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INVERSE SCATTERING PROBLEM FOR STURM–LIOUVILLE OPERATOR ON NON-COMPACT A–GRAPH. UNIQUENESS RESULT.

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Abstract. We consider a connected metric graph with the following property: each two cycles can have at most one common point. Such graphs are called A-graphs. On noncompact A-graph we consider a scattering problem for Sturm–Liouville differential operator with standard matching conditions in the internal vertices. Transport, spectral and scattering problems for differential operators on graphs appear frequently in mathematics, natural sciences and engineering. In particular, direct and inverse problems for such operators are used to construct and study models in mechanics, nano-electronics, quantum computing and waveguides. The most complete results on (both direct and inverse) spectral problems were achieved in the case of Sturm–Liouville operators on compact graphs, in the noncompact case there are no similar general results. In this paper, we establish some properties of the spectral characteristics and investigate the inverse problem of recovering the operator from the scattering data. A uniqueness theorem for such inverse problem is proved.

1. Introduction

Let Γ be a metric graph with a set of vertices $V(\Gamma)$ and a set of edges $\mathscr{E}(\Gamma)\cup\mathscr{R}(\Gamma)$, where $\mathscr{E}(\Gamma)$ is a set of compact edges and $\mathscr{R}(\Gamma)$ is a set of rays. We assume that all edges are the smooth curves which can intersect only in the vertices. We parameterize all the edges with the natural parameters and for any two points x, x' of the same edge we denote $|x-x'|$ the distance between these points along the edge (i.e., the corresponding arc length). Let $y(\cdot)$ be some function on Γ. For $x\in \operatorname{int} r$, $r\in\mathscr{E}\cup\mathscr{R}$ we define $y'(x)$ as the derivative with respect to the local natural parameter (i.e. arc length) on the edge r. Thus, we can determine the Laplacian operator $y''(x)$ for $x\in \operatorname{int} r$. Then, for $v\in V$ and $r\in\mathscr{E}(\Gamma)\cup\mathscr{R}(\Gamma)$ such that r is incident to v we define $\partial_r y(v)$ as the derivative in direction to the interior of the edge r. We denote by $I(v)$

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the set of all edges which are incident to v. The following condition in the internal vertex v is called *Kirchhoff matching condition*:

$$\sum_{r\in I(v)} \partial_r y(v) = 0. \tag{1.1}$$

Together with the following *continuity condition*:

$$\lim_{x\to v, x\in int r} y(x) = \lim_{x\to v, x\in int r'} y(x)$$

for any two edges $r, r' \in I(v)$, (1.1) forms the *standard matching conditions* which are denoted below as $MC(v)$.

Suppose that the set $\partial\Gamma$ of boundary vertices is divided into 2 parts: $\partial\Gamma = \partial_K\Gamma \cup \partial_D\Gamma$. We call the vertices from $\partial_D\Gamma$ as D*-type vertices* and the vertices from $\partial_K\Gamma$ as K*-type vertices*. For K-type vertices we assume the $MC(v)$ condition in the form (1.1) (that obviously becomes the Neumann condition), for $v \in \partial_D\Gamma$ we use the Dirichlet condition:

$$y(v) = 0 \tag{1.2}$$

in the capacity of $MC(v)$.

Now let $q(x)$ be a real-valued integrable function on Γ satisfying the following condition:

$$\int_r (1+|x|)|q(x)|\, d|x| < \infty \tag{1.3}$$

for all $r \in \mathscr{R}$, where $|x|$ is a natural parameter on r measured from the initial point of the ray. We consider the differential expression

$$\ell y := -y'' + q(x)y \tag{1.4}$$

and the Sturm–Liouville operator $L = L(\Gamma, q)$ in $L_2(\Gamma)$ which is generated by the expression (1.4) and the matching conditions $MC(v), v \in V$. More exactly, we assume $y \in domL$ iff $y \in L_2(\Gamma)$, y belongs to $W_2^2(r)$ for each $r \in \mathscr{E}$, belongs to $W_{2,loc}^2(r)$ for each $r \in \mathscr{R}$ and satisfies the matching conditions $MC(v)$ for all $v \in V$.

Transport, spectral and scattering problems for differential operators on graphs appear frequently in mathematics, natural sciences and engineering [1], [2], [3], [4], [5], [6]. In particular, direct and inverse problems for such operators are used to construct and study models in mechanics, nano-electronics, quantum computing and waveguides [7], [8].

During the last years such problems were in the focus of intensive investigations. The most complete results on (both direct and inverse) spectral problems were achieved in the case of Sturm–Liouville operators on compact graphs [9], [10], [11], [12], [13], [14], [15], where certain systems of spectra or Weyl functions were shown to be an appropriate input data for the inverse problems and where also some constructive procedures for solving these problems were developed.

In the noncompact case there are no similar general results. The presence of several noncompact edges (*rays*) and compact edges simultaneously leads to some qualitative difficulties in the investigation of the spectral problems due to the non-classical behavior of the main objects, such as Weyl–type solutions and reflection coefficients for the rays. For the first time the scattering problem on noncompact graphs was considered systematically in [16], where some useful observations were made, but complete results have been obtained only for the special case of star–type graphs. In [17] an inverse spectral problem on noncompact graphs with one ray has been investigated using Weyl functions. In [18] the authors solved a particular inverse scattering problem of recovering an operator on the ray of the simplest noncompact graph consisting of one cycle and one ray. Some results for graphs consisting of one cycle and several rays were obtained in [27], [28]. One should also mention the works [19], [20], where some non-uniqueness results were obtained for inverse scattering problems on general noncompact graphs.

In this paper we study the Sturm–Liouville operators on connected noncompact graphs with the following property: each two cycles can have at most one common point (here and everywhere in this paper cycle is a chain of different edges that forms a closed curve). Such graphs are called A-graphs. We assume that some noncompact A-graph G is given and consider the inverse problem of recovering the potential $q(x), x \in G$ of the operator $L(G, q)$. For definiteness we assume that G has at least one boundary vertex (this assumption is not necessary for the assertions of Lemma 2.7 and Corollary 2.4), let us take one of them as a root. We denote it as v^0 and the corresponding boundary edge as r^0. Our main result asserts that the potential $q(x), x \in G$ is uniquely determined by specification of (all) the following data:

- the scattering data J_r associated with each ray $r \in \mathscr{R}$;
- the Weyl functions $M_v(\cdot, G)$ associated with each boundary vertex v except the root vertex v^0;
- the Weyl functions $M_{v_\mathfrak{c}}(\cdot, G_\mathfrak{c})$ associated with each cycle $\mathfrak{c}$.

The scattering data J_r associated with the ray $r \in \mathscr{R}$ will be introduced and discussed in details in section 3. The Weyl function $M_v(\cdot, G)$ associated with the vertex $v \in V$ will be defined in section 2 (see Definition 2.1). The Weyl function $M_{v_\mathfrak{c}}(\cdot, G_\mathfrak{c})$ is defined in a similar way but for the auxiliary graph $G_\mathfrak{c}$ defined as follows (see also figure 1). Let the cycle $\mathfrak{c}$ consists of the edges (subsequently) $r_1, r_2, \dots, r_p$ that connecting v_0 with v_1, v_1 with $v_2, \dots, v_{p-1}$ with v_0, where $v_0 =: u_\mathfrak{c}$ is the vertex of $\mathfrak{c}$ nearest to the root. Then $G_\mathfrak{c}$ is a graph obtained from G by replacing the edge r_p connecting v_{p-1} and v_0 with the edge r_p' of the same length connecting

