodesic. If only the curvature κ is a non-zero constant and the torsion τ is identically zero, then the curve is called a circle. If the curvature κ and the torsion τ are non-zero constants, then the curve is called helix. If the curvature κ and the torsion τ are not constant but $\frac{\kappa}{\tau}$ is constant, then the curve is called a general helix [3, 4].

2. Introduction

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the timelike curve has vanishing second derivative. We can parallel transport an orthonormal frame along a timelike curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(N_1(s), N_2(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point. If the derivatives of $(N_1(s), N_2(s)t)$ depend only on $T(s)$ and not on each other we can make $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature [1]. Therefore, we have the alternative frame equations

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} \tag{2.1}$$

where

$$\langle T,T\rangle_L = -1, \ \ \langle N_1,N_1\rangle_L = 1, \ \ \langle N_2,N_2\rangle_L = 1.$$

One can show that

$$\kappa(s) = \sqrt{k_1^2 + k_2^2},$$
$$\theta(s) = \arctan(\frac{k_2}{k_1}),$$
$$\tau(s) = \frac{d\theta(s)}{ds}$$

[2], so that k_1 and k_2 effectively correspond to a cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation [1].

3. The slant helices according to bishop frame of the timelike curve

Definition 3.1. A curve α with $\kappa \neq 0$ E^3 is called a slant helix if the principal normal line of α make a constant angle with a fixed direction [7].

Definition 3.2. A regular timelike curve $\alpha : I \to E_1^3$ is called a slant helix provided the spacelike unit vector N_1 of the curve α has constant angle θ with some fixed spacelike unit vector u; that is, $\langle N_1(s), u\rangle = \cos\theta$ for all $s \in I$.

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

Theorem 3.1. *Let $\alpha : I \to E_1^3$ be a unit speed timelike curve with non-zero natural curvatures. Then α is a slant helix if and only if $\frac{k_1}{k_2}$ is constant.*

Proof. Let α is a slant helix in E_1^3 and $< N_1, u >= const.$. Then α is a slant helix; from the definition we have

$$\langle N_1, u\rangle = const.,$$

where the spacelike vector u is a unit vector, called the axis of the slant helix. By differentiation we get

$$\langle N_1^{'}, u\rangle = \langle k_1 T, u\rangle = k_1\langle T, u\rangle = 0.$$

Hence

$$\langle T, u\rangle = 0.$$

Again differentiating from the last equality, we can write as follows

$$\begin{aligned} \langle T^{'}, u\rangle &= \langle k_1 N_1 + k_2 N_2, u\rangle \\ &= k_1\langle N_1, u\rangle + k_2\langle N_2, u\rangle \\ &= k_1 \cos\theta + k_2 \sin\theta = 0. \end{aligned}$$

Therefore we obtain

$$\frac{k_1}{k_2} = -\tan\theta$$

as desired.

Suppose that $\frac{k_1}{k_2} = -\tan\theta$. Then we can write $u \in Sp\{N_1, N_2\}$, i.e.,

$$u = N_1 \cos\theta + N_2 \sin\theta.$$

Differentiating the last equality.

$$u' = (k_1 \cos\theta + k_2 \sin\theta) T = 0.$$

So the spacelike vector u is a constant vector. Thus, the proof is done.

Theorem 3.2. *Let $\alpha : I \to E_1^3$ be a unit speed timelike curve.Then α is a slant helix iff*

$$\det(N_1', N_1'', N_1''') = 0.$$

Proof. ($\Rightarrow$:) Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned}
N_1' &= k_1 T \\
N_1'' &= k_1' T + k_1(k_1 N_1 + k_2 N_2) = k_1' T + k_1^2 N_1 + k_1 k_2 N_2 \\
N_1''' &= k_1'' T + k_1'(k_1 N_1 + k_2 N_2) + 2k_1 k_1' N_1 + k_1^2(k_1 T) + k_1' k_2 N_2 + k_1 k_2' N_2 + k_1 k_2 (k_2 T) \\
N_1''' &= (k_1'' + k_1^3 + k_1 k_2^2) T + (3k_1 k_1') N_1 + (2k_1' k_2 + k_1 k_2') N_2
\end{aligned}$$

