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A GCD AND LCM-LIKE INEQUALITY FOR MULTIPLICATIVE LATTICES

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Abstract. Let $A_1,\ldots,A_n$ $(n\geq 2)$ be elements of an commutative multiplicative lattice. Let $G(k)$ (resp., $L(k)$) denote the product of all the joins (resp., meets) of k of the elements. Then we show that

$$L(n)G(2)G(4)\cdots G(2\lfloor n/2\rfloor)\leq G(1)G(3)\cdots G(2\lceil n/2\rceil -1).$$

In particular this holds for the lattice of ideals of a commutative ring. We also consider the relationship between

$$G(n)L(2)L(4)\cdots L(2\lfloor n/2\rfloor)\text{ and }L(1)L(3)\cdots L(2\lceil n/2\rceil -1)$$

and show that any inequality relationships are possible.

1. Introduction

Let R be a commutative ring (not necessarily with identity). Then for two ideals A_1 and A_2 of R we have

$$(A_1\cap A_2)(A_1+A_2)\subseteq A_1A_2. \qquad (\dagger_2)$$

For three ideals A_1, A_2, A_3 of R it is easily verified that we have

$$(A_1\cap A_2\cap A_3)(A_1+A_2)(A_1+A_3)(A_2+A_3)\subseteq A_1A_2A_3(A_1+A_2+A_3). \qquad (\dagger_3)$$

The purpose of this paper is to give a general containment relation $(\dagger_n)$ for n ideals $A_1,\ldots,A_n$ of R, $n\geq 2$, generalizing the previous two relations $(\dagger_2)$ and $(\dagger_3)$.

The corresponding ideal formulation is as follows. Let R be a commutative ring and let $A_1,\ldots,A_n$ $(n\geqslant 2)$ be ideals of R. For $1\leqslant k\leqslant n$ put

$$G(k):=G(k;A_1,\ldots,A_n)=\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}(A_{i_1}+\cdots+A_{i_k}),$$

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$$L(k) := L(k; A_1, \ldots, A_n) = \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (A_{i_1} \cap \cdots \cap A_{i_k})$$

(so $G(1) = L(1) = A_1 \cdots A_n$, $G(n) = A_1 + \cdots + A_n$, $L(n) = A_1 \cap \cdots \cap A_n$).

Definition 1.1. The ring R *satisfies* $(*)_n$ *for ideals* $A_1, \ldots, A_n$ *of* R $(n \geqslant 2)$ if

$$G(n) \prod_{2 \leqslant 2k \leqslant n} L(2k) = \prod_{1 \leqslant 2k+1 \leqslant n} L(2k+1), \tag{$(*)_n$}$$

satisfies $(**)_n$ *for ideals* $A_1, \ldots, A_n$ *of* R $(n \geqslant 2)$ if

$$L(n) \prod_{2 \leqslant 2k \leqslant n} G(2k) = \prod_{1 \leqslant 2k+1 \leqslant n} G(2k+1), \tag{$(**)_n$}$$

and satisfies $(\dagger_n)$ for ideals $A_1, \ldots, A_n$ of R $(n \geq 2)$ if

$$L(n) \prod_{2 \leq 2k \leq n} G(2k) \subseteq \prod_{1 \leq 2k+1 \leq n} G(2k+1). \tag{$\dagger_n$}$$

Using the ceiling function and the floor function, we may express these as follows:

$$G(n)L(2)L(4)\cdots L(2\lfloor n/2 \rfloor) = L(1)L(3)\cdots L(2\lceil n/2 \rceil - 1), \tag{$(*)_n$}$$

$$L(n)G(2)G(4)\cdots G(2\lfloor n/2 \rfloor) = G(1)G(3)\cdots G(2\lceil n/2 \rceil - 1), \tag{$(**)_n$}$$

$$L(n)G(2)G(4)\cdots G(2\lfloor n/2 \rfloor) \subseteq G(1)G(3)\cdots G(2\lceil n/2 \rceil - 1). \tag{$\dagger_n$}$$

Note that $(*)_2$ reduces to $(A_1 + A_2)(A_1 \cap A_2) = A_1 A_2$ and $(**)_2$ reduces to $(A_1 \cap A_2)(A_1 + A_2) = A_1 A_2$ while as previously mentioned $(\dagger_2)$ is $(A_1 \cap A_2)(A_1 + A_2) \subseteq A_1 A_2$. We are taking $n \geqslant 2$ as the properties $(*)_1$ and $(**)_1$ are simply $A_1 = A_1$ which is always true as is $(\dagger_1)$ $A_1 \subseteq A_1$.