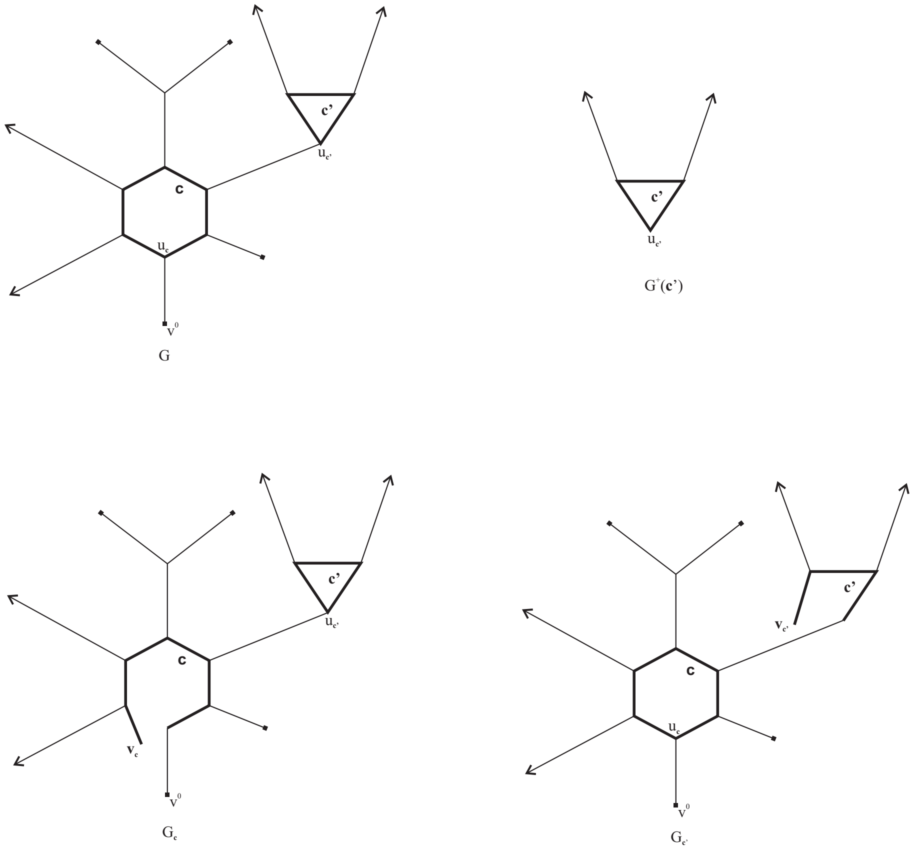

Figure 1: Auxiliary graphs $G^+(\mathfrak{c}')$, $G_\mathfrak{c}$, $G_{\mathfrak{c}'}$ for A-graph G with 2 cycles $\mathfrak{c}$ and $\mathfrak{c}'$. The rays are depicted with the arrows.

v_{p-1} and some additional vertex $v_\mathfrak{c}$. We identify the points of r'_p with the points of r_p and set $q\big|_{r'_p} = q\big|_{r_p}$, i.e. we set $q(x', G_\mathfrak{c}) := q(x, G)$ for $x' \in r'_p$, $x \in r_p$ such that $|x' - v_{p-1}| = |x - v_{p-1}|$.

The exact formulation of the above-mentioned main result is contained in Theorem 8.1. The proof of this main theorem is based on several auxiliary propositions concerning some partial inverse problems discussed in Sections 3−7.

In the rest of this section we introduce some terminology and notations that will be used in our further considerations. The edge r is called *simple edge* if it is not a part of any cycle. We agree to call simple edges and cycles as *a-edges*, i.e. a-edge is either cycle or simple edge. Let $\mathfrak{a}$ be some a-edge. The minimal number $\omega_\mathfrak{a}$ of a-edges between the rooted edge and $\mathfrak{a}$ (including $\mathfrak{a}$) is called the *order* of $\mathfrak{a}$. The order of rooted edge is equal to zero. Let $\mathscr{C}$ be the set of all cycles and $\mathscr{A}$ be the set of all a-edges. The number $\omega := \max\limits_{\mathfrak{a}\in\mathscr{A}} \omega_\mathfrak{a}$ is called the order of graph G. We denote by $\mathscr{A}^{(\mu)}$, the set of a-edges of order μ.

For given $\mathfrak{a} \in \mathscr{A}$ we define the graph $G^+(\mathfrak{a})$ as a union of all the edges r with the property:

any path containing r and the rooted edge r^0 necessarily contains some edge from $\mathfrak{a}$ (see figure 1). We call $\mathfrak{c}$ a *boundary cycle* if $G^+(\mathfrak{c}) = \mathfrak{c}$.

Also we agree that together with $L = L(G,q)$ we consider the operator $\tilde{L} = L(G,\tilde{q})$ on the same graph G but having a different potential $\tilde{q}(\cdot)$ satisfying the same conditions as $q(\cdot)$. If a certain symbol ξ denotes an object related to L, then the corresponding symbol $\tilde{\xi}$ with tilde denotes the analogous object related to $\tilde{L}$ and $\hat{\xi} := \xi - \tilde{\xi}$.

2. Eigenvalue problem Weyl functions and characteristic functions

Let Γ be an arbitrary noncompact graph (not necessarily connected and not necessarily A-graph). It is well-known that an eigenvalue problem for $L = L(\Gamma, q)$ can be reduced to some linear algebraic system by using the local fundamental system of solutions of the equation $\ell y = \lambda y$ on each edge.

Suppose that we choose and fix some (arbitrary) ordering $\prec$ on V, $\mathscr{E}$ and $\mathscr{R}$ (we agree to use the same symbol for all these three orderings). For the edge $r \in \mathscr{E}$ connecting 2 vertices u and v, where $u \prec v$ we agree to consider u as an initial vertex and v as a terminal vertex for r. Also we shall use the notation $r = [v,\infty)$ for the ray r emanating from v. Let $y(\cdot)$ be the eigenfunction corresponding to the eigenvalue $\lambda \in \mathbf{C} \setminus [0,+\infty)$. We write $y(\cdot)$ in the following form:

$$y(x) = \beta_r^1 C_r(x,\lambda) + \beta_r^2 S_r(x,\lambda), x \in r \in \mathscr{E}. \tag{2.1}$$

$$y(x) = \gamma_r e_r(x,\rho), x \in r \in \mathscr{R}, \tag{2.2}$$

Here $C_r(x,\lambda), S_r(x,\lambda)$ are the cosine- and sin- type solutions for the equation $\ell y = \lambda y$ on the edge r, i.e., $C_r(x,\lambda), S_r(x,\lambda)$ are satisfying the initial conditions:

$$C_r(u,\lambda) = \partial_r S_r(u,\lambda) = 1, \quad S_r(u,\lambda) = \partial_r C_r(u,\lambda) = 0,$$

where $u, v \in V$, $u, v \in r$, $u \prec v$. Then, $e_r(x,\rho)$ is the Jost solution for the equation $\ell y = \lambda y$ on the ray r, $\lambda = \rho^2$, $\rho \in \Omega_+ := \{\rho : \operatorname{Im}\rho > 0\}$.

In view of (2.1) and (2.2), the matching conditions $MC(v), v \in V$ reduce to some system of linear algebraic equations with respect to the values $\{\beta_r^1, \beta_r^2\}_{r\in\mathscr{E}}$, $\{\gamma_r\}_{r\in\mathscr{R}}$ and $\alpha_u := y(u), u \in V$. More precisely, we assign each compact edge r, $u, v \in r$, $u, v \in V$ with the following pair of equations:

$$\beta_r^1 C_r(u,\lambda) + \beta_r^2 S_r(u,\lambda) - \alpha_u = 0,\ \beta_r^1 C_r(v,\lambda) + \beta_r^2 S_r(v,\lambda) - \alpha_v = 0, \tag{2.3}$$

each ray $r = [v,\infty)$ with the equation:

$$\gamma_r e_r(v,\rho) - \alpha_v = 0, \tag{2.4}$$

each internal vertex v and each vertex $v \in \partial_K \Gamma$ with the equation:

$$\sum_{r \in I(v) \cap \mathscr{E}} \left(\beta_r^1 \partial_r C_r(v,\lambda) + \beta_r^2 \partial_r S_r(v,\lambda)\right) + \sum_{r \in I(v) \cap \mathscr{R}} \gamma_r \partial_r e_r(v,\rho) = 0 \tag{2.5}$$

and each vertex $v \in \partial_D \Gamma$ with the equation:

$$\alpha_v = 0. \tag{2.6}$$

We set the ordering of equations in group (2.3), (2.4) according to the ordering of edges and ordering of equations (2.5), (2.6) according to the ordering of vertices. Now we define the characteristic function $\Delta(\lambda, \Gamma, q)$ as a determinant of the system (2.3)–(2.6) with the ordering of equations described above. One can notice that, for given graph Γ and potential $q(\cdot)$, the characteristic function is uniquely determined by the ordering of the edges and vertices. Below, if the potential q is the same for all the terms in the relation, we shall often omit q in argument's list of Δ and write $\Delta(\lambda, \Gamma)$ instead of $\Delta(\lambda, \Gamma, q)$.

Let v be an arbitrary internal vertex or K-type boundary vertex. Denote $E(\Gamma, v)$ the graph Γ_0 which is constructed in the following way:

(1) replace the vertex v with the set of vertices $\{v'_r\}_{r \in I(v)}$;

(2) replace each $r \in I(v)$ connecting v and some vertex u with the edge of the same length connecting u and v'_r, replace $r = [v, \infty) \in \mathscr{R}(\Gamma)$ with the ray $[v'_r, \infty)$;

(3) add no other vertices or edges. All the $v'_r, r \in I(v)$ become boundary vertices of Γ_0, we assume that all of them are D-type vertices.