So we get

$$\begin{aligned}
\det(N_1', N_1'', N_1''') &= k_1^2 \begin{bmatrix} 1 & 0 & 0 \\ * & k_1 & k_2 \\ \star & 3k_1 k_1' & 2k_1' k_2 + k_1 k_2' \end{bmatrix} \\
\det(N_1', N_1'', N_1''') &= k_1^2 (2k_1' k_1 k_2 + k_1^2 k_2' - 3k_1 k_1' k_2) \\
&= k_1^3 (k_2' k_1 - k_2 k_1') \\
&= k_1^3 (k_1' k_2 - k_1 k_2') \\
&= k_1^3 \Big[\frac{k_1' k_2 - k_1 k_2'}{k_2^2} \Big] k_2^2 \\
&= k_1^3 k_2^2 \Big(\frac{k_1}{k_2} \Big)'
\end{aligned}$$

Since α is a slant helix, $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det(N_1', N_1'' N_1''') = 0, \ k_2 \neq 0.$$

($\Leftarrow$:) Suppose that $\det(N_1^{'}, N_1^{''} N_1^{'''}) = 0$. Then it is clear that the $\frac{k_1}{k_2} = const.$ since $(\frac{k_1}{k_2})'$ is zero.

Theorem 3.3 *Let $\alpha : I \to E_1^3$ be a unit speed timelike curve. Then α is a slant helix iff*

$$\det(N_2^{'}, N_2^{''}, N_2^{'''}) = 0.$$

Proof. ($\Rightarrow$:) Suppose that $\frac{k_1}{k_2}$ be constant. From eq. (2.1) one can find

$$N_2^{'} = k_2 T$$

and

$$\begin{aligned}
N_2^{''} &= k_2^{'} T + k_2 T^{'} = k_2^{'} T + k_2(k_1 N_1 + k_2 N_2) = (k_2^{'})T + (k_1 k_2)N_1 + (k_2^2)N_2 \\
N_2^{'''} &= (k_2^{''} T + k_2^{'} T^{'}) + (k_1^{'} k_2 N_1 + k_1 k_2^{'} N_1 + k_1 k_2 N_1^{'}) + (2k_2 k_2^{'} N_2 + k_2^2 N_2^{'}) \\
&= k_2^{''} T + k_2^{'}(k_1 N_1 + k_2 N_2) + k_1^{'} k_2 N_1 + k_1 k_2^{'} N_1 + k_1 k_2(k_1 T) + 2k_2 k_2^{'} N_2 + k_2^2(k_2 T) \\
&= k_2^{''} T + k_1 k_2^{'} N_1 + k_2 k_2^{'} N_2 + k_1^{'} k_2 N_1 + k_1 k_2^{'} N_1 + k_1^2 k_2 T + 2k_2 k_2^{'} N_2 + k_2^3 T \\
&= (k_2^{''} + k_1^2 k_2 + k_2^3)T + (2k_1 k_2^{'} + k_1^{'} k_2)N_1 + (3k_2 k_2^{'})N_2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\det(N_2^{'}, N_2^{''} N_2^{'''}) &= k_2^2 \begin{bmatrix} 1 & 0 & 0 \\ \Delta & k_1 & k_2 \\ \blacktriangleleft & 2k_1 k_2^{'} + k_1^{'} k_2 & 3k_2 k_2^{'} \end{bmatrix} \\
&= k_2^2(3k_1 k_2 k_2^{'} - 2k_1 k_2^{'} k_2 - k_1^{'} k_2 k_2) \\
&= -k_2^3(k_1^{'} k_2 - k_1 k_2^{'}) \\
&= -k_2^3 \Big[\frac{k_1^{'} k_2 - k_1 k_2^{'}}{k_2^2} \Big] k_2^2 \\
&= -k_2^5 Big(\frac{k_1}{k_2}\Big)^{'}.
\end{aligned}$$

Since α is a slant helix curve $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det(N_2^{'}, N_2^{''} N_2^{'''}) = 0, \; k_2 \neq 0$$

($\Leftarrow$:) Suppose that $\det(N_2^{'}, N_2^{''} N_2^{'''}) = 0$. Then it is clear that the $\frac{k_1}{k_2} = const.$ since$(\frac{k_1}{k_2})'$ is zero.

