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SECOND DEGREE GENERALIZED JACOBI ITERATION METHOD FOR SOLVING SYSTEM OF LINEAR EQUATIONS

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Abstract. In this paper, a Second degree generalized Jacobi Iteration method for solving system of linear equations, $Ax = b$ and discuss about the optimal values a_1 and b_1 in terms of spectral radius about for the convergence of SDGJ method of $x^{(n+1)} = b_1[D_m^{-1}(L_m + U_m)x^{(n)} + k_{1m}] - a_1 x^{(n-1)}$. Few numerical examples are considered to show that the effective of the Second degree Generalized Jacobi Iteration method (SDGJ) in comparison with FDJ, FDGJ, SDJ.

1. Introduction

Consider the linear system of :

$$Ax = b. \tag{1}$$

Where A is a nonsingular matrix of size nxn, x and b are n dimensional vectors. Splitting the matrix A as:

$$A = D - L - U. \tag{2}$$

Where D is a diagonal matrix and $-L$ and $-U$ are strictly lower and upper triangular part of A respectively. (a) First Degree of General Iteration method can be obtained in the form of:

$$x^{(n+1)} = G_1 x^{(n)} + C, \tag{3}$$

where G_1 is an iteration matrix and C is column vector

(b) First Degree of Jacobi (FDJ) can be obtained in the form of:

$$x^{(n+1)} = D^{-1}(L+U)x^{(n)} + D^{-1}b \tag{4}$$

(c) The First Degree of Generalized Jacobi (FDGJ) can be reformulated as [7]:

$$x^{(n+1)} = D_m{}^{-1}(L_m + U_m)x^{(n)} + D_m^{-1}b \tag{5}$$

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where D_m=banded diagonal matrix with band width $2m+1$, $-L_m$ and $-U_m$ are strictly lower and upper part of the matrix $A_m - D_m$.

(d) The second Degree of any iteration method has a form of

$$x^{(n+1)} = x^{(n)} + a_1(x^{(n)} - x^{(n-1)}) + b_1(x^{(n+1)} - x^{(n)}). \tag{6}$$

(e) The second Degree of Jacobi (SDJ) can be written as [4]:

$$x^{(n+1)} = b_1[D^{-1}(L+U) + k_1]x^{(n)} - a_1 x^{(n-1)} \tag{7}$$

for optimal values of a_1 and b_1, where k_1=$D^{-1}b$.

Note :- If matrix A is a row strictly diagonal dominant matrix, then the Jacobi method converges for any arbitrary choice of the initial approximation [4, 12]. In this paper, the Second degree of generalized Jacobi (SDGJ) iterative method is introduced in section 2, In section 3, I consider the comparison spectral radius of FDJ, SDJ and SDGJ methods and at the end of the section few numerical example and conclusion made.

2. Second degree generalized Jacobi (SDGJ) iterative method

Theorem. *Let A be non singular and* $A = D_m - L_m - U_m$*. If A is PD and SDD, then the Second degree of Generalized Jacobi iterative method is:*

$$x^{(n+1)} = b_1[D^{-1}(L+U) + k_1]x^{(n)} - a_1 x^{(n-1)}$$

for any initial guess and the optimal values for a1 and b1. Given:- Let A be non singular and $A = D_m - L_m - U_m$*. If A is PD and SDD. Required:- the Second degree of Generalized Jacobi iterative method is:*

$$x^{(n+1)} = b_1[D^{-1}(L+U) + k_1]x^{(n)} - a_1 x^{(n-1)}$$

Proof. Now consider Second Degree of Generalized of any iterative method from equation (1.0.6):

$$x^{(n+1)} = x^{(n)} + a_1(x^{(n)} - x^{(n-1)}) + b_1({G_1}^{(m)}x^{(n)} + {k_1}^{m} - x^{(n)}). \tag{8}$$