We assume that the additional vertices $v'_r, r \in I(v)$ are ordered according to the ordering of corresponding edges $r \in I(v)$. Clearly we can assume that we identify $\mathscr{E}(\Gamma_0)$ with $\mathscr{E}(\Gamma)$ and $\mathscr{R}(\Gamma_0)$ with $\mathscr{R}(\Gamma)$ and consider the Sturm–Liouville operator $L(\Gamma_0, q)$ with the same potential $q(\cdot)$.

Definition 2.1. Let us take an arbitrary vertex $v \in V(\Gamma)$. We call a function $\Phi_v(x, \lambda, \Gamma)$, defined at least for $x \in \Gamma$, $\lambda \in \mathbf{C} \setminus \mathbf{R}$ *the Weyl solution associated with* v iff:

(1) it satisfies standard matching conditions $MC(u)$ for all $u \in V(\Gamma) \setminus \{v\}$;

(2) it solves the differential equation $\ell \Phi_v = \lambda \Phi_v$, $x \in \operatorname{int} r$, $r \in \mathscr{E}(\Gamma) \cup \mathscr{R}(\Gamma)$;

(3) $\Phi_v(\cdot, \lambda, \Gamma) \in L_2(\Gamma)$;

(4) $\Phi_v(v, \lambda, \Gamma) = 1$.

The value

$$M_v(\lambda, \Gamma) := \sum_{r \in I(v)} \partial_r \Phi_v(v, \lambda, \Gamma)$$

is called *the Weyl function associated with* v.

We recall some facts concerning the properties of Weyl functions [27].

Lemma 2.1. *$M_v(\lambda,\Gamma)$ is a Nevanlinna function.*

Lemma 2.2. *For the Weyl function $M_v(\lambda,\Gamma)$ associated with the internal vertex or K-type boundary vertex v the following representation holds:*

$$M_v(\lambda,\Gamma) = \frac{\Delta(\lambda)}{\Delta_0(\lambda)},$$

where $\Delta(\lambda) = \Delta(\lambda,\Gamma)$, $\Delta_0(\lambda) = \Delta(\lambda,\Gamma_0)$ and $\Gamma_0 = E(\Gamma, v)$.

Lemma 2.3. *Suppose that the graph Γ is represented as $\Gamma = \Gamma_1 \cup \Gamma_2$, where the graphs Γ_1, Γ_2 are such that $\Gamma_1 \cap \Gamma_2 = v$. Let the ordering $\prec$ on Γ_1, Γ_2 be inherited from the ordering on Γ. If, further, v is boundary vertex for Γ_j we assume it to be a K-type vertex.*
Then the following representation holds:

$$\Delta(\lambda,\Gamma) = \Delta(\lambda,\Gamma_1)\Delta(\lambda,\Gamma_2') + \Delta(\lambda,\Gamma_1')\Delta(\lambda,\Gamma_2),$$

where $\Gamma_j' = E\left(\Gamma_j, v\right)$.

Proof. Using the representation from Lemma 2.2 we write:

$$M_v(\lambda,\Gamma) = \frac{\Delta(\lambda,\Gamma)}{\Delta(\lambda,\Gamma')},$$

where $\Gamma' = E(\Gamma, v)$. Since $\Gamma' = \Gamma_1' \cup \Gamma_2'$, $\Gamma_1' \cap \Gamma_2' = \emptyset$ we have $\Delta(\lambda,\Gamma') = \Delta(\lambda,\Gamma_1')\Delta(\lambda,\Gamma_2')$. Further, it's clear that $\Phi_v(x,\lambda,\Gamma) = \Phi_v(x,\lambda,\Gamma_j)$ for $x \in \Gamma_j$. Thus, one can easily obtain that $M_v(\lambda,\Gamma) = M_v(\lambda,\Gamma_1) + M_v(\lambda,\Gamma_2)$. Substituting here the representations from Lemma 2.2 we obtain the required relation. □

Corollary 2.1. *Let $\Gamma = \bigcup\limits_{j=1}^{p} \Gamma_j$, where $\Gamma_j \cap \Gamma_k = v$ for any $j \neq k$ and $v \in \partial_K \Gamma_j$ if $v \in \partial \Gamma_j$. Then*

$$\Delta(\lambda,\Gamma) = \sum_{k=1}^{p} \Delta(\lambda,\Gamma_k) \prod_{j\neq k} \Delta(\lambda,\Gamma_j'),$$

where $\Gamma_j' = E\left(\Gamma_j, v\right)$.

Under the conditions of Lemma 2.3 we define $C_K(\Gamma,\Gamma_1)$ as $C_K(\Gamma,\Gamma_1) := \Gamma_2$ and $C_D(\Gamma,\Gamma_1) := E(\Gamma_2, v)$. In particular, if Γ_1 is the graph consisting of one edge $r \in \mathscr{E} \cup \mathscr{R}$ then we denote the obtaining graphs as $C_K(\Gamma, r)$ and $C_D(\Gamma, r)$ and say that $C_K(\Gamma, r)$ is the graph obtaining from Γ by K-cutting-off the edge r and $C_D(\Gamma, r)$ is the graph obtaining from Γ by D-cutting-off r.

Corollary 2.2. *Let* $r = [v, \infty) \in \mathscr{R}(\Gamma)$, $v \in V(\Gamma)$. *Then*

$$\Delta(\lambda) = d_r(\rho)\Delta^r(\lambda) + d^r(\rho)\Delta_r(\lambda),$$

where $\Delta(\lambda) = \Delta(\lambda, \Gamma)$, $\Delta^r(\lambda) = \Delta(\lambda, C_K(\Gamma, r))$, $\Delta_r(\lambda) = \Delta(\lambda, C_D(\Gamma, r))$, $d_r(\rho) = e_r(v, \rho)$, $d^r(\rho) = \partial_r e_r(v, \rho)$.

Remark 2.1. It is often convenient to use both spectral parameters λ and ρ in the same formula like it has been done in Corollary 2.2. Here and everywhere below we assume $\lambda = \rho^2$ and if $\rho \in \mathbf{R} \setminus \{0\}$ we agree that $\lambda = \rho^2 + \operatorname{sgn}\rho \cdot i0$ (i.e. on the boundary of the cut in $\mathbf{C} \setminus [0, +\infty)$ one should take here and below the corresponding limit).

Corollary 2.3. *Let* $r \in \mathscr{E}(\Gamma)$ *be the edge connecting the vertices* u *and* v, *where* $u \in \partial\Gamma$. *Then*

$$\Delta(\lambda) = d_r(\lambda)\Delta^r(\lambda) + d^r(\lambda)\Delta_r(\lambda),$$

where $\Delta(\lambda) = \Delta(\lambda, \Gamma)$, $\Delta^r(\lambda) = \Delta(\lambda, C_K(\Gamma, r))$, $\Delta_r(\lambda) = \Delta(\lambda, C_D(\Gamma, r))$, $d^r(\lambda) = \Delta(\lambda, r^*)$, $d_r(\lambda) = \Delta(\lambda, r_*)$. r^*, r_* *are the graphs with one edge* r, $V(r^*) = V(r_*) = \{u, v\}$. *Further,* v *is the* D*-type boundary vertex for* r_* *and* K*-type boundary vertex for* r^*. *The type of boundary vertex* u *for both* r_* *and* r^* *is the same as in* Γ. *The orderings of the vertices* u, v *in both these graphs are the same as in* Γ *as well.*

Let us consider the zeros of the characteristic function $\Delta(\lambda, \Gamma)$. First we note that the set $\Lambda^-(\Gamma, q)$ of all negative eigenvalues of L coincides with the set of all negative zeros of $\Delta(\lambda, \Gamma)$. Let $\Lambda_0^-(\Gamma, q)$ be the set of zeros of $\Delta(\lambda, \Gamma, q)$ in $\mathbf{C} \setminus [0, +\infty)$ counted with their multiplicities. Denote $N_-(\Gamma, q) = \operatorname{card}(\Lambda_0^-(\Gamma, q))$. Proceeding as in [27] we arrive at the following assertion.

Lemma 2.4. *All zeros of* $\Delta(\lambda, \Gamma, q)$ *in* $\mathbf{C} \setminus [0, +\infty)$ *are real. Their total number admits the following estimate:*

$$N_-(\Gamma, q) \le N_0 + Q,$$

where

$$Q = \sum_{r \in \mathscr{R}} \int_r |x| \cdot |q(x)| \cdot d|x|$$

and N_0 *depends only upon* $q(x)$, $x \in \bigcup_{r \in \mathscr{E}} r$, *i.e. upon the values of* $q(\cdot)$ *on the compact part of* Γ.