A commutative ring R is called a *chained* ring (resp., *arithmetical ring*) if the lattice of ideals of R is a chain (resp., distributive). So an integral domain is a chained ring if and only if it is a valuation domain. It is well known that R is an arithmetical ring if and only if R_M is a chained ring for each maximal ideal M of R. An integral domain is a *Prüfer domain* if every nonzero finitely generated ideal is invertible. R is a Prüfer domain if and only if R_M is a valuation domain for each maximal ideal M of R if and only if R_M is a chained ring for each maximal ideal M of R. Thus a Prüfer domain is an arithmetical ring that is an integral domain. Finally, R is a *Prüfer ring* if every finitely generated regular ideals is invertible. Here, an element is *regular* if it is not a zero-divisor and an ideal is *regular* if it contains a regular element.

We showed [1, Theorem 2.4] if R is an arithmetical ring, then $(*)_n$ and $(**)_n$ hold for all ideals $A_1, \ldots, A_n$ of R and that, R is a Prüfer ring if and only if $(**)_n$ hold for some $n \geq 2$

(equivalently, for all $n \geq 2$) for all ideals $A_1, \ldots, A_n$ of R when at least $n-1$ of them are regular [1, Theorem 2.6]. We also proved that $(\mathrm{GCD})_n$ and $(\mathrm{LCM})_n$ hold for any GCD domain [1, Theorem 2.8]:

$$\gcd(a_1,\ldots,a_n) \prod_{2\leqslant 2k\leqslant n} \prod_{1\leqslant i_1<\cdots<i_{2k}\leqslant n} \operatorname{lcm}(a_{i_1},\ldots,a_{i_{2k}}) = a_1\cdots a_n \prod_{2\leqslant 2k+1\leqslant n} \prod_{1\leqslant i_1<\cdots<i_{2k+1}\leqslant n} \operatorname{lcm}(a_{i_1},\ldots,a_{i_{2k+1}}) \tag{(GCD)$_n$}$$

$$\operatorname{lcm}(a_1,\ldots,a_n) \prod_{2\leqslant 2k\leqslant n} \prod_{1\leqslant i_1<\cdots<i_{2k}\leqslant n} \gcd(a_{i_1},\ldots,a_{i_{2k}}) = a_1\cdots a_n \prod_{2\leqslant 2k+1\leqslant n} \prod_{1\leqslant i_1<\cdots<i_{2k+1}\leqslant n} \gcd(a_{i_1},\ldots,a_{i_{2k+1}}). \tag{(LCM)$_n$}$$

Note that for a PID R, GCD_n (resp., LCM_n) may be obtained from $(*)_n$ (resp., $(**)_n$) by taking $A_1 = (a_1), \ldots, A_n = (a_n)$.

Thus neither $(*)_n$ nor $(**)_n$ always holds. In Section 2, however, we show that the one-sided inclusion

$$L(n) \prod_{2\leqslant 2k\leqslant n} G(2k) \subseteq \prod_{1\leqslant 2k+1\leqslant n} G(2k+1) \tag{$\dagger_n$}$$

holds for general commutative rings (which may not have an identity). Indeed, this holds not only for ideal lattices of commutative rings, but in the quite general setting of a (commutative) multiplicative lattice. In Section 3 we give some examples to illustrate results from Section 2.

