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THE JACOBSON SEMIRADICAL OVER A CERTAIN SEMIRING

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Abstract. The concept of the semiradical class of semirings was introduced in [3]. The purpose of this paper is to study one such semiradicals, the Jacobson semiradicals, over certain semirings. We generalize the concept of the Jacobson radical of a ring to a semiring. Some properties of the Jacobson semiradical $JS(R)$ of the semiring R parallel those of ring theory. In Section 1 we describe some preliminary definitions. In Section 2 we define regular strongly austere semimodulse. Theorem 1 characterizes a regular strongly austere semimodule in terms of a regular modular maximal subtractive left ideal. We define $JS(R)$ and derive some properties of this structure. In Section 3 we show that the $JS(R)$ of a semiring has many representations as the intersection of left ideals. One of the more important of these is that $JS(R)$ is the intersection of all left weakly primitive subtractive ideals. Proposition 6 characterizes a semiweekly primitive semiring in terms of a weakly primitive semiring. The interrelationships of strongly austere, weakly primitive and semiweekly primitive semirings are examined in Theorem 3. In Section 4, Proposition 7 shows that the $JS(R)$ of a semiring with identity is the intersection of all regular maximal modular subtractive left ideals. Corollary 3 shows that $JS(R)$ is the unique largest superfluous left ideal of R. Proposition 8 shows that the class of Jacobson semiradical of semirings is closed under direct sum. We conclude with Section 5, a consideration of a certain restricted class of semirings. We show that the Jacobson semiradical for the semirings belonging to this class constitutes a semiradical class. Finally, Example 1 shows that a semiradical class need not be a radical class.

1. Introduction

Recall ([1], [2], [3], [4] & [6]) the following:

1.1. Let $\alpha : M \to N$ be a homomorphism of semimodules. The subsemimodule $\mathrm{Im}(\alpha)$ of N is defined as

$$\mathrm{Im}(\alpha) = \{n \in N : n + \alpha(m') = \alpha(m) \text{ for some } m,\, m' \in M\}.$$

Then α is i-regular if $\alpha(m) = \mathrm{Im}(\alpha)$; α is k-regular if for any $m,\, m' \in M$, $\alpha(m) = \alpha(m')$ implies $m + k = m' + k'$ for some $k,\, k' \in Ker\,\alpha$; and α is regular if it is both i-regular and k-regular.

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1.2. Let μ be a class of semimodules and M a left semimodule. The reject of μ in M and the strong reject of μ in M are defined as $\mathrm{Re}j_M(\mu) = \cap\{Ker\, h | h : M \to U$ for some $U \in \mu$ and h is a homomorphism$\}$, and $S\mathrm{Re}j_M(\mu) = \cap\{Ker\, h | h : M \to U$ for some $U \in \mu$ and h is a regular homomorphism$\}$, respectively.

1.3. Let M be a left R-semimodule. An element $m \in M$ is regular if the homormorphism $\theta : R \to M$ defined by $r \mapsto rm$ is a regular homomorphism.

1.4. If M is a left R-semimodule then its left annihilator is $L_R(M) = \{r \in R : rm = 0$ for every element $m \in M\}$ and its left strong annihilator is $SL_R(M) = \{r \in R : rm = 0$ for every regular element $m \in M\}$.

1.5. A non-empty subset N of a left semimodule M is subtractive if and only if for all m, $m' \in M$, $m + m' \in N$ and $m \in N$ imply that $m' \in N$.

1.6. A left R-semimodule M is austere if $\{0\}$ and M are the only two subtractive subsemimodules of M and strongly austere if M is austere and $SL_R(M) \neq R$.

1.7. An ideal I of a semiring R determines an equivalence relation $\equiv_I$ on I, the Bourne relation, defined as $r \equiv_I r'$ if and only if there exist elements a, $a' \in I$ satisfying $r + a = r' + a'$. If $r \in R$ then we write r/I instead of $r/\equiv_I$. The factor semiring $R/\equiv_I$ is denoted by R/I.

1.8. A left ideal I in a semiring R is modular if there exists $e \in R$ such that $r/I = re/I$ for every $r \in R$ and $e/I \neq 0$.

