TAMKANG JOURNAL OF MATHEMATICS
Volume 37, Number 1, 93-103, Spring 2006

A NOTE ON THE GENERALIZED NEUTRAL ORTHOGONAL GROUP IN DIMENSION FOUR

RYAD GHANAM

Abstract. We study the main properties of the generalized neutral orthogonal group $O(2,2)$ and its Lie algebra $o(2,2)$. We also give an explicit isomorphism between the Lie algebras $su(1,1)\oplus su(1,1)$ and $o(2,2)$. We use this isomorphism to classify the subalgebras of $o(2,2)$.

1. Introduction

In this paper we study the generalized orthogonal group $O(2,2)$ of a neutral metric in dimension four. In Section 2 we state and prove several theorem regarding the group $O(2,2)$. In Section 3 we study the Lie algebra of the group $O(2,2)$ which denoted by $o(2,2)$. We also show the relationship between the Lie algebras $su(1,1)\oplus su(1,1)$ and $o(2,2)$ by constructing an explicit isomorphism between them. In Section 4 we use that isomorphism to obtain a classification of the Lie algebras of $o(2,2)$. We use this isomorphism to classify the subalgebras of $o(2,2)$.

As regards notation, elements of $su(1,1)\oplus su(1,1)$ will be thought of as 2×2 block diagonal matrices. We shall denote the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ by I, K, L, J, respectively. Also $SO_o(2,1)$ will denote the connected component of the identity in $SO(2,1)$.

2. The generalized orthogonal group $O(2,2)$

Let g be the inner product on $\mathbb{R}^4$ of the form $g=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. If $u=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$

Received and revised September 1, 2004.
2000 *Mathematics Subject Classification*. 70G65.
Key words and phrases. .

and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ are two vectors in $\mathbb{R}^4$, then $g(u, v)$ is given by

$$g(u, v) = u^t g u = u_1 v_1 + u_2 v_2 - u_3 v_3 - u_4 v_4.$$

We define the following set Ω:

$$\Omega = \{f : \mathbb{R}^4 \mapsto \mathbb{R}^4 | g(f(u), f(v)) = g(u, v)\}.$$

This means that Ω is the set of all linear transformations on $\mathbb{R}^4$ that preserve length.

Theorem 2.1. *If f is in Ω, then f^{-1}, the inverse of f, is in Ω.*

Proof. Let f be in Ω such that A is the matrix representation of f. Then A^{-1} is the matrix representation of f^{-1}, i.e

$$f^{-1}(u) = A^{-1} u \tag{1}$$

for all $u \in \mathbb{R}^4$. Since f is in Ω we have

$$g(u, v) = g(A(A^{-1}u), A(A^{-1}v) = g(A^{-1}u, A^{-1}v), \tag{2}$$

and so f^{-1} is in Ω.

Theorem 2.2. *If f and h are in Ω, then $f \circ g$ is in Ω.*

Proof. For any u and v in $\mathbb{R}^4$, we have the following:

$$g((f \circ h)(u), (f \circ h)(v)) = g(f(h(u)), f(h(v))) = g(h(u), h(v)) = g(u, v) \tag{3}$$

Theorem 2.3. *Ω is a group, where the group mutiplication is the composition of functions.*

Theorem 2.4. *If $f \in \Omega$ and A is the matrix of f, then $AgA^t = g$, where A^t is the transpose of A.*

Proof. Let $f : \mathbb{R}^4 \mapsto \mathbb{R}^4$ be a linear transformation in Ω, then for any $u \in \mathbb{R}^4$, we have $f(u) = Au$, where A is the matrix representation of f. Now, for $u \neq 0$, we have the following:

$$g(f(u), f(u)) = g(u, u) \tag{4}$$

and so

$$(f(u))^t g f(u) = u^t g u. \tag{5}$$

Now, we replace $f(u)$ by Au to obtain:

$$(Au)^t g(Au) = u^t gu \tag{6}$$

and so

$$u^t A^t gAu = u^t gu \tag{7}$$

hence

$$u^t(A^t gA - g)u = 0. \tag{8}$$

Since equation (8) is true for all non-zero vectors u, then we must have

$$A^t gA - g = 0, \tag{9}$$

and so

$$A^t gA = g. \tag{10}$$

Let $M_4(\mathbb{R})$ be the set of all 4×4 matirces with real entries, we define the generalized neutral orthogonal group $O(2,2)$ to be