2. Inclusion Formula for Multiplicative Lattices

We have noted in the Introduction that the identity $(*)_n$ or $(**)_n$ holds for all ideals of special rings. However one inclusion formula holds for a general commutative ring as follows. Using the expression in the former section, it is expressed as

$$L(n;A_1,A_2,\ldots,A_n) \prod_{2\leqslant 2k\leqslant n} G(2k;A_1,A_2,\ldots,A_n) \subseteq \prod_{1\leqslant 2k+1\leqslant n} G(2k+1;A_1,A_2,\ldots,A_n)$$

or equivalently,

$$L(n)G(2)G(4)\cdots G(2\lfloor n/2\rfloor) \subseteq G(1)G(3)\cdots G(2\lceil n/2\rceil - 1).$$

This is the only inclusion formula concerning both sides of $(*)_n$ and $(**)_n$ which holds for all ideals of a general commutative ring. For, it is shown by Example 3.1, that the opposite inclusion does not always hold and by Example 3.2 that neither of the inclusions between

$G(n)L(2)L(4)\cdots L(2\lfloor n/2\rfloor)$ and $L(1)L(3)\cdots L(2\lceil n/2\rceil-1)$ always holds. We prove the above inclusion in a more generalized form: inequality in a (commutative) multiplicative lattice.

By a *multiplicative lattice* we mean a lattice with a commutative, associative product that distributes over finite joins. Observe that $A \le B$ implies $AC \le BC$ for elements A, B, and C of a multiplicative lattice. The ideals of a commutative ring (or even a semiring) or commutative multiplicative semigroup with 0 forms a complete multiplicative lattice with $A \vee B = A + B$ for a ring or semiring ($A \vee B = A \cup B$ for a semigroup), $A \wedge B = A \cap B$, and AB as the usual ideal product.

Given elements $A_1, \dots, A_n$ of a multiplicative lattice $\mathscr{I}$, we can define $G(k)$, $L(k) \in \mathscr{I}$:

$$G(k) := \prod_{1\leqslant i_1<\cdots<i_k\leqslant n} (A_{i_1} \vee \cdots \vee A_{i_k}),$$
$$L(k) := \prod_{1\leqslant i_1<\cdots<i_k\leqslant n} (A_{i_1} \wedge \cdots \wedge A_{i_n}),$$

as in Section 1, replacing $+$ and $\cap$ respectively by $\vee$ and $\wedge$. The identities $(*)_n$, $(**)_n$ and $(\dagger_n)$ are defined in the same way as Definition 1.1. Then we can prove the following generalization of [1, Lemma 2.1]. The proof is similar.

Proposition 2.1. *Let $\mathscr{I}$ be a multiplicative lattice and take $A_1, \dots, A_n \in \mathscr{I}$ $(n \geqslant 2)$. Suppose that $\{A_1, \dots, A_n\}$ has a maximum (resp., minimum) element. Then $(*)_n$ (resp., $(**)_n$) holds for $A_1, \dots, A_n \in \mathscr{I}$.*

In this general setting, we do not know any other meaningful sufficient condition for the identities $(*)_n$ and $(**)_n$ to hold. Thus we content ourselves with a one-sided inequality as follows. Note that it implies the one-sided inclusion formula for ideals of a general commutative semiring (and hence ring) which may not have an identity.

Theorem 2.2. *Let $\mathscr{I}$ be a multiplicative lattice. For $A_1, \dots, A_n \in \mathscr{I}$ $(n \in \mathbb{N})$, we always have the following:*

$$L(n)G(2)G(4)\cdots G(2\lfloor n/2\rfloor) \leqslant G(1)G(3)\cdots G(2\lceil n/2\rceil-1). \qquad (\dagger_n)$$

Proof. In this theorem, $(\dagger_1)$ should be interpreted as the trivial assertion $A \leqslant A$. The assertion $(\dagger_2)$ follows from

$$(A_1 \wedge A_2)(A_1 \vee A_2) \leqslant A_2A_1 \vee A_1A_2 = A_1A_2.$$

Assume that, for some $n \geqslant 3$, we have proved $(\dagger_k)$ $(k < n)$. Let us put

$$G(p; q, r; A_1, \dots, A_n) := \prod_{q<i_1<\cdots<i_{p-2}<r} (A_q \vee A_{i_1} \vee \cdots \vee A_{i_{p-2}} \vee A_r)$$

$$(1 \leqslant q \leqslant r \leqslant n,\ 1 \leqslant p \leqslant r-q+1).$$

Here, in the case $r = q$,

$$G(1; q, q; A_1, \ldots, A_n) = A_q,$$

and in the case $r = q+1$,

$$\begin{aligned} G(1; q, q+1; A_1, \ldots, A_n) &= A_q A_{q+1}, \\ G(2; q, q+1; A_1, \ldots, A_n) &= A_q \vee A_{q+1}. \end{aligned}$$

We also have

$$G(p; A_1, \ldots, A_n) = \prod_{\substack{1 \leqslant q \leqslant n \\ p+q-1 \leqslant r \leqslant n}} G(p; q, r; A_1, \ldots, A_n)$$

for $1 \leqslant p \leqslant n$.