1.9. A semiring R is the semisubdirect product of the family of semiring $\{R_i; i \in I\}$ if for each $i \in I$, there exists a surjective k-regular semiring homomorphism $\Pi_i : R \to R_i$, such that $\bigcap_{i\in I} Ker\, \Pi_i = 0$.

1.10. A class ρ of semirings is a radical class whenever the following three conditions are satisfied:

(a) ρ is homomorphically closed; i.e. if R' is a hormomorphic image of a ρ-semiring R, then R' is also a ρ-semiring

(b) Every semiring R contains a ρ-ideal $\rho(R)$ which in turn contains every other ρ-ideal of R

(c) The factor semiring $R/\rho(R)$ does not contain any nonzero ρ-ideal; i.e. $\rho(R/\rho(R)) = 0$.

1.11. A class $\bar{\rho}$ of semiring is said to be a semiradicals class whenever the following three conditions are satisfied:

(a$'$) $\bar{\rho}$ is k-homomorphic closed; i.e. if R' is an k-homomorphic image of a $\bar{\rho}$-semiring R, then R' is also a $\bar{\rho}$-semiring.

(b$'$) Every semiring R contains a $\bar{\rho}$-subtractive ideal $\bar{\rho}(R)$ which in turn contains every other $\bar{\rho}$-subtractive ideal of R.

(c′) The factor semiring $R/\bar{\rho}(R)$ does not contain any non-zero $\bar{\rho}$-subtractive ideals; i.e. $\bar{\rho}(R/\bar{\rho}(R)) = 0$.

Remark 1. For potential applications, we note that semimodules over semirings are important in studying the properties of the semiring, and the latter arise in diverse areas of applied mathematics, including optimization theory, automata theory, mathematical modeling and parallel computation systems (c.f. [4], pp. iv, 138).

2. The Definition of the Jacobson Semiradical $JS(R)$

Definition 1. A left R-semimodule M is regular if for every $m \in M$, m is regular. A left ideal I in a semiring R is regular if the left R-semimodule R/I is regular.

Remark 2. Note that if M is a module, then M is regular. Similarly, if R is a ring, then every left ideal is regular.

Theorem 1. *A left R-semimodule M over a semiring R is regular and strongly austere if and only if M is isomorphic to R/I for some regular modular maximal subtractive left ideal I.*

Proof. Let M be a regular strongly austere semimodule. Since every $m \in M$ is regular. Rm is subtractive. Hence $Rm = M$. Define $\theta : R \to Rm$ as $\theta(r) = rm$. Since M is regular, θ is a k-regular surjective homomorphism. Consider $\phi : R/Ker\,\theta \to Rm$, where $\phi(r/Ker\,\theta) = rm$. Clearly ϕ is an isomorphism, and since M is austere, $Ker\,\theta$ is a maximal subtractive left ideal. Because M is regular, $Ker\,\theta$ is a regular modular maximal subtractive left ideal. Conversely, let I be a regular modular maximal subtractive left ideal of R such that $M \simeq R/I$. Clearly M is austere, and since I is modular, $L_R(M) \neq R$. Thus M is a regular strongly austere semimodule.

Definition 2. For a semiring R the *Jacobson semiradical* $JS(R)$ is the annihilator of R in χ; i.e., $JS(R) = L_R(\chi)$ where χ is the class of all regular strongly austere left R-semimodules.

Proposition 1. *For a semiring R, $JS(R)$ is a subtractive ideal.*

Lemma 1. *Let R and R' be semirings and let $\phi : R \to R'$ be a surjective k-regular homomorphism. Then every regular strongly austere R'-semimodules is a regular strongly austere R-semimodule.*