$$O(2,2) = \{A \in M_4(\mathbb{R}) | A^t gA = g\}.$$

Theorem 2.5. *If $A \in O(2,2)$, then $det(A) = \pm 1$.*

Proof. Since $A^t gA = g$, we take the the deteminant of both sides to obtain:

$$det(A^t gA) = det(g) \tag{11}$$

and so

$$det(A^t)det(g)det(A) = det(g) \tag{12}$$

since $det(A^t) = det(A)$, we have:

$$det(A)^2 = 1 \tag{13}$$

and so

$$det(A) = \pm 1 \tag{14}$$

Theorem 2.6. *If $A \in O(2,2)$, then $A^t \in O(2,2)$.*

Proof. Since $A^t gA = g$, then

$$(A^t gA)^{-1} = g^{-1} \tag{15}$$

and so

$$(gA^t)^{-1}A^{-1} = g^{-1} \tag{16}$$

hence

$$(A^t)^{-1}g^{-1}A^{-1} = g^{-1} \tag{17}$$

but since the inverse commutes with the transpose we have:

$$(A^{-1})^t g^{-1} A^{-1} = g^{-1} \tag{18}$$

since $g = g^{-1} = g^t$, we have

$$(A^{-1})^t g((A^{-1})^t)^t = g. \tag{19}$$

We conclude that $(A^{-1})^t \in O(2,2)$, and so $((A^{-1})^t)^{-1} = A^t \in O(2,2)$.

Theorem 2.7. *If $\alpha \in \mathbb{C}$ is an eigenvalue of $A \in O(2,2)$, then:*

1. $\alpha \neq 0$
2. α^{-1} *is an eigenvalue of* A, A^t *and* A^{-1}.

Proof. If $\alpha \in \mathbb{C}^4$ and since $det(A) \neq 0$, then $\alpha \neq 0$. There exits a non-zero vector $v \in \mathbb{R}^4$ such that

$$Av = \alpha v \tag{20}$$

and so

$$A^{-1}Av = A^{-1}(\alpha v) \tag{21}$$

hence

$$v = \alpha A^{-1} v. \tag{22}$$

We divide both sides of equation (22) by α to obtain

$$\alpha^{-1} v = A^{-1} v. \tag{23}$$

This proves that α^{-1} is an eigenvalue of A^{-1}. Now, we want to show that α^{-1} is an eigenvalue of A^t. Since $AgA^t = g$, we have:

$$gA^t = A^{-1} g \tag{24}$$

and so

$$gA^t g^{-1} = A^{-1} \tag{25}$$

hence

$$gA^t g^{-1} x = A^{-1} x \tag{26}$$

and so

$$gA^t g^{-1} x = \alpha^{-1} x \tag{27}$$

and so

$$A^t g^{-1} x = g^{-1} \alpha^{-1} x \tag{28}$$

hence

$$A^t (g^{-1} x) = \alpha^{-1} (g^{-1} x). \tag{29}$$

This proves that α^{-1} is an eigenvalue of A^t.

Theorem 2.8. *If $\alpha \in \mathbb{C}$ is an eigenvalue of $A \in O(2,2)$ and with a corresponding non-null eigenvector, then $|\alpha| = 1$.*

Proof. Let x be the corresponding non-null eigenvector with $g(x,x) \neq 0$, then

$$g(x,x) = g(Ax,Ax) = g(\alpha x, \alpha x) = \alpha\bar{\alpha} g(x,x) = |\alpha|^2 g(x,x)$$

hence

$$|\alpha|^2 = 1 \tag{30}$$

and so $|\alpha|$=1.

3. The isomorphism between $su(1,1) \oplus su(1,1)$ and $o(2,2)$

Let g be an inner product on $\mathbb{R}^4$ of the form $g = \mathrm{diag}(1,1,-1,-1)$. Then $A \in O(2,2)$ if and only if $A^T gA = g$ where A^T is the transpose of A.