If n is even: $n = 2m \geqslant 4$, we have to prove

$$\begin{aligned} \text{(Left)} := L(2m; A_1, \ldots, A_{2m}) &\prod_{1 \leqslant p \leqslant m} G(2p; A_1, \ldots, A_{2m}) \\ \leqslant &\prod_{1 \leqslant p \leqslant m} G(2p-1; A_1, A_2, \ldots, A_{2m}) =: \text{(Right)}. \end{aligned}$$

The expression (Left) contains the factor $(A_{2m-1} \vee A_{2m})$. Let $(\text{Left})_{2m-1}$ (resp., $(\text{Left})_{2m}$) denote the expression obtained by substitution of this factor $(A_{2m-1} \vee A_{2m})$ by A_{2m-1} (resp. by A_{2m}) in (Left). Since $(\text{Left}) = (\text{Left})_{2m-1} \vee (\text{Left})_{2m}$, we only have to prove

$$(\text{Left})_{2m-1} \leqslant (\text{Right}), \quad (\text{Left})_{2m} \leqslant (\text{Right}).$$

By symmetry, we only have to prove the latter. Since

$$\begin{aligned} \text{(Left)} &= L(2m; A_1, \ldots, A_{2m}) \prod_{1 \leqslant p \leqslant m-1} \Big(G(2p; A_1, \ldots, A_{2m-1}) \\ &\quad \prod_{1 \leqslant q \leqslant 2m-2p+1} G(2p; q, 2m; A_1, \ldots, A_{2m}) \Big) \\ &= L(2m; A_1, \ldots, A_{2m}) \Big(\prod_{1 \leqslant p \leqslant m-1} G(2p; A_1, \ldots, A_{2m-1}) \Big) \\ &\quad \cdot \Big(\prod_{1 \leqslant q \leqslant 2m-1} (A_q \vee A_{2m}) \Big) \Big(\prod_{\substack{2 \leqslant p \leqslant m \\ 1 \leqslant q \leqslant 2m-2p+1}} G(2p; q, 2m; A_1, \ldots, A_{2m}) \Big) \\ &= L(2m; A_1, \ldots, A_{2m}) \Big(\prod_{1 \leqslant p \leqslant m-1} G(2p; A_1, \ldots, A_{2m-1}) \Big) \cdot \Big(\prod_{1 \leqslant q \leqslant 2m-1} (A_q \vee A_{2m}) \Big) \\ &\quad \cdot \Big(\prod_{\substack{2 \leqslant p \leqslant m \\ 1 \leqslant q \leqslant 2m-2p+1}} G(2p-2; A_q \vee A_{q+1} \vee A_{2m}, \end{aligned}$$

$$A_q \vee A_{q+2} \vee A_{2m}, \dots, A_q \vee A_{2m-1} \vee A_{2m})\Big),$$

if we put

$$B_{q,\nu} := A_q \vee A_\nu \vee A_{2m} \quad (q < \nu < 2m),$$

we have

$$\begin{aligned}(\text{Left})_{2m} = {} & L(2m; A_1, \dots, A_{2m})\Big(\prod_{1 \leqslant p \leqslant m-1} G(2p; A_1, \dots, A_{2m-1})\Big) \cdot \Big(\prod_{1 \leqslant q \leqslant 2m-2} (A_q \vee A_{2m})\Big) A_{2m} \\ & \Big(\prod_{\substack{2 \leqslant p \leqslant m \\ 1 \leqslant q \leqslant 2m-2p+1}} G(2p-2; B_{q,q+1}, B_{q,q+2}, \dots, B_{q,2m-1})\Big).\end{aligned}$$