Proof. Let M be a regular strongly austere R'-semimodule. Define $\phi : R \times M \to M$ by the rule $rm = \phi(r)m$. Clearly M is an R-semimodule. Let K be a subtractive R-semimodule. Since ϕ is surjective, $R'K = \phi(R)K = RK \subset K$ and thus K is a subtractive R'-subsemimodule. Therefore M is an austere R-semimodule. Now M is a regular strongly austere R'-semimodules, so $0 \neq m \in M$ is regular. If $r_1m = r_2m$, then $\phi(r_1)m = \phi(r_2)m$. Because m is regular, $\phi(r_1) + r_1' = \phi(r_2) + r_2'$, where $r_1'm = r_2'm = 0$

and r'_1, $r'_2 \in R'$. Since ϕ is onto, $r'_1 = \phi(r_3)$, $r'_2 = \phi(r_4)$. Hence $\phi(r_1 + r_3) = \phi(r_2 + r_4)$. And ϕ is k-regular, so $r_1+r_3+k_1 = r_2+r_4+k_2$, where r_1, r_2, r_3, $r_4 \in R$ and k_1, $k_2 \in Ker\,\phi$. Clearly $(r_3+k_1)m = \phi(r_3+k_1)m = \phi(r_3)m+\phi(k_1)m = \phi(r_3)m+0 = \phi(r_3)m = r'_1m = 0$. Similarly $(r_4+k_2)m = 0$. Now if $r_1m+b = r_2m$, then $\phi(r_1)m+b = \phi(r_2)m$. Since m is a regular element, we have $b = r'_2m$. Now ϕ is onto, so $r'_2 = \phi(r_3)$. Thus $b = \phi(r_3)m = r_3m$. Therefore M is a regular strongly austere R-semimodule.

Proposition 2. *Let R and R' be semirings and let $\phi : R \to R'$ be a surjective k-regular homomorphism. Then $\phi(JS(R)) \subseteq JS(R')$ with equality if $Ker\,\phi \subseteq JS(R)$.*

Proof. Let χ be the class of all regular strongly austere R'-semimodules. By Lemma 1 every regular strongly austere R'-semimodule is a strongly austere R-semimodule. If $r \in JS(R)$, then $rM = 0$ for all $M \in \chi$. But $rM = \phi(r)M = 0$. Hence $\phi(r) \in JS(R')$. Thus $\phi(JS(R)) \subset JS(R')$.

Corollary 1. *Let I be a subtractive ideal of a semiring R.*

(1) $JS(R/I) \supseteq (JS(R) + I)/I$.
(2) *If $I \subseteq JS(R)$, then $JS((R)/I) = JS(R)/I$. In particular $JS(R/J(R)) = 0$.*
(3) *If $JS(R/I) = 0$, then $JS(R) \subseteq I$.*
(4) *$I = JS(R)$ if and only if $I \subseteq JS(R)$ and $JS(R/I) = 0$.*

Proof. For (1) and (2), apply Proposition 2 with $\Pi : R \to R/I$ being the homomorphism. Statement (3) is a direct consequence of (1), and (4) follows from (2) and (3).

3. Characterizations of the Jacobson Semradical $JS(R)$

In this section we define a weakly primitive ideal. We characterize it in terms of the annihilator of a regular strongly austere left R-semimodule, and we obtain some more of its properties. Theorem 2 characterizes the Jacobson semiradical $JS(R)$ in terms of the intersection of weakly primitive ideals.

Definition 3. A left semimodule M is *faithful* if its left annihilator is 0.

Definition 4. A semiring R is *left weakly primitive* if there exists a faithful regular strongly austere left R-semimodule. An ideal P of a semiring R is *left weakly primitive* if the quotient R/P is a left weakly primitive semiring.

Proposition 3. *A subtractive ideal P of a semiring is R weakly primitive if and only if P is a left annihilator of a regular strongly austere left R-semimodule.*

Proof. If P is a left weakly primitive ideal, let M be a regular strongly austere faithful left R/P-semimodule. M is an R-semimodule with rm ($r \in R, m \in M$) defined to be $(r/P)m$. Then $RM = (R/P)M \neq 0$ and every R-subtractive subsemimodule of M is an

R/P subtractive subsemimodule of M, and hence M is a strongly austere R-semimodule. Clearly M is a regular R-semimodule. If $r \in R$, then $rM = 0$ if and only if $(r/P)M = 0$. But $(r/P)M = 0$ if and only if $r \in P$, since M is a faithful R/P-semimodule. Therefore P is the left annihilator of the regular strongly austere R-semimodule M. Conversely, suppose that P is the left annihilator of a regular strongly austere left R-semimodule M. M is a regular strongly austere R/P-semimodule with $(r/P)m = rm$ for $r \in R$, $m \in M$. Furthermore, if $(r/P)M = 0$ then $rM = 0$. Hence $r \in L_R(M) = P$ and $r/P = 0$ in R/P. Consequently, M is a faithful R/P-semimodule. Therefore R/P is a left weakly primitive semiring, and thus P is a left weakly primitive ideal of R.