Theorem 3.1. *M belongs to the Lie algebra $o(2,2)$ if it satisfies $gM + (gM)^T = 0$.*

Proof. Let $\alpha : (a,b) \to O(2,2)$ be a smooth curve in $O(2,2)$ such that $0 \in (a,b)$ and $\alpha(0) = I$, where I is the 4×4 identity matrix. Then

$$(\alpha(t))^T g\alpha(t) = g. \tag{31}$$

Now, we differentiate both sides of equation (31) with respect to t to obtain

$$((\alpha(t))^T)' g\alpha(t) + (\alpha(t))^T g\alpha'(t) = 0. \tag{32}$$

It is not difficult to show that the derivative commutes with the transpose, and so

$$((\alpha(t))')^T g\alpha(t) + (\alpha(t))^T g\alpha'(t) = 0. \tag{33}$$

Taking $t = 0$ and substituting $\alpha(0) = \alpha(0)^T = I$ imply

$$((\alpha(0))')^T g + g\alpha'(0) = 0. \tag{34}$$

Let $\alpha(0)' = M$ to obtain

$$M^T g + gM = 0. \tag{35}$$

Hence

$$(gM)^T + gM = 0. \tag{36}$$

Lemma 3.1. *A 2×2 matrix M belongs to the Lie algebra of $SU(1,1)$ if and only if gM is skew-Hermitian and trace M is zero, where $g = \mathrm{diag}(1,-1)$.*

Proof. Similar to the proof of Theorem 3.1.

We now define a map $h : SU(1,1) \mapsto O(2,1)$

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \mapsto \begin{bmatrix} Re(a^2 - b^2) & Im(a^2 + b^2) & 2Im\ ab \\ -Im(a^2 - b^2) & Re(a^2 + b^2) & 2Re\ ab \\ 2Im\ \bar{a}b & 2Re\ \bar{a}b & |a|^2 + |b|^2 \end{bmatrix}$$

where a and b are in $\mathbb{C}$ with $|a|^2 - |b|^2 = 1$. A lengthy but routine calculation proves that h is a homomorphism. If we choose iK, iJ and L as a basis for $su(1,1)$, then the map h induces a Lie algebra homomorphism

$$h_* : su(1,1) \mapsto o(2,1)$$

given by

$$\begin{bmatrix} xi & a+bi \\ a-bi & -xi \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} 0 & 2x & 2b \\ -2x & 0 & 2a \\ 2b & 2a & 0 \end{bmatrix}.$$

One can easily verify that h_* is an isomorphism.

Corollary 3.2. *h is a covering map with kernel $\{\pm I_2\}$. Hence*

$$SU(1,1)/\mathbb{Z}_2 \approx SO_o(2,1).$$

Using the adjoint representation we can find a simple Jordan normal form for a given element M in $o(2,1)$ and hence by applying $(h_*)^{-1}$ a simple form for any element in $su(1,1)$. The eigenvalues of $M \in o(2,1)$ are of the form:

1. $\lambda = 0, \quad \lambda = \pm\alpha, \quad \alpha > 0$
2. $\lambda = 0, \quad \lambda = \pm\beta i, \quad \beta > 0$
3. $\lambda = 0$ with multiplicity 3 (nilpotent).

The corresponding Jordan forms are

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \beta & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

respectively. Now let g be a 3×3 real symmetric matrix relative to which M is skew-adjoint, i.e.,

$$gM + (gM)^t = 0. \tag{37}$$

We reduce g to a simple form by means of a transformation that leaves the forms of M given above invariant thereby obtaining simultaneous normal forms for g and M. In fact we find that in all three cases g can be reduced to the form

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In $su(1,1)$ the corresponding three forms for a Lie algebra element are $\alpha iJ, \beta iK$ and $iK + L$, respectively, and the hermitian form can be brought into the form of the matrix K. Now we consider the group O(2,2) with the inner product g of the form $\begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$. We define a map $f : SU(1,1) \times SU(1,1) \longrightarrow SO_o(2,2)$ by

$$\left(\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} c & d \\ \bar{d} & \bar{c} \end{bmatrix} \right) \mapsto \begin{bmatrix} Re(a\bar{c} + bd) & -Im(a\bar{c} - bd) & -Im(ad - b\bar{c}) & Re(ad + b\bar{c}) \\ Im(a\bar{c} + bd) & Re(a\bar{c} - bd) & Re(ad - b\bar{c}) & Im(ad + b\bar{c}) \\ Im(a\bar{d} + bc) & Re(a\bar{d} - bc) & Re(ac - b\bar{d}) & Im(ac + b\bar{d}) \\ Re(a\bar{d} + bc) & -Im(a\bar{d} - bc) & -Im(ac - b\bar{d}) & Re(ac + b\bar{d}) \end{bmatrix}$$