Now we consider the positive zeros of $\Delta(\lambda, \Gamma)$. More exactly, let $\Lambda_0^+(\Gamma)$ be the set of all positive zeros of the function $\Delta(\Gamma, \lambda + i0)$. First we need the following estimates that can be obtained in a similar way as Lemma 2.4 in [27].

Lemma 2.5. *In terms of Corollary 2.2 the following estimates hold*

$$|\Delta(\lambda)| \geq |\Delta_r(\lambda)| \cdot \left|d_r(\rho)\right| \cdot |Im\, m_r(\lambda)|,$$
$$|\Delta(\lambda)| \geq \left|\Delta^r(\lambda)\right| \cdot \left|d^r(\rho)\right| \cdot \left|Im\, \frac{1}{m_r(\lambda)}\right|,$$

where $\rho \in \overline{\Omega}_+ \setminus \{0\}$, $m_r(\lambda) = d^r(\rho)(d_r(\rho))^{-1}$ *is the classical Weyl function for* r.

Now we can obtain the following result

Lemma 2.6. $\Lambda_0^+(\Gamma)$ *is at most countable set. The set* $Z_0^+(\Gamma) := \{\rho : \rho^2 \in \Lambda_0^+(\Gamma)\}$ *has the following property: for any segment* $[t, t+1]$ *the number of elements of* $Z_0^+(\Gamma)$ *lying in this segment is bounded by some constant which does not depend on* t.

Proof. Since for any positive ρ one has $d_r(\rho) \neq 0$ and $\mathrm{Im} m_r(\lambda + i0) > 0$ we conclude that $\Delta(\lambda + i0, \Gamma) = 0$ implies $\Delta(\lambda + i0, \Gamma') = 0$, where $\Gamma' = C_K(\Gamma, r)$. We can repeat this and cut-off subsequently all the rays. Thus, any $\lambda_0 \in \Lambda_0^+(\Gamma)$ must be a zero of $\Delta(\lambda, \Gamma_c)$, where Γ_c is the compact graph obtained from Γ by cutting-off all the rays. For compact graphs validity of assertion of the Lemma is well-known [15]. □

Now we consider the characteristic function $\Delta(\lambda, G)$ of (arbitrary) A-graph G. Denote $|r|$ the length of the edge $r \in \mathscr{E}(G)$ and $|G| := \sum_{r \in \mathscr{E}(G)} |r|$. Define the set $\mathscr{E}^{\pm} = \{\sum_{r \in \mathscr{E}} \varepsilon_r |r| : \varepsilon_r \in \{-1, 0, 1\}\}$.

Lemma 2.7. *For* $\lambda = \rho^2$, $\rho \to \infty$, $\rho \in \overline{\Omega}_+$ *the following asymptotical representation holds:*

$$\Delta(\lambda, G) = \left(\frac{i}{2\rho}\right)^{N(G)-1} \Big(\sum_{l \in \mathscr{E}^{\pm}} B_l(G) \exp(-i\rho l) + O(\rho^{-1} \exp(\tau |G|))\Big),$$

where $N(G) = N_D(G) + N_{\mathscr{C}}(G)$, $N_D(G)$ *is the number of the* D*-type boundary vertices,* $N_{\mathscr{C}}(G)$ *is the number of cycles,* $\tau = Im\rho$ *and* $B_l(G)$ *are the constants that do not depend upon the potential* $q(\cdot)$. *Moreover, all the* $B_l(G)$, $l \in \mathscr{E}^{\pm}$ *are real and* $B_{|G|}(G) \neq 0$.

Proof. We use the induction with respect to the number of edges. For any one-edge graph (i.e. graph consisting of one simple edge or one one-edge cycle) the required assertion can be obtained via the direct calculation. Now we assume the assertion to be true for any A-graph with less than n edges and consider an arbitrary A-graph G with n edges. Let us take some internal vertex v such that $I(v)$ contains at least 3 edges (if we could not find such vertex the situation is actually equivalent to the case of one-edge graph mentioned above). Then we can represent G as $G = \bigcup_{k=1}^{p} G_k$, where:

- all G_j are A-graphs with less then n edges;

- for any $j \neq k$ $G_j \cap G_k = v$;
- if $v \in \partial G_j$ then $v \in \partial_K G_j$;
- each G_j has exactly 1 a-edge containing vertex v.

The last requirement guarantees, in particular, that all $G'_j := E(G_j, v)$ are A-graphs as well. Thus we can use the representation from Corollary 2.1 for $\Delta(\lambda, G)$ and (by the inductive assumption) assertion of Lemma for each of $\Delta(\lambda, G_j)$, $\Delta(\lambda, G'_j)$.

Let us consider the values of $N(G_j)$ and $N(G'_j)$. Note that v is either the boundary vertex for G_j or the vertex belonging to some cycle of G_j. In the first case we have $N_{\mathscr{C}}(G'_j) = N_{\mathscr{C}}(G_j)$, $N_D(G'_j) = N_D(G_j)+1$. In the second case we have $N_{\mathscr{C}}(G'_j) = N_{\mathscr{C}}(G_j)-1$, $N_D(G'_j) = N_D(G_j)+2$ and in both cases we obtain $N(G'_j) = N(G_j)+1$. Since $\sum\limits_{j=1}^{p} N(G_j) = N(G)$ the calculation described above yields the required representation for $\Delta(\lambda, G)$ with the constants $B_l(G)$ that are real and independent of $q(\cdot)$ (because this was true for all $B_l(G_k)$, $B_l(G'_k)$). Now we are to control the value of $B_{|G|}(G)$. Simple algebra yields:

$$B_{|G|}(G) = \sum_{k=1}^{p} B_{|G_k|}(G_k) \prod_{j\neq k} B_{|G_j|}(G'_j).$$

By the inductive assumption we have $B_{|G_j|}(G'_j) \neq 0$ and we can rewrite the last relation as follows:

$$B_{|G|}(G) = \prod_{j=1}^{p} B_{|G_j|}(G'_j) \sum_{k=1}^{p} \frac{B_{|G_k|}(G_k)}{B_{|G_k|}(G'_k)}. \tag{2.7}$$

Let us consider the Weyl functions $M_v(\lambda, G_j)$. The representation from Lemma yields the following asymptotics for $\rho \to \infty$, $0 < \alpha < \arg\rho < \beta < \pi/2$:

$$M_v(\lambda, G_j) = \frac{\Delta(\lambda, G_j)}{\Delta(\lambda, G'_j)} = -2i\rho \frac{B_{|G_j|}(G_j)}{B_{|G_j|}(G'_j)}(1+o(1)).$$

Since $M_v(\lambda, G_j)$ are Nevanlinna functions we conclude that all $B_{|G_j|}(G_j)\left(B_{|G_j|}(G'_j)\right)^{-1}$ are real and negative. This means that the sum in right-hand side of (2.7) is nonzero and consequently $B_{|G|}(G) \neq 0$. □

Let us agree to use the notation A_ε, $\varepsilon > 0$ for (different) sets of the form $A_\varepsilon = \{\rho \in \overline{\Omega}_+ : \operatorname{dist}(\rho, Z) \geq \varepsilon\}$, where $Z \subset \{\rho : 0 \leq \operatorname{Im}\rho \leq \tau_0\}$ is some at most countable set with the property: for any real t the number of elements of Z lying in the rectangle $\{\operatorname{Re}\rho \in [t, t+1], \operatorname{Im}\rho \in [0, \tau_0]\}$ is bounded by some constant which does not depend on t.

From Lemma 2.7 using standard methods [31] one can deduce the following result.

Corollary 2.4. *For $|\rho| > \rho_*$, $\rho \in A_\varepsilon$ the following estimates hold:*

$$C_1|\rho|^{1-N(G)}\exp(\tau|G|) < |\Delta(\lambda, G)| < C_2|\rho|^{1-N(G)}\exp(\tau|G|).$$

3. Partial inverse scattering problem on the ray

Let us take an arbitrary ray $r \in \mathscr{R}(G)$. We call the function $\psi_r(x,\rho)$, $x \in G$, $\rho \in \Omega_+$ *the Weyl–type solution associated with* r iff:

(1) it satisfies standard matching conditions $MC(v)$ for all $v \in V$;

(2) it solves the differential equation $\ell\psi_r = \rho^2\psi_r$, $x \in \operatorname{int} r'$, $r' \in \mathscr{E}(G) \cup \mathscr{R}(G)$;

(3) $\psi_r(x,\rho) = O\big(\exp(i\rho|x|)\big)$ as $x \to \infty$, $x \in r'$, $r' \in \mathscr{R} \setminus \{r\}$;

(4) $\psi_r(x,\rho) = \exp(-i\rho|x|)(1 + o(1))$ as $x \to \infty$, $x \in r$.

Proceeding in a similar way as in [27] one can obtain the following results.