This also can be written after some computation one can get:

$$X^{(n+1)} = G_m x^{(n)} + H_m x^{(n-1)} + K_m. \tag{9}$$

Where $G_m = (1 + a_1 - b_1)I + b_1{G_1}^{(m)}$, $H_m = -a_1 I$ and $K_m = b_1{k_1}^{(m)} = b_1 D_m^{-1} b$. By using of Golub and Varga [4]

$$\begin{pmatrix} x^{(n)} \\ x^{(n+1)} \end{pmatrix} = \begin{pmatrix} 0 & I \\ H_m & G_m \end{pmatrix} \begin{pmatrix} x^{(n-1)} \\ x^{(n)} \end{pmatrix} + \begin{pmatrix} 0 \\ K_m \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x^{(n)} \\ x^{(n+1)} \end{pmatrix} = \widehat{G_m} \begin{pmatrix} x^{(n-1)} \\ x^{(n)} \end{pmatrix} + \begin{pmatrix} 0 \\ K_m \end{pmatrix}, where \widehat{G_m} = \begin{pmatrix} 0 & I \\ H_m & G_m \end{pmatrix}. \tag{10}$$

The above equation converges to the exact solution if $\sigma(\widehat{G_m}) < 1$, (i.e. The spectra radius of $\widehat{G_m}$ less than one). In order to solve the spectra radius of $\widehat{G_m}$, first we have to solve the eigen values λ_m of$\widehat{G_m}$.

$$\Rightarrow \widehat{G_m}\widehat{x} = \lambda_m \widehat{x}, Let \widehat{x} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \widehat{x} \neq 0, i.e. zero matrix \tag{11}$$

$\Rightarrow \widehat{G_m} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \widehat{x} = \lambda_m \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & I \\ H_m & G_m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda_m \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$
where $u_1 \in \Re, H_m, G_m, I \in \Re^{nxn}$.

$$\Rightarrow u_2 = \lambda_m u_1 and H_m u_1 + G_m u_2 = \lambda_m u_2 I \tag{12}$$

but $u_2 = \lambda_m u_1$ substitute into (12) we get

$\Rightarrow H_m u_1 + G_m \lambda_m u_1 = \lambda_m \lambda_m u_1 I$

$\Rightarrow H_m u_1 + G_m \lambda_m u_1 - \lambda_m{}^2 u_1 I = 0, 0 \in \Re^{nxn}$

$\Rightarrow u_1(H_m + G_m \lambda_m - \lambda_m{}^2 I) = 0$. We know that

$$u_1 \neq 0 hence H_m + G_m \lambda_m - \lambda_m{}^2 I = 0 \tag{13}$$

$\Rightarrow \lambda_m{}^2 I - G_m \lambda_m - H_m = 0.$

Substitute: $G_m = (1 + a_1 - b_1)I + b_1 G_1{}^{(m)}, H_m = -a_1 I, K_m = b_1 k_1{}^m$

$\Rightarrow \lambda_m{}^2 I - [(1 + a_1 - b_1)I + b_1 G_1{}^{(m)}]\lambda_m + a_1 I = 0$

$\Rightarrow \lambda_m{}^2 I - (1 + a_1 - b_1)\lambda_m I - b_1 G_1{}^{(m)} \lambda_m + a_1 I = 0$

$\Rightarrow -b_1 \lambda_m [-\frac{\lambda_m{}^2 I}{b_1 \lambda_m} + \frac{(1+a_1-b_1)\lambda_m I}{b_1 \lambda_m} + \frac{b_1 G_1{}^{(m)} \lambda_m}{b_1 \lambda_m} - \frac{a_1 I}{b_1 \lambda_m}] = 0$

$\Rightarrow -\frac{\lambda_m{}^2 I}{b_1 \lambda_m} + \frac{(1+a_1-b_1)\lambda_m I}{b_1 \lambda_m} + \frac{b_1 G_1{}^m \lambda_m}{b_1 \lambda_m} - \frac{a_1 I}{b_1 \lambda_m} = 0$ but $-b_1 \lambda_m \neq 0$

$\Rightarrow G_1{}^m + \frac{(1+a_1-b_1)I}{b_1} - \frac{(a_1+\lambda_m{}^2)I}{b_1 \lambda_m} = 0, 0 \in \Re^{nxn}$.