Corollary 2. *Every regular modular maximal subtractive left ideal of R contains a weakly primitive subtractive ideal and every primitive subtractive ideal is the intersection of the regular modular maximal subtractive left ideals containing it.*

Proof. Let K be a regular modular maximal subtractive left ideal of R. By Theorem 1, R/K is a regular strongly austere R-semimodule. Hence the annihilator of R/K is a weakly primitive subtractive ideal contained in K. Now let P be a weakly primitive subtractive ideal of R and let M be a regular strongly austere left R-semimodule whose annihilator is P. Given a non zero $m \in M$, let $L_R(m) = \{r \in R : rm = 0\}$. Clearly $L_R(m)$ is a subtractive left ideal. Since M is regular and strongly austere, $M = Rm \cong R/L_R(m)$ and $L_R(m)$ is a regular modular maximal subtractive left ideal of R. Since

$$P = \bigcap_{0 \neq m \in M} L_R(m),$$

the result follows.

Definition 5. A left R-semimodule M is said to be *semi-regular and strongly austere* if M is a direct sum of regular strongly austere subsemimodules.

Proposition 4. *$JS(R)$ is the annihilator of a semi-regular and strongly austere R-semimodule and $JS(R) \subseteq I$ if I is the annihilator of a regular strongly austere R-semimodule.*

Proof. Let $\{M_i : i \in A\}$ be a full set of representatives of the isomorphism classes of a regular strongly austere R-semimodule. Then by Proposition 3, $\{L_R(M_i); i \in A\}$ is the set of all weakly primitive ideals of R. Also,

$$L_R\Big(\bigoplus_{i \in A} M_i\Big) = \bigcap_{i \in A} L_R(M_i) = JS(R),$$

proving the first assertion. The second statement is obvious.

Definition 6. Let I be a left ideal of R. Define $(I : R) = \{r \in R \mid rR \subset I\}$.

Proposition 5. *Let I be a left ideal. Then $(I : R)$ is an ideal of R. If I is modular and subtractive, then $(I : R)$ is the largest ideal of R that is contained in I.*

Now we come to an important characterization of $JS(R)$. In order to simplify the statement of several results, we adopt the following convention: if the class μ of those subsets of a semiring R that satisfy a property is empty then $\bigcap_{I\in\mu} I$ is defined to be R.

Theorem 2. *If R is a semiring, then there is a subtractive ideal $JS(R)$ of R such that*

(1) *$JS(R)$ is the intersection of all of the left strong annihilators of regular, strongly austere left R-semimodules.*
(2) *$JS(R)$ is the intersection of all of the left weakly primitive subtractive ideals.*
(3) *$JS(R) = \cap(I : R)$ where I runs over all the regular modular maximal subtractive left ideals of R.*

Proof. Let $JS(R)$ be the intersection of all of the left annihilators of regular, strongly austere left R-semimodules. If R has no regular, strongly austere left R-semimodules, then $JS(R) = R$ by the convention adopted above. $JS(R)$ is a subtractive ideal. We now show that statements (1), (2) and (3) are true for all subtractive left ideals. First observe that R itself cannot be the annihilator of a regular strongly austere left R-semimodule M because otherwise $L_R(M) = R$. This fact, together with Theorem 1 and Proposition 3, implies that the following conditions are equivalent:

- $JS(R) = R$.
- R has no regular strongly austere left R-semimodules.
- R has no left weakly primitive subtractive ideals.
- R has no regular modular maximal subtractive left R-ideals.