Next we consider general slant helices in the Lorentzian manifold M_1. Then we have equalities

$$\begin{cases} \alpha^{'}(s) = T \\ D_T T = k_1 N_1 + k_2 N_2 \\ D_T N_1 = k_1 T \\ D_T N_2 = k_2 T \end{cases} \tag{3.1}$$

for any $s \in I$, where N_1 and N_2 are vector fields and k_1 and k_2 are functions of parameter s.

Theorem 3.4. *Let $\alpha : I \to E_1^3$ be a unit speed timelike curve on M_1 is a general slant helix iff*

$$D_T(D_T D_T N_1) = A D_T N_1 + 3k_1^{'} D_T T. \tag{3.2}$$

where

$$A = \kappa^2 + \frac{k_1^{''}}{k_1}, \quad k_1^2 + k_2^2 = \kappa^2.$$

Proof. Suppose that α is a general slant helix. Then, from (3.1), we have

$$\begin{aligned} D_T(D_T N_1) &= D_T(k_1 T) = k_1^{'} T + k_1 D_T T \\ &= k_1^{'} T + k_1(k_1 N_1 + k_2 N_2) \\ D_T(D_T N_1) &= k_1^{'} T + k_1^2 N_1 + k_1 k_2 N_2 \end{aligned} \tag{3.3}$$

and

$$D_T(D_T D_T N_1) = k_1^{''} T + k_1^{'} D_T T + 2k_1 k_1^{'} N_1 + k_1^2 D_T N_1 + k_1^{'} k_2 N_2 + k_1 k_2^{'} N_2 + k_1 k_2 (k_2 T)$$

$$D_T(D_T D_T N_1) = k_1^{''} T + k_1^{'} D_T T + 2k_1 k_1^{'} N_1 + k_1^2 D_T N_1 + k_1^{'} k_2 N_2 + k_1 k_2^{'} N_2 + k_1 k_2^2 T$$

$$D_T(D_T D_T N_1) = (k_1^{''} + k_1 k_2^2) T + k_1^2 D_T N_1 + 2k_1 k_1^{'} N_1 + (k_1^{'} k_2 + k_1 k_2^{'}) N_2 + k_1^{'} D_T T \tag{3.4}$$

Now, since α is a general slant helix, we have

$$\frac{k_1}{k_2} = \text{constant}$$

and this upon the derivation gives rise to

$$k_1^{'} k_2 = k_1 k_2^{'}.$$

If we substitute the values

$$T = \frac{1}{k_1} D_T N_1 \tag{3.5}$$

and

$$(k_1 k_2)^{'} = 2k_1^{'} k_{2,}$$

in (3.4) we obtain

$$\begin{aligned} D_T(D_T D_T N_1) &= (k_1^{''} + k_1 k_2^2)\Big(\frac{1}{k_1} D_T N_1\Big) + k_1^2 D_T N_1 + (2k_1 k_1^{'} N_1 + 2k_1^{'} k_2 N_2) + k_1^{'} D_T T \\ &= \Big(\frac{k_1^{''}}{k_1} + k_2^2\Big) D_T N_1 + k_1^2 D_T N_1 + (2k_1 k_1^{'} N_1 + 2k_1^{'} k_2 N_2) + k_1^{'} D_T T \end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{k_1''}{k_1} + \kappa^2\Big) D_T N_1 + 2k_1'(k_1 N_1 + k_2 N_2) + k_1' D_T T \\
&= \Big(\frac{k_1''}{k_1} + \kappa^2\Big) D_T N_1 + 2k_1' D_T T + k_1' D_T T \\
D_T(D_T D_T N_1) &= \Big(\frac{k_1''}{k_1} + \kappa^2\Big) D_T N_1 + 3k_1' D_T T.
\end{aligned}
$$

$$
D_T(D_T D_T N_1) - \Big(\kappa^2 + \frac{k_1''}{k_1}\Big) D_T N_1 = 3k_1' D_T T.
$$

So we get as desired.