If $\lambda_m$is the eigen value of $\widehat{G}_m$ and μ_m is the eigen valu of $G_1{}^m$, then
$\det(G_1{}^m + \frac{(1+a_1-b_1)I}{b_1} - \frac{(a_1+\lambda_m{}^2)I}{b_1 \lambda_m}) = 0$, has a solution

$$\mu_m + \frac{(1 + a_1 - b_1)}{b_1} - \frac{(a_1 + \lambda_m{}^2)}{b_1 \lambda_m} = 0, i.e., 0 \in \Re \tag{14}$$

Let the eigen value$\lambda_m of \widehat{G}_m be \lambda_m = v e^{i\theta} = v(\cos\theta + i \sin\theta)$, then the spectral radius of

$$\widehat{G}_m = \max_{i=0}^{n} |(\lambda_m)_i| = |v e^{i\theta}| = |v| = \sqrt{a_1}$$

So when we insert $\lambda_m = ve^{i\theta}$ in the above equation. $i.e.\mu_m + \frac{(1+a_1-b_1)}{b_1} - \frac{(a_1+ve^{2i\theta})}{b_1 ve^{i\theta}} = 0$

$\Rightarrow \mu_m + \frac{1+a_1-b_1}{b_1} - \frac{v^2e^{2i\theta}}{b_1 ve^{i\theta}} - \frac{a_1}{b_1 ve^{i\theta}} = 0$

$\Rightarrow \mu_m + \frac{1+a_1-b_1}{b_1} - \frac{v(\cos\theta + i\sin\theta)}{b_1} - \frac{a_1(\cos\theta - i\sin\theta)}{b_1 v} = 0$

$\Rightarrow \mu_m + \frac{1+a_1-b_1}{b_1} - \frac{v\cos\theta}{b_1} - i\frac{v\sin\theta}{b_1} - \frac{a_1\cos\theta}{b_1 v} + i\frac{a_1\sin\theta}{vb_1} = 0$

$$\therefore \mu_m = \frac{1}{b_1}(v + \frac{a_1}{v})\cos\theta - \frac{1+a_1-b_1}{b_1} + i\frac{1}{b_1}(v - \frac{a_1}{v})\sin\theta = 0 \tag{15}$$

$Re\mu_m = \frac{1}{b_1}(v + \frac{a_1}{v})\cos\theta - \frac{1+a_1-b_1}{b_1}$

$$\Rightarrow \cos\theta = \frac{Re\mu_m + \frac{1+a_1-b_1}{b_1}}{\frac{1}{b_1}(v + \frac{a_1}{v})} \tag{16}$$

$Im\mu_m = \frac{1}{b_1}(v - \frac{a_1}{v})\sin\theta = 0$

$$\Rightarrow \sin\theta = \frac{IM\mu_m}{\frac{1}{b_1}(v + \frac{a_1}{v})} \tag{17}$$

We know that $\cos^2\theta + \sin^2\theta = 1$

$$\left[\frac{Re\mu_m + \frac{1+a_1-b_1}{b_1}}{\frac{1}{b_1}(v + \frac{a_1}{v})}\right]^2 + \left[\frac{IM\mu_m}{\frac{1}{b_1}(v + \frac{a_1}{v})}\right]^2 = 1 \quad \text{is ellipse} \tag{18}$$

$$\text{centre} = c(h,k) = \left(-\frac{1+a_1-b_1}{b_1}, 0\right) \tag{19}$$

$$\text{Length of semi-major axis} = a' = \frac{1}{b_1}(v + \frac{a_1}{v}) \tag{20}$$

$$\text{Length of semi-minor axis} = b' = \frac{1}{b_1}(v - \frac{a_1}{v}) \tag{21}$$

$$\text{Foci} = F_1 = (h-c,0) = \left(-\frac{1+a_1-b_1}{b_1} - \frac{2\sqrt{a_1}}{b_1}, 0\right) = (\alpha, 0) \tag{22}$$

$$\text{Foci} = F_2 = (h+c,0) = \left(-\frac{1+a_1-b_1}{b_1} + \frac{2\sqrt{a_1}}{b_1}, 0\right) = (\beta, 0) \tag{23}$$

$v_1 = (h-a',0) = (-\frac{1+a_1-b_1}{b_1} - \frac{1}{b_1}(v + \frac{a_1}{v}), 0)$

$v_2 = (h+a',0) = (-\frac{1+a_1-b_1}{b_1} + \frac{1}{b_1}(v + \frac{a_1}{v}), 0)$

$v_3 = (h, k+b') = (-\frac{1+a_1-b_1}{b_1}, \frac{1}{b_1}(v - \frac{a_1}{v}))$

$v_4 = (h, k-b') = (-\frac{1+a_1-b_1}{b_1}, -\frac{1}{b_1}(v - \frac{a_1}{v}))$

Before we proof theorem let us proof the following Lemmas.