where a, b, c and d are in $\mathbb{C}$ with $|a|^2 - |b|^2 = 1$ and $|c|^2 - |d|^2 = 1$. Again, a long computation verifies that f is a homomorphism with kernel $\{I_2 \times I_2, -I_2 \times -I_2\}$ and so f induces an isomorphism $f_* : su(1,1) \oplus su(1,1) \longrightarrow$ o(2,2). An explicit form of f_* mapping generators to generators is as follows:

$$f_* \begin{bmatrix} iK & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix} \qquad f_* \begin{bmatrix} 0 & 0 \\ 0 & iK \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$$

$$f_* \begin{bmatrix} iJ & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad f_* \begin{bmatrix} 0 & 0 \\ 0 & iJ \end{bmatrix} = \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$$

$$f_* \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}, \qquad f_* \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix} = \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix}$$

4. Classification of the Subalgebras of $o(2,2)$

To obtain the classification of the subalgebras of $o(2,2)$ we now have the apparently simple task of finding subalgebras in $su(1,1) \oplus su(1,1)$ and pushing forward with f_*.

Lemma 4.1. *There is a two-dimensional Lie subalgebra denoted by A in $su(1,1)$, which is unique up to isomorphism.*

Proof. Note that iJ and $iK + L$ generate a 2-dimensional Lie subalgebra, since

$$[iJ, iK + L] = 2(iK + L).$$

As for uniqueness, if M and N generate a 2-dimensional Lie subalgebra, then we can put M in canonical form, and so we discuss the following cases.

Case 1. If $M = iJ$, we may assume $N = \alpha iK + \beta L$, when $\alpha, \beta \in \mathbb{R}$. Now $[M, N] = 2\beta iK + 2\alpha L$, and this has to be a constant multiple of N. It easily follows that $\alpha = \pm\beta$ and by change of basis we can reduce to $\alpha = \beta$. This is the subalgebra A.

Case 2. If $M = iK$, we may assume $N = \alpha iJ + \beta L$, where $\alpha, \beta \in \mathbb{R}$, and so $[M, N] = 2\beta iJ - 2\alpha L$. This has to be a constant multiple of N, and so $2\beta iJ - 2\alpha L = c(\alpha iJ + \beta L)$. This easily leads to a contradiction.

Case 3. Similar to Case (2) starting from $M = iK + L$.

We obtained the classification by considering elements of $su(1,1) \oplus su(1,1)$ as 2×2 block diagonal matrices and letting the restriction of a subalgebra to each block have dimensions m and n, respectively . Of course the primary invariant of a subalgebra is its dimension which can vary from zero to six. We note also that the matrix $\begin{bmatrix} K & 0 \\ 0 & I \end{bmatrix}$ defines an automorphism of $o(2,2)$ which maps $\begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}$ to $\begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$, $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ to $\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$ to $\begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix}$, respectively. It follows that the cases (m, n) and (n, m) give rise to isomorphic subalgebras in $o(2,2)$. The classification for one-dimensional algebras is obtained immediately and, allowing for change of basis, agrees with results obtained previously in [4]. For the remaining algebras, one simply checks for a given dimension d with $2 \leq d \leq 5$, all the possible subalgebras of $su(1,1) \oplus su(1,1)$ that can arise for particular values of m and n such that $m \geq n$ and $2 \leq m + n \leq 5$ and one realizes the corresponding subalgebra of $o(2,2)$ by applying f_*. Let us outline the method for three-dimensional sualgebras. First of all $(3,0)$ corresponds to $su(1,1) \oplus \{0\}$ and the algebra in $o(2,2)$ has generators $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, $\begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}$, $\begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}$. For the case (3,1) we may assume that our generators are of the form $\begin{bmatrix} iJ & 0 \\ 0 & aN \end{bmatrix}$, $\begin{bmatrix} iK & 0 \\ 0 & bN \end{bmatrix}$ and $\begin{bmatrix} L & 0 \\ 0 & cN \end{bmatrix}$ where $a, b, c \in \mathbb{R}$ and the precise form of N is unknown. The Lie bracket of the first two must be twice the third, which implies that $c = 0$. Likewise from the brackets of the first and third and second and third , respectively we deduce that $a = b = 0$. and hence no such algebra of this kind is possible. Similarly one may argue that the case $(3,2)$ is impossible and that $(3,3)$ corresponds precisely to the diagonal subalgebra in $su(1,1) \oplus su(1,1)$. For the case $(2,1)$ we may assume that the generators have the form $\begin{bmatrix} iJ & 0 \\ 0 & aM \end{bmatrix}$, $\begin{bmatrix} iK + L & 0 \\ 0 & bM \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}$ where $a, b \in \mathbb{R}$. We can now put M into canonical form and we obtain three classes of algebra in $o(2,2)$ depending on a parameter α. Similarly for the case $(2,2)$ we obtain a single class of Lie algebras in $o(2,2)$ that depends on a parameter α. Table 1 gives our list of the subalgebras of $o(2,2)$ in terms of a basis for each subalgebra. We state the dimension of the subalgebra and have numbered the subalgebras from 1 to 32 for future reference, though certain of these numbers pertain to classes of subalgebras depending on parameters α and β.