Therefore by the convention adopted above (l), (2) and (3) are true if $JS(R) = R$. Now assume $JS(R) \neq R$. Statement (1) is trivial. Statement (2) is an immediate consequence of Proposition 3. For (3), let I be a regular modular maximal subtractive left ideal. Then $r \in (I : R)$ iff $rR/I = I$ iff $r \in L_R(R/I)$.

Definition 7. A semiring R is *semiweakly primitive if $JS(R) = 0$.* The semiring R is a *Jacobson semiring if $JS(R) = R$.*

Proposition 6. *A semiring R is semiweakly primitive if and only if R is a semisubdirect product of weakly primitive semirings.*

Proof. Let $\{P_i; i \in I\}$ be the set of all weakly primitive subtractive ideals of R, let $R_i = R/P_i$ and let $\Pi_i : R \to R_i$ be the natural homomorphism with $Ker\,\Pi_i = P_i$. Since $\bigcap_{i\in I} P_i = 0$, R is a semisubdirect product of R_i, $i \in I$. Conversely, assume that R is a semisubdirect product of R_i, $i \in I$, where each R_i is a weakly primitive semiring. By definition, for each $i \in I$, there exists a surjective k-regular semiring homomorphism $\Pi_i : R \to R_i$ with $Ker \underset{i\in I}{\Pi_i} = 0$. Thus each $Ker\,\Pi_i$ is a weakly primitive ideal and hence R is semiweakly primitive.

Definition 8. A semiring R is *strongly austere* if $R^2 \neq 0$ and R has no proper subtractive ideals.

Theorem 3. *Let R be a semiring:*

(1) *If R is weakly primitive, then R is semiweakly primitive.*
(2) *If R is a strongly austere left R-semimodule and semiweakly primitive, then R is weakly primitive.*
(3) *If R is strongly austere, then R is either a weakly primitive semiweakly primitive or a Jacobson semiring.*

Proof. (1): R has a faithful regular strongly austere left R-semimodule M, so $JS(R) \subseteq L_R(M) = 0$. (2): Since R is strongly austere; $R \neq 0$. Since $JS(R) \neq R$, there exists a regular, strongly austere left R-semimodule M. The left annihilator $L_R(M)$ is a subtractive ideal of R by Proposition 2.2 in [2] and $L_R(M) \neq R$ (since $RM \neq 0$). But M is austere, so $L_R(M) = 0$ and thus M is a regular, strongly austere faithful R-semimodule. Therefore R is weakly primitive. (3): If R is strongly austere then the subtractive ideal $JS(R)$ is either R or zero. In the former case R is a Jacobson semiring and in the latter case R is semiweakly primitive and weakly primitive by (2).

4. The Jacobson Semiradical $JS(R)$ over Semirings with Identity

Throughout this section R will be a semiring with identity 1. Proposition 7 shows that $JS(R)$ is the intersection of all regular maximal subtractive left ideals. Theorem 4 shows that $JS(R)$ is the unique largest superfluous left ideal of R.

Proposition 7. *For any semiring R the following properties hold:*

(1) *$JS(R)$ is the intersection of all of the rejects of regular strongly austere left R-semimodules.*
(2) *$JS(R)$ is the intersection of all of the strong rejects of regular strongly austere left R-semimodules.*
(3) *$JS(R)$ is the intersection of all of the regular maximal subtractive left ideals of R.*

Proof. We obtain (l) by using Corollary 2.4 in [1]. Similarly, (2) follows from Proposition 3.4 in [2], and comes from (1).

Definition 9. A left R-semimodule M is *regular with respect to the subsemimodule K if* M/K is regular. A left ideal H is *strongly regular* if H is regular and the semimodule R/H is regular with respect to every subtractive maximal subsemimodule.

Definition 10. A left ideal I of R is *superfluous* if $I+K$ is subtractive for all regular maximal subtractive left ideals K and if $I+H = R$ implies $H = R$ for all strongly regular subtractive left ideals H of R.