Lemma 1. *If μ_m is real, then $\alpha \le \mu \le \beta < 1$,for any foci α and β which are real.*

Given:-μ_m is real number

Required:- $\alpha \le \mu \le \beta < 1$,for any foci α and β which are real

Proof. $\Rightarrow \mu_m = \frac{1}{b_1}(v + \frac{a_1}{v})\cos\theta - \frac{1+a_1-b_1}{b_1}$, since μ_m is real number.In this equation θ varies.
$\Rightarrow -\frac{1}{b_1}(v + \frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1} \le \mu_m \le \frac{1}{b_1}(v + \frac{a_1}{v})\cos\theta - \frac{1+a_1-b_1}{b_1}$, since $-1 \le \cos\theta \le 1$

$$\Rightarrow \alpha \le \mu_m \le \beta < 1. \tag{24}$$

Because α and β from equation (22) and (23) and to be convergent $\rho({G_1}^{(m)}) < 1$ so all the eigen values must be less than 1.

Lemma 2. *If the eigen values μ_m of ${G_1}^{(m)} < 1$ are real and lie in the interval $\alpha \le \mu \le \beta < 1$, then the optimal choices of a_1 and b_1 must satisfy the following conditions:*

(a) $v^2 = a_1$. (25)

(b) $\dfrac{\alpha+\beta}{2} = -\dfrac{1+a_1-b_1}{b_1}$. (26)

(c) $\dfrac{\beta-\alpha}{2} = \dfrac{2v}{b_1}$. (27)

(d) $2v = \dfrac{\beta-\alpha}{2-(\alpha+\beta)}(1+v^2)$. (28)

Given:-The eigen values μ_m of ${G_1}^{(m)} < 1$ are real and lie in the interval $\alpha \le \mu \le \beta < 1$.

Required:- proof a-d.

Proof. (a) we know know μ_m is real, then $\frac{1}{b_1}(v - \frac{a_1}{v})\sin\theta = 0$, we have $\frac{1}{b_1}\sin\theta = 0$ or $(v - \frac{a_1}{v}) = 0$, so we get $v^2 = a_1$ or $\sin\theta = 0$,from the second equation we have $\theta = 2\pi, n = 0, 1, 2, \ldots$
Therefore $V^2 = a_1$.

(b) From the (22) and (23) and from (a), we get:-

$$\alpha = \frac{-2v}{b_1} - \frac{1+a_1-b_1}{b_1} \quad \text{and} \quad \beta = \frac{2v}{b_1} - \frac{1+a_1-b_1}{b_1}$$

$\Rightarrow \frac{\alpha+\beta}{2} = -\frac{1+a_1-b_1}{b_1}$ mid point formula.

(c) From (b), we have $\frac{\beta-\alpha}{2} = \frac{2v}{b_1}$.

(d) From (b), we have $\frac{\alpha+\beta}{2} = -\frac{1+a_1-b_1}{b_1}$

$$\Rightarrow 1 - \frac{\alpha+\beta}{2} = 1 - (-\frac{1+a_1-b_1}{b_1}) \Rightarrow \frac{2-\alpha+\beta}{2} = 1 + (\frac{1+a_1-b_1}{b_1}) = \frac{1+a_1}{b_1} \tag{$*$}$$

Divide equation (27) by (∗), we get

$\Rightarrow \frac{\frac{\beta-\alpha}{2}}{\frac{2-(\alpha+\beta)}{2}} = \frac{\frac{2v}{b_1}}{\frac{1+a_1}{b_1}}$

$\Rightarrow \frac{\beta-\alpha}{2-(\alpha+\beta)} = \frac{2v}{1+a_1}$

$\Rightarrow 2v = \frac{\beta-\alpha}{2-(\alpha+\beta)}(1+v^2).$

Lemma 3. *If $\overline{\mu}_m$ is the spectral radius of $G_1{}^{(m)}$, then*

$$\overline{\mu}_m = \frac{\beta-\alpha}{2-(\alpha+\beta)} \tag{29}$$

Given :- μ_m is the spectral radius of $G_1{}^{(m)}$.