Lemma 3.1. *For $x \in r$ $\psi_r(x,\rho)$ is meromorphic with respect to ρ in Ω_+ with possible poles on the imaginary axis.*

We denote the set of poles of $\psi_r(x,\rho)$, $x \in r$ as Z_r^-.

Lemma 3.2. *Z_r^- is a finite set. If $\rho_0 \in Z_r^-$ then $\lambda_0 = \rho_0^2 \in \Lambda^-$.*

Lemma 3.3. *All poles of $\psi_r(x,\rho)$, $x \in r$ are simple. For the residue* $\operatorname{res}_{\rho=\rho_0}\psi_r(x,\rho)$, $\rho_0 \in Z_r^-$ *the following representation holds:*

$$\operatorname{res}_{\rho=\rho_0}\psi_r(x,\rho) = i\alpha_r(\rho_0)e_r(x,\rho_0).$$

The values $\alpha_r(\rho_0)$ are all real and positive.

Denote Z_0^+ the set of all $\rho \in \mathbf{R}$ such that $\lambda = \rho^2 \in \Lambda_0^+ := \Lambda_0^+(G)$.

Lemma 3.4. *If $\rho_0 \in \mathbf{R} \setminus \big(\{0\} \cup Z_0^+\big)$ then there exists the limit $\psi_r(x,\rho_0) := \lim\limits_{\rho\to\rho_0,\rho\in\Omega_+}\psi_r(x,\rho)$. If $\rho_0 \in Z_0^+$ then $\psi_r(x,\rho)$ and $\psi_r'(x,\rho)$ are bounded as $\rho \to \rho_0$, $\rho \in \Omega_+$.*

Lemma 3.5. *For $\psi_r(x,\rho)$, $\rho \in \mathbf{R} \setminus \big(\{0\} \cup Z_0^+\big)$ the following representation holds:*

$$\psi_r(x,\rho) = e_r(x,-\rho) + s_r(\rho)e_r(x,\rho), \quad x \in r.$$

We call the function $s_r(\cdot)$, *the reflection coefficient associated with* r.

Lemma 3.6. *For all $\rho \in \mathbf{R} \setminus \big(\{0\} \cup Z_0^+\big)$ one has $s_r(-\rho) = \overline{s_r(\rho)}$ and $|s_r(\rho)| \le 1$.*

Lemma 3.7. *$\psi_r(x,\rho)$, $\psi_r'(x,\rho)$, $x \in r$ are bounded as $\rho \to 0$, $\rho \in \overline{\Omega}_+$.*

Lemma 3.8. *For $x \in r$, $\rho \to \infty$, $\rho \in A_\varepsilon$ the following estimates hold:*

$$\psi_r(x,\rho) = O\big(\exp(-i\rho|x|)\big),\ \psi_r'(x,\rho) = O\big(\rho\exp(-i\rho|x|)\big),$$

$$\hat{\psi}_r(x,\rho) = O\big(\rho^{-1}\exp(-i\rho|x|)\big).$$

Proof. In order to obtain the asymptotics for $\psi_r(x,\rho)$ it is convenient to use the following representation, that can be obtained by direct calculation:

$$\psi_r(x,\rho) = \gamma_r(\rho)e_r(x,\rho) + \delta_r(\rho)S_r(x,\lambda), \quad x \in r, \tag{3.1}$$

where

$$\delta_r(\rho) = -\frac{2i\rho}{d_r(\rho)}, \tag{3.2}$$

$$\gamma_r(\rho) = \frac{2i\rho}{d_r(\rho)} \cdot \frac{\Delta_r(\lambda)}{\Delta(\lambda)} \tag{3.3}$$

and $\Delta_r(\lambda)$ is the characteristic function for $G_r := C_D(G,r)$ (we recall that $d_r(\rho) = e_r(v,\rho)$).

First we estimate $\gamma_r(\rho)$. Using Corollary 2.4 and taking into account that $|G_r| = |G|$ and $N(G_r) \ge N(G)+1$ we obtain

$$\frac{\Delta_r(\lambda)}{\Delta(\lambda)} \le \frac{C}{|\rho|}$$

that yields

$$\gamma_r(\rho) = O(1),\ \rho \to \infty,\ \rho \in A_\varepsilon. \tag{3.4}$$

Now consider $\hat{\gamma}_r(\rho)$. From Lemma 2.7 and Corollary 2.4 one can deduce the following estimates that hold for $|\rho| > \rho_*$, $\rho \in A_\varepsilon$:

$$\frac{\hat{\Delta}(\lambda)}{\Delta(\lambda)} = O\left(\frac{1}{\rho}\right),\ \frac{\hat{\Delta}_r(\lambda)}{\Delta_r(\lambda)} = O\left(\frac{1}{\rho}\right).$$

This yields

$$\hat{\gamma}_r(\rho) = O\big(\rho^{-1}\big) \tag{3.5}$$

for $|\rho| > \rho_*$, $\rho \in A_\varepsilon$.

To complete the proof it is sufficient to use the estimates (3.4), (3.5), the obvious estimates:

$$\delta_r(\rho) = O(\rho),\ \hat{\delta}_r(\rho) = O(1)$$

and the classical asymptotics:

$$e_r^{(\nu)}(x,\rho) = (i\rho)^\nu e^{i\rho|x|}\big(1 + O(\rho^{-1})\big),\ \hat{e}_r(x,\rho) = O\left(\rho^{-1}e^{i\rho|x|}\right),$$
$$S_r^{(\nu)}(x,\lambda) = O\left(\rho^{\nu-1}e^{-i\rho|x|}\right),\ \hat{S}_r(x,\lambda) = O\left(\rho^{-2}e^{-i\rho|x|}\right). \qquad \square$$

Definition 3.2. The data $J_r := \{s_r(\cdot), Z_r^-, \alpha_r(\rho), \rho \in Z_r^-\}$ are called *the scattering data, associated with* r.

Problem IP1(r). Given J_r, recover the potential $q(x)$ for $x \in r$.

Theorem 3.1. *If $J_r = \tilde{J}_r$ then $q = \tilde{q}$ a.e. on r, i.e. the potential on the ray r is uniquely determined by the scattering data, associated with r. Moreover, $M_v(\cdot, G) = \tilde{M}_v(\cdot, G)$.*

Proof. Consider for $x \in r$, $\lambda \in \mathbf{C} \setminus [0, +\infty)$ the following functions:

$$\varphi_1(x,\lambda) := \psi_r(x,\rho), \quad \varphi_2(x,\lambda) := e_r(x,\rho), \quad \lambda = \rho^2, \rho \in \Omega_+.$$

Let us define the matrices

$$\Psi(x,\lambda) := \begin{bmatrix} \varphi_1(x,\lambda) & \varphi_2(x,\lambda) \\ \varphi_1'(x,\lambda) & \varphi_2'(x,\lambda) \end{bmatrix}$$

and $\tilde{\Psi}(x,\lambda)$ and introduce the *spectral mapping matrix*:

$$P(x,\lambda) := \Psi(x,\lambda)\tilde{\Psi}^{-1}(x,\lambda).$$

It follows from Lemma 3.5 that for the limit-value matrices $\Psi^{\pm}(x,\lambda) := \Psi(x, \lambda \pm i0)$, $\lambda \in (0, +\infty) \setminus \Lambda_0^+$ the following relation holds:

$$\Psi^-(x,\lambda) = \Psi^+(x,\lambda)w(\lambda),$$

where

$$w(\lambda) = \begin{bmatrix} \overline{s_r(\rho)} & 1 \\ 1 - |s_r(\rho)|^2 & -s_r(\rho) \end{bmatrix}, \quad \lambda = \rho^2, \rho \in (0, +\infty).$$

Suppose that $s_r = \tilde{s}_r$. Then $w = \tilde{w}$ and consequently $P^+(x,\lambda) = P^-(x,\lambda)$, $\lambda \in (0,+\infty) \setminus (\Lambda_0^+ \cup \tilde{\Lambda}_0^+)$. This means that $P(x,\lambda)$ is holomorphic in $\lambda \in \mathbf{C} \setminus \left(\{0\} \cup \Lambda_0^+ \cup \tilde{\Lambda}_0^+ \cup \Lambda_r^- \cup \tilde{\Lambda}_r^-\right)$, where $\Lambda_r^- = \{\lambda = \rho^2, \rho \in Z_r^-\}$. Take an arbitrary $\lambda_0 \in (0, +\infty) \cap (\Lambda_0^+ \cup \tilde{\Lambda}_0^+)$. It follows from Lemma 3.4 that $P(x,\lambda)$ is bounded in the neighborhood of λ_0, so λ_0 is a removable singularity for $P(x,\lambda)$.