Theorem 4. *Let R be a semiring such that $JS(R) \neq R$. For any left ideal I of R, the following conditions are equivalent:*

(1) $I \subseteq JS(R)$.
(2) *I is superfluous.*
(3) *For any finitely generated left R-semimodule M which is regular with respect to every maximal subsemimodule, $IM = M$ implies that $M = 0$.*

Proof. (1) $\Longrightarrow$ (3): Assume that $M \neq 0$. Since M is finitely generated, it follows that M has a maximal subsemimodule K, in which case M/K is regular. Clearly $R(M/K) \neq 0$. Therefore M/K is regular and strongly austere. Hence $JS(R)M/K = 0$, i.e. $JS(R)M \subseteq K$. Thus $IM \subseteq JS(R)M \subseteq K \neq M$ as required. (3) $\Longrightarrow$ (2): Suppose that $I + H = R$ and H is a strongly regular subtractive left ideal of R. Setting $M = R/H$, it follows that M is a finitely generated R-semimodule which is regular with respect to every maximal subsemimodule and $IM = (I + H)/H = M$. Hence, by hypothesis $M = 0$. Since H is subtractive. $H = R$. Therefore I is superfluous. (2) $\Longrightarrow$ (1): If I is not contained in K for some regular maximal subtractive left ideal K of R, then $I+K = R$ and hence $K = R$ contradiction. Thus I is contained in every strongly regular maximal subtractive left ideal K of R and therefore $I \subseteq JS(R)$.

Corollary 3. *If R is a semiring such that $JS(R) \neq R$, then $JS(R)$ is the unique largest superfluous left ideal of R.*

Proof. Use Theorem 4.

We now show that the class of Jacobson semiradical semirings is closed under direct sum.

Proposition 8. *Let $\{R_i : i \in I\}$ be a family of semirings. Then*

$$JS\Big(\bigoplus_{i\in I} R_i\Big) = \bigoplus_{i\in I} JS(R_i).$$

Proof. Let $R = \bigoplus_{i\in I} R_i$. If M is a left R_i-semimodule, then M is a left R-semimodule with $\sum_{i=1}^{n} r_i m = r_i m$. It is clear that if M is a regular strongly austere left R_i-semimodule then M is a regular strongly austere left R-semimodule. If $r \in JS(R)$, then $rM = 0$ for all regular strongly austere left R_i-semimodules M. Thus $r \in \bigoplus_{i\in I} JS(R_i)$. Conversely if M is a left R-semimodule, then M is a left R_i-semimodule with $r_i m = \sum_{j=1}^{n} r_j m$ such that $r_i = r_j$ if $j = i$ and $r_j = 0$ for $j \neq i$. It is clear that if M is a regular strongly austere left R-semimodule then M is a regular strongly austere left R_i-semimodule. If $r \in \bigoplus_{i\in I} JS(R_i)$, then $rM = 0$ for all regular strongly austere R-semimodule M. Thus $r \in JS(R)$.

5. The Jacobson Semiradical over a Restricted Class

In this section we shall examine the semiradicals $JS(R)$ over a restricted class ρ of semirings. If we restrict our class of semirings to the class of rings, we see that $JS(R)$ coincides with classic Jacobson radical. So we call $JS(R)$ over the class ρ the Jacobson semiradical over that class.

Let ρ be the class of semirings R for which every subtractive ideal I of R satisfies the following conditions:

(1) If $i \underset{L_R(r/K)}{\equiv} i'$ then $i \underset{L_R(r/K)\cap I}{\equiv} i'$ where K is a regular modular maximal subtractive left ideal R, i, $i' \in I$ and $r \in R$.
(2) $I(r/K)$ is subtractive for every regular modular maximal subtractive left ideal K of R.
(3) If $re/L_I(i/H) = r'e/L_I(i/H)$ then $re/L_R(i/H) = r'e/L_R(i/H)$ where H is a regular modular maximal subtractive left ideal of I, $ei/H = i/H$, and e, $i \in I$.

Proposition 9. *Let $R \in \rho$. If I is a subtractive ideal of R, then $JS(I) \subseteq JS(R) \cap I$.*

Proof. If M is a regular strongly austere R-semimodule then by Lemma 4 in [5] we have either $IM = 0$ or M is an austere I-semimodule. In the latter case, if $IM \neq 0$ then M is a regular strongly austere I-semimodule. So in both cases $JS(I)M = 0$. Thus $JS(I) \subseteq JS(R) \cap I$.