Required:- $\mu_m = \frac{\beta-\alpha}{2-(\alpha+\beta)}$.

Proof. $\mu_m = \frac{1}{b_1}(v+\frac{a_1}{v})\cos\theta - \frac{1+a_1-b_1}{b_1}$.

By definition and derivative of functions in calculus

$$\frac{d\mu_m}{d\theta} = \frac{d}{d\theta}[\frac{1}{b_1}(v+\frac{a_1}{v})\cos\theta - \frac{1+a_1-b_1}{b_1}] = -\frac{1}{b_1}(v+\frac{a_1}{v})\sin\theta.$$

To calculate the maximum and minimum value the above equation equate to zero.

$-\frac{1}{b_1}(v+\frac{a_1}{v})\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, \ldots$. When $\theta = 0$, then $\mu_m = \frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1}$.

When $\theta = \pi$, then $\mu_m = -\frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1}$.

When $\theta = 2\pi$, then $\mu_m = \frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1}$.

From the above the maximum value occurs at $\theta = 0$ and 2π

$$\Rightarrow \max_{i=1}^{n}(\mu_m) = \frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1} = \beta$$

The minimum value occurs at

$$\theta = \pi \Rightarrow \min_{i=1}^{n}\mu_m = -\frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1} = \alpha$$

$\Rightarrow \overline{\mu}_m = \max_{i=1}^{n} |\mu_m| = \max_{i=1}^{n} |\frac{1}{b_1}(v+\frac{a_1}{v})\cos\theta - \frac{1+a_1-b_1}{b_1}| = \frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1}$,

since $-1 \le \cos\theta \le 1$ and

$\Rightarrow -\overline{\mu}_m = \min_{i=1}^{n}(-|\mu_m|) = \min -|-\frac{1}{b_1}(v+\frac{a_1}{v})\cos\theta - \frac{1+a_1-b_1}{b_1}| = \frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1}$

$\Rightarrow \overline{\mu}_m = \frac{2v}{b_1}(-\frac{1+a_1-b_1}{b_1} = \frac{2v}{b_1} - \frac{1+a_1}{b_1} + 1 = \frac{\beta-\alpha}{2} - \frac{2-(\alpha+\beta)}{2} + 1 = \beta$ by equation (27) and (*)

Therefore $\overline{\mu}_m = \beta$ (i)

$$\Rightarrow -\overline{\mu}_m = -\frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1} = -\frac{2v}{b_1} - \frac{1+a_1}{b_1} + 1 = \alpha.$$

Therefore $\overline{\mu}_m = -\alpha$ (ii)

By (i) and (ii) $2\overline{\mu}_m = \beta - \alpha$ (iii)

and $2 = 2-(\alpha+\beta)$ (iv)

and divide equation (iii) by (iv), we get

$$\therefore \overline{\mu}_m = \frac{\beta-\alpha}{2-(\alpha+\beta)}.$$

Now let us determine the values of a_1 and b_1. First, let us find a_1 from Lemma3d equation (28)

$$\Rightarrow\ 2v = \frac{\beta-\alpha}{2-(\alpha+\beta)}(1+v^2) \quad \text{by Lemma 3d, we have}$$
$$\Rightarrow\ 2v = \overline{\mu}_m(1+v^2)$$
$$\Rightarrow\ \overline{\mu}_m v^2 - 2v + \overline{\mu}_m = 0.$$

This is the equation of quadratic whose graph is a parabola and the minimum value occurs at p=$(\frac{1}{\overline{\mu}_m}, \frac{\overline{\mu}_m{}^2-1}{\overline{\mu}_m})$ since $\overline{\mu}_m > 0$.
One can solve by quadratic formula of the above equation:

$$v = \frac{2\pm\sqrt{4-4\overline{\mu}_m{}^2}}{2\mu_m} = \frac{1\pm\sqrt{1-\overline{\mu}_m{}^2}}{\overline{\mu}_m}$$
$$\Rightarrow \quad v_1 = \frac{1+\sqrt{1-\overline{\mu}_m{}^2}}{\overline{\mu}_m} \quad \text{and} \quad v_2 = \frac{1-\sqrt{1-\overline{\mu}_m{}^2}}{\overline{\mu}_m}.$$