Then, $J_r = \tilde{J}_r$ means in particular that $Z_r^- = \tilde{Z}_r^-$. Taking an arbitrary $\lambda_0 = \rho_0^2$, $\rho_0 \in Z_r^-$ we can conclude that λ_0 is either a pole or a removable singularity for $P(x,\lambda)$. Let us consider the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$. One has:

$$P_{11}(x,\lambda) = \frac{1}{2i\rho}\left(\psi_r(x,\rho)\tilde{e}_r'(x,\rho) - \tilde{\psi}_r'(x,\rho)e_r(x,\rho)\right),$$
$$P_{12}(x,\lambda) = \frac{1}{2i\rho}\left(\tilde{\psi}_r(x,\rho)e_r(x,\rho) - \psi_r(x,\rho)\tilde{e}_r(x,\rho)\right).$$

Substituting here the representations

$$\psi_r(x,\rho) = \frac{i\alpha_r(\rho_0)}{\rho-\rho_0} e_r(x,\rho_0) + O(1), \quad \rho \to \rho_0,$$
$$\tilde{\psi}_r(x,\rho) = \frac{i\tilde{\alpha}_r(\rho_0)}{\rho-\rho_0} \tilde{e}_r(x,\rho_0) + O(1), \quad \rho \to \rho_0,$$

and taking into account that $\alpha_r(\rho_0) = \tilde{\alpha}_r(\rho_0)$ we obtain $P_{11}(x,\lambda) = O(1)$, $P_{12}(x,\lambda) = O(1)$ in a neighborhood of λ_0. Thus λ_0 is a removable singularity.

Then, using Lemma 3.8 and the classical asymptotics for the Jost solution $e_r(x,\rho)$, one can obtain the estimates:

$$P_{11}(x,\lambda) - 1 = O\left(\frac{1}{\rho}\right), \quad P_{12}(x,\lambda) = O\left(\frac{1}{\rho}\right), \quad \lambda \to \infty, \rho^2 = \lambda, \rho \in A_\varepsilon.$$

On the other hand Lemma 3.7 yields:

$$P_{11}(x,\lambda) - 1 = O\left(\frac{1}{\rho}\right), \quad P_{12}(x,\lambda) = O\left(\frac{1}{\rho}\right), \quad \lambda \to 0, \rho^2 = \lambda.$$

These estimates together mean that actually $P_{11}(x,\lambda) - 1 = 0$, $P_{12}(x,\lambda) = 0$, i.e. $\varphi_\nu(x,\lambda) = \tilde{\varphi}_\nu(x,\lambda)$, $\nu = 1,2$ and, consequently, $q(x) = \tilde{q}(x)$ for a.e. $x \in r$. Notice, that in particular we have $\psi_r(x,\rho) = \tilde{\psi}_r(x,\rho)$, $x \in r$, $\rho \in \overline{\Omega}_+ \setminus Z_r^-$.

Since we have $\psi_r(x,\rho) = \psi_r(v,\rho)\cdot\Phi_v(x,\lambda,G^r)$, $x \in G^r := C_K(G,r)$ the matching conditions $MC(v)$ for $\Phi_v(x,\lambda,G^r)$ yield:

$$\frac{\partial_r\psi_r(v,\rho)}{\psi_r(v,\rho)} + M_v(\lambda,G^r) = 0$$

and we obtain $M_v(\lambda,G^r) = \tilde{M}_v(\lambda,G^r)$. Finally, since we have

$$M_v(\lambda,G) = M_v(\lambda,G^r) + m_r(\lambda)$$

we can conclude now that $M_v(\lambda,G) = \tilde{M}_v(\lambda,G)$. □

4. Partial inverse spectral problem for compact boundary edge

Let us consider some edge $r \in \mathscr{E}$ connecting the vertices u and v, where v is a boundary vertex.

Problem IP2(r). Given the Weyl function $M_v(\cdot,G)$, recover the potential $q(x)$ for $x \in r$.

In our studying this problem we follow the standard scheme of the spectral mapping method [14], [26]. First we need some asymptotics for Weyl solution $\Phi_v(x,\lambda)$, $x \in r$.

Lemma 4.1. *For $\lambda = \rho^2$, $\rho \to \infty$, $\rho \in A_\varepsilon$ with any $\varepsilon > 0$ the following asymptotics hold:*

$$\Phi_v(x,\lambda) = O\big(\exp(-\tau|x-v|)\big),\ \Phi'_v(x,\lambda) = O\big(\rho\exp(-\tau|x-v|)\big),$$
$$\hat{\Phi}_v(x,\lambda) = O\big(\rho^{-1}\exp(-\tau|x-v|)\big),$$

where $\tau = Im\rho$ and the derivative $\Phi'_v(x,\lambda)$ is considered with respect to the natural parameter measured along the edge r from the vertex v.

Proof. For definiteness we assume that v is of D-type (otherwise the representations below using the characteristic functions require slight modifications but the result remains the same).

We use the representation:

$$\Phi_v(x,\lambda) = \gamma_r(\lambda)S_{r,v}(x,\lambda) + \delta_r(\lambda)S_{r,u}(x,\lambda),\ x \in r, \tag{4.1}$$

where $S_{r,v}(x,\lambda)$, $S_{r,u}(x,\lambda)$ are the (local) solutions for the equation $\ell y = \lambda y$ on the edge r normalized by the initial conditions: $S_{r,v}(v,\lambda) = S_{r,u}(u,\lambda) = 0$, $\partial_r S_{r,v}(v,\lambda) = \partial_r S_{r,u}(u,\lambda) = 1$. Direct calculation yields the following representations for the coefficients $\gamma_r(\lambda)$, $\delta_r(\lambda)$:

$$\gamma_r(\lambda) = -\frac{1}{d_r(\lambda)}\frac{\Delta_r(\lambda)}{\Delta(\lambda)},\ \delta_r(\lambda) = \frac{1}{d_r(\lambda)}, \tag{4.2}$$

where we use the same notations as in Corollary 2.3.

First we estimate $\gamma_r(\lambda)$. Using Corollary 2.4 and taking into account that $|G_r| = |G| - |r|$ and $N(G_r) \geq N(G)$ we obtain

$$\frac{\Delta_r(\lambda)}{\Delta(\lambda)} \leq C\exp(-\tau|r|).$$

Together with the classical estimate for $d_r(\lambda) = S_{r,v}(u,\lambda)$:

$$|d_r(\lambda)| \geq C|\rho|^{-1}\exp(\tau|r|)$$

that yields

$$\big|\gamma_r(\lambda)\big| \leq C|\rho|\exp(-2\tau|r|). \tag{4.3}$$

Now consider $\hat{\gamma}_r(\lambda)$. From Lemma 2.7 and Corollary 2.4 one can deduce the following estimates that hold for $|\rho| > \rho_*$, $\rho \in A_\varepsilon$:

$$\frac{\hat{\Delta}(\lambda)}{\Delta(\lambda)} = O\left(\frac{1}{\rho}\right),\ \frac{\hat{\Delta}_r(\lambda)}{\Delta_r(\lambda)} = O\left(\frac{1}{\rho}\right).$$

This yields

$$\big|\hat{\gamma}_r(\lambda)\big| \leq C\exp(-2\tau|r|). \tag{4.4}$$

for $|\rho| > \rho_*$, $\rho \in A_\varepsilon$.

Next, for δ_r, $\hat{\delta}_r$ we obtain from (4.2) and classical asymptotics the following estimates:

$$|\delta_r(\lambda)| \le C|\rho|\exp(-\tau|r|),\ \left|\hat{\delta}_r(\lambda)\right| \le C\exp(-\tau|r|). \tag{4.5}$$

In order to complete the proof it is sufficient now to use the representation (4.1), the estimates (4.4), (4.5) and the following classical asymptotics for the local solutions:

$$\begin{aligned}
S_{r,v}(x,\lambda) &= \rho^{-1}\sin\rho|x-v| + O(\rho^{-2}\exp(\tau|x-v|)),\\
S'_{r,v}(x,\lambda) &= \cos\rho|x-v| + O(\rho^{-1}\exp(\tau|x-v|)),\\
S_{r,u}(x,\lambda) &= \rho^{-1}\sin\rho|x-u| + O(\rho^{-2}\exp(\tau|x-u|)),\\
S'_{r,u}(x,\lambda) &= -\cos\rho|x-u| + O(\rho^{-1}\exp(\tau|x-u|)).
\end{aligned}$$

□

Theorem 4.1. *If $M_v(\cdot,G) = \tilde{M}_v(\cdot,G)$ then $q = \tilde{q}$ a.e. on r. Moreover, $M_u(\cdot,G) = \tilde{M}_u(\cdot,G)$.*

Proof. Proceeding as in proof of Theorem 3.1 with the conventional arguments of spectral mapping method we define the matrices:

$$\Psi(x,\lambda) := \begin{bmatrix} \Phi_v(x,\lambda) & S_{r,v}(x,\lambda) \\ \Phi'_v(x,\lambda) & S'_{r,v}(x,\lambda) \end{bmatrix}$$

and $\tilde{\Psi}(x,\lambda)$ and introduce the spectral mappings matrix:

$$P(x,\lambda) := \Psi(x,\lambda)\tilde{\Psi}^{-1}(x,\lambda),\ x \in r.$$

Here, as in previous Lemma the derivatives are considered with respect to the natural parameter measured along the edge r from the vertex v.