Table 1

Number	Basis for the Lie Subalgebra	Dimension
1	$\begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{bmatrix}$	1
2	$\begin{bmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix}$	1
3	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	1
4	$\begin{bmatrix} 0 & -\beta & \alpha & 0 \\ \beta & 0 & 0 & -\alpha \\ \alpha & 0 & 0 & \beta \\ 0 & -\alpha & -\beta & 0 \end{bmatrix}$	1
5	$\begin{bmatrix} 0 & 1-\alpha & 0 & 1 \\ \alpha-1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \alpha+1 \\ 1 & 0 & -(\alpha+1) & 0 \end{bmatrix}$	1
6	$\begin{bmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{bmatrix}$	1
7	$\begin{bmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{bmatrix}$	1
8	$\begin{bmatrix} 0 & 1 & -\alpha & 1 \\ -1 & 0 & 1 & -\alpha \\ -\alpha & 1 & 0 & 1 \\ 1 & -\alpha & -1 & 0 \end{bmatrix}$	1
9	$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	1

Table 1 (Continue)

Number	Basis for the Lie Subalgebra	Dimension
10	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}$	2
11	$\begin{bmatrix} 0 & I+\alpha K \\ I+\alpha K & 0 \end{bmatrix}, \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}$	2
12	$\begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} \alpha J & I \\ I & \alpha J \end{bmatrix} (\alpha \neq 0)$	2
13	$\begin{bmatrix} J & I+L \\ I+L & J \end{bmatrix}, \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}$	2
14	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$	2
15	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$	2
16	$\begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$	2
17	$\begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} J & L \\ L & J \end{bmatrix}$	2
18	$\begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$	2
19	$\begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$	2
20	$\begin{bmatrix} 0 & I+K \\ I+K & 0 \end{bmatrix}, \begin{bmatrix} (\beta-1)J & J+\beta L \\ -J+\beta L & (\beta+1)J \end{bmatrix} (\beta = \pm 1)$	2
21	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$	3
22	$\begin{bmatrix} 0 & I+K \\ I+K & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2J \end{bmatrix}, \begin{bmatrix} 0 & J+L \\ -J+L & 0 \end{bmatrix}$	3
23	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$	3
24	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$	3
25	$\begin{bmatrix} 0 & I+\alpha L \\ I+\alpha L & 0 \end{bmatrix}, \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} J & L \\ L & J \end{bmatrix}$	3
26	$\begin{bmatrix} 0 & I+\alpha K \\ I+\alpha K & 0 \end{bmatrix}, \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} J & L \\ L & J \end{bmatrix}$	3
27	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}, \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$	4
28	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$	4
29	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}, \begin{bmatrix} J & L \\ L & J \end{bmatrix}$	4
30	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}, \begin{bmatrix} J & L \\ L & J \end{bmatrix}$	4
31	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}, \begin{bmatrix} J & L \\ L & J \end{bmatrix}, \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$	5
32	$\begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}, \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}, \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix}$	6

References

[1] S. Helgason , Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press 1978.

[2] R. Ghanam, The holonomy groups of four dimensional neutral metrics, Ph.D. dissertation, The University of Toledo, May, 2000.

[3] R. Ghanam and G. Thompson, *The Holonomy Lie Algebras of neutral metrics in dimension four*, J. Math. Phys. **42**, (2001). 2266-2284.

[4] G. Thompson, *Normal forms for elements of* $o(p, q)$ *and Hamiltonians with integrals linear in momenta*, J. Austral. Math. Soc. Ser. B **33**(1992), 486-507.

Department of Mathematics, University of Pittsburgh at Greensburg, 1150 Mt. Pleasant Rd., Greensburg, PA 15601, U.S.A.

E-mail: ghanam@pitt.edu