Note that

$$A_q \vee A_{2m} \leqslant L(2m-q-1; B_{q,q+1}, B_{q,q+2}, \dots, B_{q,2m-1}).$$

Thus we have

$$\begin{aligned}(\text{Left})_{2m} = {} & L(2m; A_1, \dots, A_{2m})\Big(\prod_{1 \leqslant p \leqslant m-1} G(2p; A_1, \dots, A_{2m-1})\Big) \cdot (A_{2m-2} \vee A_{2m}) A_{2m} \\ & \cdot \prod_{1 \leqslant q \leqslant 2m-3} \Big((A_q \vee A_{2m}) \prod_{2 \leqslant p \leqslant m-(q-1)/2} G(2p-2; B_{q,q+1}, B_{q,q+2}, \dots, B_{q,2m-1})\Big) \\ \leqslant {} & L(2m-1; A_1, \dots, A_{2m-1})\Big(\prod_{1 \leqslant p \leqslant m-1} G(2p; A_1, \dots, A_{2m-1})\Big) \cdot (A_{2m-2} \vee A_{2m}) A_{2m} \\ & \cdot \prod_{1 \leqslant q \leqslant 2m-3} \Big(L(2m-q-1; B_{q,q+1}, \dots, B_{q,2m-1}) \\ & \prod_{2 \leqslant p \leqslant m+(1-q)/2} G(2p-2; B_{q,q+1}, B_{q,q+2}, \dots, B_{q,2m-1})\Big).\end{aligned}$$

By the inductive assumption and by

$$A_{2m-2} \vee A_{2m} \leqslant A_{2m-2} \vee A_{2m-1} \vee A_{2m},$$

the expression $(\text{Left})_{2m}$ is majorized by

$$\begin{aligned}& \Big(\prod_{1 \leqslant p \leqslant m} G(2p-1; A_1, \dots, A_{2m-1})\Big)(A_{2m-2} \vee A_{2m-1} \vee A_{2m}) A_{2m} \\ & \quad \cdot \prod_{1 \leqslant q \leqslant 2m-3} \prod_{1 \leqslant p \leqslant m-q/2} G(2p-1; B_{q,q+1}, B_{q,q+2}, \dots, B_{q,2m-1}) \\ & \leqslant \Big(\prod_{1 \leqslant p \leqslant m} G(2p-1; A_1, \dots, A_{2m})\Big) = (\text{Right}).\end{aligned}$$

Here note that $A_{2m-2} \vee A_{2m-1} \vee A_{2m}$ is the only join of three A_i's which contains A_{2m} and is not equal to some $B_{q,r}$.

In the case $n = 2m+1 \geqslant 3$, putting $C_{q,r} := A_q \vee A_r \vee A_{2m+1}$ and replacing the factor $A_{2m} \vee A_{2m+1}$ of

$$(\text{Left}) := L(2m+1; A_1, \ldots, A_{2m+1}) \prod_{1 \leqslant p \leqslant m} G(2p; A_1, \ldots, A_{2m+1})$$

by A_{2m+1}, we similarly have the following.

$$\begin{aligned}
(\text{Left})_{2m+1} = {} & L(2m; A_1, \ldots, A_{2m+1}) \Big(\prod_{1 \leqslant p \leqslant m} G(2p; A_1, \ldots, A_{2m}) \Big) \\
& \cdot (A_{2m-1} \vee A_{2m+1}) A_{2m+1} \\
& \cdot \prod_{1 \leqslant q \leqslant 2m-2} \Big(L(2m-q; C_{q,q+1}, \ldots, C_{q,2m}) \\
& \prod_{2 \leqslant p \leqslant m+1-q/2} G(2p-2; C_{q,q+1}, C_{q,q+2}, \ldots, C_{q,2m}) \Big) \\
\leqslant {} & \Big(\prod_{1 \leqslant p \leqslant m} G(2p-1; A_1, \ldots, A_{2m}) \Big) (A_{2m-1} \vee A_{2m} \vee A_{2m+1}) A_{2m+1} \\
& \cdot \prod_{1 \leqslant q \leqslant 2m-2} \prod_{1 \leqslant p \leqslant m} G(2p-1; C_{q,q+1}, C_{q,q+2}, \ldots, C_{q,2m}) \\
= {} & \Big(\prod_{1 \leqslant p \leqslant m} G(2p-1; A_1, \ldots, A_{2m+1}) \Big) = (\text{Right}).
\end{aligned}$$

This completes the mathematical induction. □

3. Examples

In this section we give some examples which illustrate the results of Section 2.