Conversely let us assume that (3.1) holds. We show that the timelike curve α is a general slant helix. Differentiating covariantly (3.5) we obtain

$$
\begin{aligned}
D_T T &= D_T\Big(\frac{1}{k_1} D_T N_1\Big) \\
D_T T &= -\frac{k_1'}{k_1^2} D_T N_1 + \frac{1}{k_1} D_T D_T N_1
\end{aligned}
$$

and so,

$$
D_T D_T T = \Big(-\frac{k_1'}{k_1^2}\Big)' D_T N_1 - \frac{k_1'}{k_1^2} D_T D_T N_1 - \frac{k_1'}{k_1^2} D_T D_T N_1 + \frac{1}{k_1} D_T D_T D_T N_1 \tag{3.6}
$$

If we use (3.1) in (3.6), we get

$$
\begin{aligned}
D_T D_T T &= \Big(-\frac{k_1'}{k_1^2}\Big)' D_T N_1 - \frac{2k_1'}{k_1^2} D_T D_T N_1 + \frac{1}{k_1} A D_T N_1 + \frac{3k_1'}{k_1} D_T T \\
&= \Big[\Big(-\frac{k_1'}{k_1^2}\Big)' + \frac{A}{k_1}\Big] D_T N_1 - \frac{2k_1'}{k_1^2} D_T D_T N_1 + \frac{3k_1'}{k_1} D_T T
\end{aligned}
$$

$$
\begin{aligned}
D_T D_T T &= \Big[\Big(-\frac{k_1'}{k_1^2}\Big)' + \frac{A}{k_1}\Big] D_T N_1 - \frac{2k_1'}{k_1^2}(k_1' T + k_1^2 N_1 + k_1 k_2 N_2) + \Big(\frac{3k_1'}{k_1} k_1 N_1 + \frac{3k_1'}{k_1} k_2 N_2\Big) \\
&= \Big[\Big(-\frac{k_1'}{k_1^2}\Big)' + \frac{A}{k_1}\Big] D_T N_1 - 2\Big(\frac{k_1'}{k_1}\Big)^2 T - 2k_1' N_1 - \frac{2k_1' k_2}{k_1} N_2 + 3k_1' N_1 + \frac{3k_1' k_2}{k_1} N_2.
\end{aligned}
$$

Substituting (3.3) and (3.4) in this last equality we have

$$
D_T D_T T = \Big[\Big(-\frac{k_1'}{k_1^2}\Big)' + \frac{A}{k_1}\Big] D_T N_1 - 2\Big(\frac{k_1'}{k_1}\Big)^2 T + k_1' N_1 + \frac{k_1' k_2}{k_1} N_2. \tag{3.7}
$$

On the other hand we can write $D_T(D_T T)$ as follows

$$
D_T(D_T T) = k_1 D_T N_1 + k_2^2 T + k_1' N_1 + k_2' N_2 \tag{3.8}
$$

From comparision the (3.7) and (3.8) we obtain the equalities below

$$\frac{k_1^{'} k_2}{k_1} = k_2^{'}$$

and so

$$\frac{k_1^{'}}{k_2} = \frac{k_2^{'}}{k_2}. \tag{3.9}$$

Integrating (3.9), we get

$$\frac{k_1}{k_2} = const.$$

Thus α is a general slant helix. Hence, the proof is done.

References

[1] L.R. Bishop, *There is more than one way to frame a curve*, Amer. Math. Monthly **82**(1975), 246–251.

[2] B. Bukcu and M. K. Karacan, Bishop Frame of The Timelike Curve in Minkowski 3-Space, Süleyman Demirel University,Faculty of Science and Art, Journal of Science, (Accepted for Publication) 2008.

[3] N. Ekmekci and H. H. Hacisalihoglu, *On helices of a lorentzian manifold*, Commun. Fak. Sci. Univ. Ankara, Series A1 **45**(1996), 4-5-50.

[4] K. Ilarslan, *Characterizations of spacelike general helices in Lorentzian manifold*, Kragujevac J. **25**(2003), 209–218.

[5] T. Ikawa, *On Curves and Submanifolds in an Indefinite-Riemannian Manifold*, Tsukuba J. Math. **9**(1985), 353–371.

[6] J. Oprea, Differential Geometry and Its Applications, Prentice Hall, N.J. 07458, 1997.

[7] L. Kula and Y. Yayli, *On slant helix and its spherical indicatrix*, Applied Mathematics and Computation **169**(2005), 600–607.

[8] M. Petrovic-Torgasev and E. Sucurovic, *Some characterizations of the lorentzian spherical timelike and null*, Matematiqki Vesnik, **53**(2001), 21–27.

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