Using the representations

$$\begin{aligned}
P_{11}(x,\lambda) &= \Phi_v(x,\lambda)\tilde{S}'_{r,v}(x,\lambda) - \tilde{\Phi}'_v(x,\lambda)S_{r,v}(x,\lambda),\\
P_{12}(x,\lambda) &= \tilde{\Phi}_v(x,\lambda)S_{r,v}(x,\lambda) - \Phi_v(x,\lambda)\tilde{S}_{r,v}(x,\lambda),
\end{aligned}$$

and Lemma 4.1 we obtain the estimates:

$$P_{11}(x,\lambda) - 1 = O\left(\rho^{-1}\right),\ P_{12}(x,\lambda) = O\left(\rho^{-1}\right),\ \rho \to \infty, \rho \in A_\varepsilon. \tag{4.6}$$

On the other hand from the same representations and $M_v(\cdot,G) = \tilde{M}_v(\cdot,G)$ it follows that $P_{11}(x,\lambda) - 1$ and $P_{12}(x,\lambda)$ are entire functions with respect to λ. In view of (4.6) we conclude that actually $P_{11}(x,\lambda) - 1 \equiv 0$ and $P_{12}(x,\lambda) \equiv 0$. Thus, we have $\Phi_v(x,\lambda) = \tilde{\Phi}_v(x,\lambda)$ and consequently $q = \tilde{q}$ a.e. on r.

Further, it is clear that $\Phi_v(x,\lambda,G) = \Phi_v(u,\lambda,G)\cdot\Phi_u(x,\lambda,G^r)$, $x \in G^r := C_K(G,r)$. Thus the matching conditions $MC(u)$ for $\Phi_v(x,\lambda,G)$ yield:

$$\frac{\partial_r \Phi_v(u,\lambda,G)}{\Phi_v(u,\lambda,G)} + M_u(\lambda,G^r) = 0$$

and we obtain $M_u(\lambda,G^r) = \tilde{M}_u(\lambda,G^r)$. On the other hand the same considerations yield

$$M_u(\lambda,G) = M_u(\lambda,r^*) + M_u(\lambda,G^r)$$

(r^* is the same one-edge graph as in Corollary 2.3). Since (as it has been already proven) $q\,|_r = \tilde{q}\,|_r$ we obtain finally: $M_u(\lambda,G) = \tilde{M}_u(\lambda,G)$. □

5. Partial inverse spectral problem for internal simple edge

Let r be internal simple edge connecting the vertices u and v, where u is nearer to the root than v.

Problem IP3(r). Given the Weyl function $M_v(\cdot,G)$, and $q\,\big|_{G^+(r)\setminus r}$, recover $q\,|_r$.

Theorem 5.1. *If $M_v(\cdot,G) = \tilde{M}_v(\cdot,G)$ and $q\,\big|_{G^+(r)\setminus r} = \tilde{q}\,\big|_{G^+(r)\setminus r}$ then $q\,|_r = \tilde{q}\,|_r$. Moreover, $M_u(\cdot,G) = \tilde{M}_u(\cdot,G)$.*

Proof. Define $G_0^+(r) := C_K\big(G^+(r), r\big)$, $G_0^-(r) := C_K\big(G, G_0^+(r)\big)$. Since

$$M_v(\lambda,G) = M_v(\lambda,G_0^+(r)) + M_v(\lambda,G_0^-(r)),$$

under the conditions of Theorem we have $M_v(\lambda,G_0^-(r)) = \tilde{M}_v(\lambda,G_0^-(r))$ that by virtue of Theorem 4.1 yields $q\,|_r = \tilde{q}\,|_r$. This means, in turn that $q\,\big|_{G^+(r)} = \tilde{q}\,\big|_{G^+(r)}$ and

$$M_u(\lambda,G_0^+(r)) = \tilde{M}_u(\lambda,G_0^+(r)). \tag{5.1}$$

Further, $M_v(\lambda,G_0^-(r)) = \tilde{M}_v(\lambda,G_0^-(r))$ implies

$$\Phi_v(x,\lambda,G_0^-(r)) = \tilde{\Phi}_v(x,\lambda,G_0^-(r)).\ x \in r \tag{5.2}$$

Notice that the matching conditions $MC(u)$ for $\Phi_v(x,\lambda,G_0^-(r))$ yield:

$$\frac{\partial_r \Phi_v(u,\lambda,G_0^-(r))}{\Phi_v(u,\lambda,G_0^-(r))} + M_u(\lambda,G^-(r)) = 0.$$

In view of (5.2) this means that

$$M_u(\lambda,G^-(r)) = \tilde{M}_u(\lambda,G^-(r)).$$

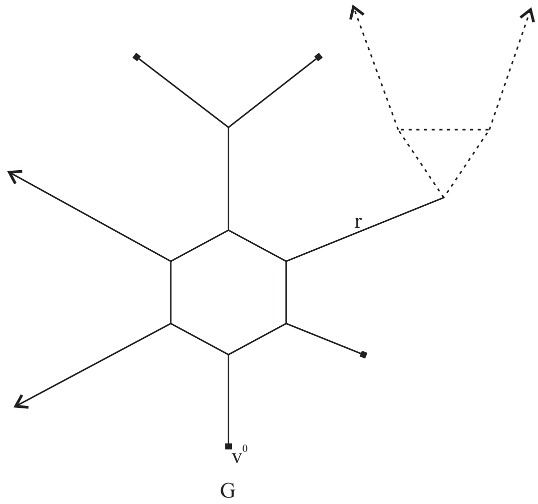

Figure 2: To the formulation of the Problem IP3(r) for internal simple edge r. The dashed lines depict the edges with a priori known potential.

From this, taking into account (5.1) and the relation

$$M_u(\lambda, G) = M_u(\lambda, G^-(r)) + M_u(\lambda, G^+(r))$$

we obtain $M_u(\lambda, G) = \tilde{M}_u(\lambda, G)$ and this completes the proof. $\square$

6. Partial inverse spectral problem for boundary cycle

Now we consider some boundary cycle $\mathfrak{c} \in \mathscr{C}$.

Problem IP4($\mathfrak{c}$)**.** Given the Weyl function $M_{v_\mathfrak{c}}(\cdot, G_\mathfrak{c})$, recover the potential $q|_\mathfrak{c}$.

Theorem 6.1. *If $M_{v_\mathfrak{c}}(\cdot, G_\mathfrak{c}) = \tilde{M}_{v_\mathfrak{c}}(\cdot, G_\mathfrak{c})$ then $q = \tilde{q}$ a.e. on $\mathfrak{c}$. Moreover, $M_{u_\mathfrak{c}}(\cdot, G) = \tilde{M}_{u_\mathfrak{c}}(\cdot, G)$.*

Proof. First, we can use Theorem 4.1 and conclude that $q\Big|_{r'_p} = \tilde{q}\Big|_{r'_p}$ and $M_{v_{p-1}}(\lambda, G_\mathfrak{c}) = \tilde{M}_{v_{p-1}}(\lambda, G_\mathfrak{c})$. Then, using Theorem 5.1 we obtain for $j = p-1, \ldots, 1$ subsequently: $q\Big|_{r_j} = \tilde{q}\Big|_{r_j}$ and $M_{v_{j-1}}(\lambda, G_\mathfrak{c}) = \tilde{M}_{v_{j-1}}(\lambda, G_\mathfrak{c})$. Finally we conclude that $q|_\mathfrak{c} = \tilde{q}|_\mathfrak{c}$ and $M_{u_\mathfrak{c}}(\cdot, G_\mathfrak{c}) = \tilde{M}_{u_\mathfrak{c}}(\cdot, G_\mathfrak{c})$.