Example 3.1 (A ring with $L(n) \prod_{2 \leqslant 2k \leqslant n} G(2k) \subsetneq \prod_{1 \leqslant 2k+1 \leqslant n} G(2k+1)$ for all $n \geqslant 2$)**.** Put $R = k[X_1, X_2, \ldots]$ for a field k and take $A_i = (X_i)$. Here $L(n) = (X_1) \cap \cdots \cap (X_n) = (X_1 \cdots X_n)$, $G(1) = (X_1 \cdots X_n)$ and $G(n) = (X_1, \ldots, X_n)$. We can quote Theorem 2.2 to get

$$L(n) \prod_{2 \leqslant 2k \leqslant n} G(2k) \subseteq \prod_{1 \leqslant 2k+1 \leqslant n} G(2k+1).$$

However, we sketch a proof. This example shows that the inequality given in Theorem 2.2 may be strict, or equivalently, that the reverse inequality need not hold.

Note that each of the ideals $L(n)$ and $G(l)$ $(1 \leqslant l \leqslant n)$ is generated by monomials in $X_1, \ldots, X_n$. Of course $L(n)$ and $G(1)$ are generated by $X_1 \cdots X_n$ and $G(n)$ is generated by $X_1, \ldots, X_n$. Suppose $1 < k < n$, $G(k) = \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (X_{i_1}, \ldots, X_{i_n})$, a product of $\binom{n}{k}$ ideals and hence is generated by monomials in $X_1, \ldots, X_n$ of degree $\binom{n}{k}$. In fact $G(k)$ is generated by

$$\{X_1^{\alpha_1} \cdots X_n^{\alpha_n} : 0 \leqslant \alpha_i \leqslant \binom{n-1}{k-1},\ \alpha_1 + \cdots + \alpha_n = \binom{n}{k}\}.$$

Suppose n is even. Then $L(n)\prod G(2k)$ is generated by the monomials

$$X_1\cdots X_n \prod_{1\leqslant k\leqslant n/2-1} X_1^{\alpha_{1,2k}}\cdots X_n^{\alpha_{n,2k}} X_1^{\beta_1}\cdots X_n^{\beta_n}$$

where

$$0\leqslant \alpha_{i,2k}\leqslant \binom{n-1}{2k-1},\quad \alpha_{1,2k}+\cdots+\alpha_{n,2k}=\binom{n}{2k},$$
$$0\leqslant \beta_i\leqslant 1,\quad \beta_1+\cdots+\beta_n=1.$$

This may be rewritten as $\prod_{1\leqslant i\leqslant n} X_i^{1+\alpha_{i,2}+\cdots+\alpha_{i,n-2}+\beta_i}$ and has degree $n+\sum_{1\leqslant k\leqslant n/2}\binom{n}{2k}$. Similarly one can write down the generators for $G(2k+1)$ which are monomials of degree $\sum_{0\leqslant k\leqslant n/2-1}\binom{n}{2k+1}$. Now

$$n+\sum_{1\leqslant k\leqslant n/2}\binom{n}{2k}=\sum_{0\leqslant k\leqslant n/2-1}\binom{n}{2k+1},$$

so the monomial generators for $L(n)\prod G(2k)$ have degree $n-1$ greater than the ones for $G(2k+1)$, which rules out equality of $L(n)\prod G(2k)$ and $\prod G(2k+1)$. A rather messy argument shows that each of the monomial generators for $L(n)\prod G(2k)$ is a multiple of a monomial generator for $\prod G(2k+1)$. Thus we have $L(n)\prod G(2k)\subsetneq \prod G(2k+1)$. The case for n odd is similar.

Example 3.2 (Any relationship between $G(3)L(2)$ and $L(1)L(3)$ is possible)**.** Take $R=k[X,Y,Z]$, k a field.