Proposition 10. *Let I be a subtractive ideal of $R \in \rho$. Then $JS(I) = I \cap JS(R)$.*

Proof. By Proposition 9, $JS(I) \subseteq I \cap JS(R)$. Let M be a regular strongly austere I-semimodule, so $IM = M$. Hence $RM = RIM \subseteq IM = M$, and M is an R-semimodule. Moreover, M is a regular strongly austere left R-semimodule. Therefore $JS(R)M = 0$. Hence $I \cap JS(R) \subseteq JS(I)$ and thus $I \cap JS(R) = JS(I)$.

Corollary 4. *Let I be a subtractive ideal of R. Then*

(1) *If $JS(R) = 0$, then $JS(I) = 0$.*
(2) *$JS(JS(R)) = JS(R)$.*

Proof. (1) By Proposition 10 we have $JS(I) = JS(R) \cap I = 0 \cap I = 0$. (2) By Proposition 9 $JS(JS(R)) = JS(R) \cap JS(R) = JS(R)$.

Theorem 5. *The class $JS = \{JS(R) : R \in \rho\}$ forms a semiradical class.*

Proof. Let I be an ideal of the semiring R such that $I \in JS$ and I is not contained in $JS(R)$. Since $I \in JS$, we have $I = JS(R')$ for some semiring $R' \in \rho$. By Proposition 10 and Corollary 4, $I = JS(JS(R')) = JS(R) \cap JS(R')$. Hence $I \subseteq JS(R)$. Thus every semiring R contains a JS-ideal $JS(R)$ that contains every other JS-ideal of R. By using Proposition 2, JS is k-homomorphically closed, so by Corollary 1, $JS(R/JSR)) = 0$.

The next example describes a semiradical class σ that cannot be a radical class.

Example 1. Let $\sigma = \rho \cup R$, where R is the semiring consisting of

$$\{(\bar{0}, \{\bar{0}\}), (\bar{0}, Z/<2>), (\bar{1}, Z/<2>)\}$$

with $\bar{0}$, $\bar{1} \in Z/<2>$ and with addition $\oplus$ and multiplication $\otimes$ defined by

$$(\bar{a}, Z/<2>) \oplus (\bar{b}, Z/<2>) = (\bar{a} + \bar{b}, Z/<2>)$$

and

$$(\bar{a}, Z/<2>) \otimes (\bar{b}, Z/<2>) = (\bar{a}\bar{b}, Z/<2>).$$

Let $R' \in \rho$ be the semiring consisting of the two elements 0 and 1 with the two operations given by the tables

$\oplus$	0	1
0	0	1
1	1	1

$\otimes$	0	1
0	0	0
1	0	1

It is easy to see that the class σ is a Jacobson semiradicals class. Clearly $JS(R') = 0$ and $JS(R) = \{(\bar{0}, \{\bar{0}\}), (\bar{0}, Z/<2>)\}$. Let $f : R \to R'$ be the semiring homomorphism defined by $f(\bar{0}, \{\bar{0}\}) = 0$, $f(\bar{1}, Z/<2>) = f(\bar{0}, Z/<2>) = 1$. Clearly f is onto and $f\{(\bar{0}, \{\bar{0}\}), (\bar{0}, Z/<2>)\} = \{0, 1\}$, whence $f(JS(R))$ is not contained in $JS(R')$. Thus the class σ is semiradical but not radical.

References

[1] Huda M. J. AL-Thani, *Cogenerators of semimodules and the notion of rejects II: Regects of semimodules*, AMSE Journal, to appear.

[2] Huda M. J. AL-Thani, *The rejects and the semicogenerators*, AMSE Journal, to appear.

[3] Huda M. J. AL-Thani, *Semiradical classes*, preprint.

[4] J. S. Golan. The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science. Pitman, monographs and surveys in pure and applied mathematics. Longman Scientific & Technical, Harlow UK, 1992.

[5] Kenzo Lizuka, *On the Jacobson radical of a semiring*, Tohoku Math. J. **2**(1959), 409-421.

[6] Michihiro Takahashi, *On the bordism categories II, elementary properties of semimodules*, Math. Sem. Notes **9**(1981), 495-550.

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