Define $G^-(\mathfrak{c}) := C_K(G, G^+(\mathfrak{c}))$. Since

$$M_{u_\mathfrak{c}}(\lambda, G_\mathfrak{c}) = M_{u_\mathfrak{c}}(\lambda, G^+_\mathfrak{c}(r_0)) + M_{u_\mathfrak{c}}(\lambda, G^-(\mathfrak{c})).$$

and (as it has been actually proven) $q\Big|_{G^+_\mathfrak{c}(r_0)} = \tilde{q}\Big|_{G^+_\mathfrak{c}(r_0)}$ we have:

$$M_{u_\mathfrak{c}}(\lambda, G^-(\mathfrak{c})) = \tilde{M}_{u_\mathfrak{c}}(\lambda, G^-(\mathfrak{c})).$$

Taking into account that

$$M_{u_{\mathfrak{c}}}(\lambda, G) = M_{u_{\mathfrak{c}}}(\lambda, G^-(\mathfrak{c})) + M_{u_{\mathfrak{c}}}(\lambda, G^+(\mathfrak{c}))$$

and $q\big|_{G^+(\mathfrak{c})} = \tilde{q}\big|_{G^+(\mathfrak{c})}$ we obtain finally $M_{u_{\mathfrak{c}}}(\lambda, G) = \tilde{M}_{u_{\mathfrak{c}}}(\lambda, G)$. □

7. Partial inverse spectral problem for internal cycle

Consider some internal cycle $\mathfrak{c} \in \mathscr{C}$.

Problem IP5($\mathfrak{c}$)**.** Given $M_{v_{\mathfrak{c}}}(\cdot, G_{\mathfrak{c}})$ and $q(x)$, $x \in G^+(\mathfrak{c}) \setminus \mathfrak{c}$, recover the potential $q|_{\mathfrak{c}}$.

Theorem 7.1. *If $M_{v_{\mathfrak{c}}}(\cdot, G_{\mathfrak{c}}) = \tilde{M}_{v_{\mathfrak{c}}}(\cdot, G_{\mathfrak{c}})$ and $q(x) = \tilde{q}(x)$, $x \in G^+(\mathfrak{c}) \setminus \mathfrak{c}$ then $q|_{\mathfrak{c}} = \tilde{q}|_{\mathfrak{c}}$. Moreover, $M_{u_{\mathfrak{c}}}(\cdot, G) = \tilde{M}_{u_{\mathfrak{c}}}(\cdot, G)$.*

Proof. It is sufficient to repeat the arguments from the proof of Theorem 6.1. □

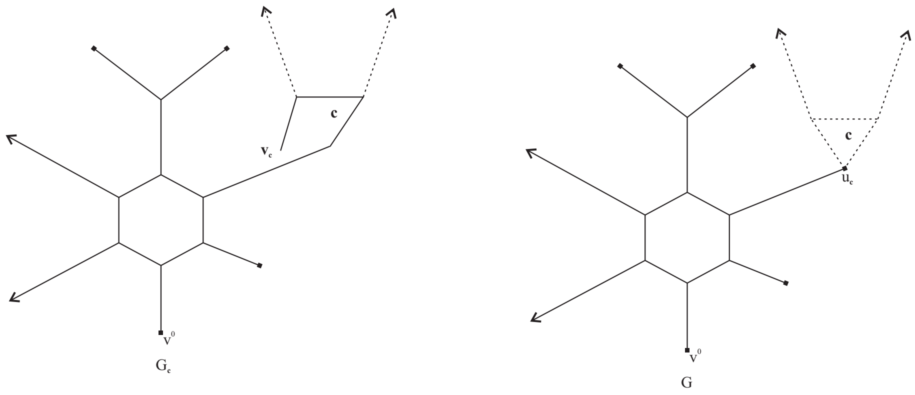

Figure 3: To the formulation of the Problem IP5($\mathfrak{c}$) for internal cycle $\mathfrak{c}$. The dashed lines on the left picture depict the edges with a priori known potential. The dashed lines on the right picture depict the edges with a potential known after the Problem IP5($\mathfrak{c}$) is solved.

8. Global inverse scattering problem

Problem IP(G). Given J_r, $r \in \mathscr{R}$, $M_v(\cdot, G)$, $v \in \partial G \setminus \{v^0\}$, $M_{v_{\mathfrak{c}}}(\cdot, G_{\mathfrak{c}})$, $\mathfrak{c} \in \mathscr{C}$, recover $q(x)$, $x \in G$.

Theorem 8.1. *Problem IP(G) has at most one solution, i.e., the specified data uniquely determine the potential $q(x)$, $x \in G$.*

Proof. For each fixed ray $r = [v, \infty)$, $r \in \mathscr{A}^{(\omega)}$ we apply Theorem 3.1 and get $q|_r = \tilde{q}|_r$, $M_v(\cdot, G) = \tilde{M}_v(\cdot, G)$.

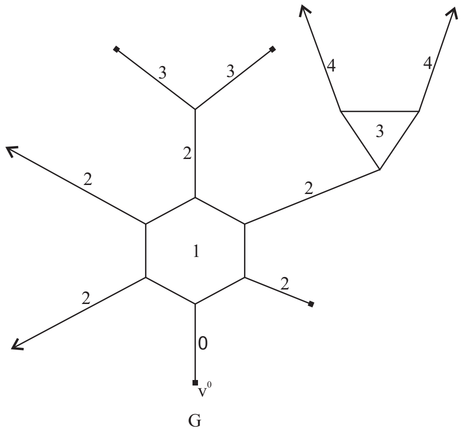

Figure 4: To the proof of Theorem 8.1. All the a-edges from the set $\mathscr{A}^{(\mu)}$ are marked with μ. The uniqueness of recovering of the potential is established subsequently for a-edges from $\mathscr{A}^{(\mu)}$ in order of decreasing of μ.

For each fixed boundary edge r, $r \in \mathscr{A}^{(\omega)}$ connecting vertex $v \in \partial G$ with the vertex u we apply Theorem 4.1 and get $q|_r = \tilde{q}|_r$, $M_u(\cdot, G) = \tilde{M}_u(\cdot, G)$.

For each fixed boundary cycle $\mathfrak{c} \in \mathscr{A}^{(\omega)}$ we apply Theorem 6.1 and get $q|_{\mathfrak{c}} = \tilde{q}|_{\mathfrak{c}}$, $M_{u_{\mathfrak{c}}}(\cdot, G) = \tilde{M}_{u_{\mathfrak{c}}}(\cdot, G)$.

Thus, we have proved that $q|_{\mathfrak{a}} = \tilde{q}|_{\mathfrak{a}}$ for all a-edges $\mathfrak{a} \in \mathscr{A}^{(\omega)}$.

Fix $\mu \in \{\omega - 1, \ldots, 0\}$ and suppose that we have proved that $q|_{\mathfrak{a}} = \tilde{q}|_{\mathfrak{a}}$ for all a-edges $\mathfrak{a} \in \mathscr{A}^{(\omega)} \cup \ldots \cup \mathscr{A}^{(\mu+1)}$. Then

(1) For each fixed ray $r = [v, \infty)$, $r \in \mathscr{A}^{(\mu)}$ we apply Theorem 3.1 and get $q|_r = \tilde{q}|_r$, $M_v(\cdot, G) = \tilde{M}_v(\cdot, G)$.

(2) For each fixed boundary edge r, $r \in \mathscr{A}^{(\mu)}$ connecting vertex $v \in \partial G$ with the vertex u we apply Theorem 4.1 and get $q|_r = \tilde{q}|_r$, $M_u(\cdot, G) = \tilde{M}_u(\cdot, G)$.

(3) For each fixed boundary cycle $\mathfrak{c} \in \mathscr{A}^{(\mu)}$ we apply Theorem 6.1 and get $q|_{\mathfrak{c}} = \tilde{q}|_{\mathfrak{c}}$, $M_{u_{\mathfrak{c}}}(\cdot, G) = \tilde{M}_{u_{\mathfrak{c}}}(\cdot, G)$.

(4) For each fixed internal simple edge r, $r \in \mathscr{A}^{(\mu)}$ connecting vertex $v \in G^+(r)$ with the vertex $u \in \partial G^+(r)$ we apply Theorem 5.1 and get $q|_r = \tilde{q}|_r$, $M_u(\cdot, G) = \tilde{M}_u(\cdot, G)$.

(5) For each fixed internal cycle $\mathfrak{c} \in \mathscr{A}^{(\mu)}$ we apply Theorem 7.1 and get $q|_{\mathfrak{c}} = \tilde{q}|_{\mathfrak{c}}$, $M_{u_{\mathfrak{c}}}(\cdot, G) = \tilde{M}_{u_{\mathfrak{c}}}(\cdot, G)$.

Thus, we have proved that $q|_{\mathfrak{a}} = \tilde{q}|_{\mathfrak{a}}$ for all a-edges $\mathfrak{a} \in \mathscr{A}^{(\mu)}$.

Using the above–mentioned arguments successively for $\mu = \omega - 1, \ldots, 1, 0$ we get $q = \tilde{q}$ a.e. on G. $\square$

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