1. $G(3)L(2)=L(1)L(3)$: Take $A_1=(X)$, $A_2=(Y)$, $A_3=(X,Y)$. Since $A_1,A_2\subseteq A_3$, $G(3)L(2)=L(1)L(3)$ by Proposition 2.1 or [1, Lemma 2.1].

2. $G(3)L(2)\subsetneq L(1)L(3)$: Take $A_1=(X)$, $A_2=(Y)$, $A_3=(Z)$. By [1, Example 3.3], $G(3)L(2)\subsetneq L(1)L(3)$.

3. $G(3)L(2)\supsetneq L(1)L(3)$: Take

$$A_1=(X,Y),\quad A_2=(Y,Z),\quad A_3=(Z,X).$$

So $G(3)=(X,Y,Z)$ and

$$L(2)=((X,Y)\cap(X,Z))((X,Y)\cap(X,Z))((X,Z)\cap(Y,Z))$$
$$=(X,YZ)(Y,XZ)(Z,XY).$$

So

$$G(3)L(2)=(X,Y,Z)(X,YZ)(Y,XZ)(Z,XY)$$
$$=(X^3Y^2,X^2Y^3,Y^3Z^2,Y^2Z^3,X^2Z^3,X^3Z^2,X^2YZ,XY^2Z,XYZ^2).$$

On the other hand,

$$\begin{aligned} L(1)L(3) &= (X,Y)(Y,Z)(Z,X)((X,Y)\cap(Y,Z)\cap(Z,X)) \\ &= (X,Y)(Y,Z)(Z,X)(XY,YZ,ZX) \\ &= (X^3Y^2, X^2Y^3, Y^3Z^2, Y^2Z^3, X^2Z^3, X^3Z^2, \\ &\quad X^2Y^2Z, X^2YZ^2, XY^2Z^2, X^3YZ, XY^3Z, XYZ^3), \end{aligned}$$

which is easily checked to be a proper subset of $G(3)L(2)$.

4. $G(3)L(2)$ and $L(1)L(3)$ are incomparable: Take ideals

$$A_1 = (X^2, Y), \quad A_2 = (Y^3, Z), \quad A_3 = (XY, Z).$$

First we show $XY^5Z \in G(3)L(2)$ but $XY^5Z \notin L(1)L(3)$. The first inclusion follows from

$$\begin{aligned} XY^5Z &= Y\cdot Y^3\cdot Z\cdot XY \\ &\in (A_1+A_2+A_3)(A_1\cap A_2)(A_2\cap A_3)(A_1\cap A_3). \end{aligned}$$

On the other hand,

$$\begin{aligned} A_1\cap A_2\cap A_3 &= (A_1\cap A_2)\cap A_3 \\ &= (X^2Z, Y^3, YZ)\cap(XY,Z) = (X^2Z, XY^3, YZ). \end{aligned}$$

If

$$XY^5Z \in L(1)L(3) = A_1A_2A_3(A_1\cap A_2\cap A_3),$$

it follows that $XY^5Z \in A_1A_2A_3(XY^3, YZ)$. So $Y^2Z \in A_1A_2A_3$ or $XY^4 \in A_1A_2A_3$. This implies $Y^2Z \in (Y)(Z)(Z)$ or $XY^4 \in (X^2,Y)(Y^3)(XY)$, each of which obviously yields a contradiction. Thus we have proved that $XY^5Z \notin L(1)L(3)$.

Next we prove that $Y^2Z^3 \in L(1)L(3)$ but $Y^2Z^3 \notin G(3)L(2)$. The first inclusion follows from

$$Y^2Z^3 = Y\cdot Z\cdot Z\cdot YZ \in A_1A_2A_3(A_1\cap A_2\cap A_3).$$

Since each nonzero element of

$$L(2) = (X^2Z, Y^3, YZ)(XY^3, Z)(X^2Z, XY, YZ)$$
$$(\text{resp., } G(3) = (X^2, Y, Z))$$

has degree at least 5 (resp., 1), any nonzero element of $G(3)L(2)$ has degree at least 6. Hence Y^2Z^3 can not be contained in $G(3)L(2)$.

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References

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