The smallest value is $\Rightarrow v_2 = \frac{1-\sqrt{1-\overline{\mu}_m{}^2}}{\overline{\mu}_m}$. Let $1+v^2 = \omega \Rightarrow a_1 = \omega - 1$

$$\Rightarrow\ 1+v^2 = \frac{2}{1+\sqrt{1-\overline{\mu}^2_m}}$$
$$\therefore\ a_1 = \frac{\overline{\mu}^2_m}{(1+\sqrt{1-\overline{\mu}^2_m})^2}, \quad \text{since } a_1 = v^2.$$

Secondly,let us find $b_1 \Rightarrow b_1 = \frac{4v}{\beta-\alpha}$ by equation (27) $\Rightarrow b_1 = \frac{4v}{\beta-\alpha} = \frac{2\mu_m(1+v^2)}{\beta-\alpha}$

$$\therefore\quad b_1 = \frac{4}{(1+\sqrt{1-\mu^2_m})(2-(\alpha+\beta))}.$$

Lemma 4. *If matrix A is positive definite matrix and if* $G_1{}^{(m)}$ *is Jacobi iterative matrix, then* $\beta = -\alpha = \overline{\mu}_m = \sigma(G_1{}^m)$.
Given :- matrix A is positive definite matrix and if $G_1{}^{(m)}$ *is Jacobi iterative matrix*
Required:- $\beta = -\alpha = \overline{\mu}_m = \sigma(G_1{}^m)$.

Proof. In order to proof this Lemma, we have to use Lema 3

$$\Rightarrow \overline{\mu}_m = \max_{i=1}^{n} | \mu_m |= \frac{1}{b_1}(v + \frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1} = \beta$$
$$\Rightarrow -\overline{\mu}_m = -\frac{1}{b_1}(v + \frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1} = \alpha$$
$$\therefore \beta = -\alpha = \overline{\mu}_m.$$

Now we can find the optimal value of a_1 and b_1

$$\text{i.e. } a_1 = \frac{\overline{\mu}_m^2}{(1+\sqrt{1-\overline{\mu}_m^2})^2}$$
$$b_1 = \frac{2}{1+\sqrt{1-\overline{\mu}_m^2}},$$

since $\beta = -\alpha \Rightarrow \alpha + \beta = 0$.

Now let us find second degree of Generalized Jacobi(SDGJ) method:-

$$\Rightarrow \frac{1+a_1-b_1}{b_1} = \frac{\alpha+\beta}{2}$$
$$\Rightarrow \frac{1+a_1-b_1}{b_1} = \frac{\alpha-\alpha}{2} \quad \text{since } \beta = -\alpha$$
$$\Rightarrow (1+a_1-b_1) = 0.$$

From the second degree

$$\Rightarrow \quad x^{(n+1)} = G_m x^{(n)} + H_m x^{(n-1)} + k_m$$
$$\Rightarrow \quad x^{(n+1)} = [(1-b_1+a_1)I + b_1 G_1{}^m]x^{(n)} + (-a_1 I)x^{(n-1)} + b_1 k_1^m$$
$$\Rightarrow \quad x^{(n+1)} = b_1 G_1 m x^{(n)} - a_1 x^{(n-1)} + b_1 k_1{}^m$$
$$\therefore \quad x^{(n+1)} = b_1[D_m{}^{-1}(L_m + U_m)x^{(n)} + k_1{}^m] - a_1 x^{(n-1)},$$

where

$$a_1 = \frac{\overline{\mu}_m^2}{(1+\sqrt{1-\overline{\mu}_m^2})^2}, \quad b_1 = \frac{2}{1+\sqrt{1-\overline{\mu}_m^2}}.$$

Note:

1. $x^{(n)} = G_1{}^{(m)} x^{(0)} + k_1{}^{(m)}),\ x^{n+1} = b_1(G_1 m x^{(n)} + k_1{}^m) - a_1 x^{(n-1)}$.
2. So far, One can seen from the above that all the eigenvalues of the Jacobi iterative matrix are real numbers. Next let us see the eigen values of the iterative matrix are complex. In the real eigen value case, optimum convergence is obtained by setting α and β equal to the smallest and largest eigen values respectively. But in complex eigen value case, the situation is different by [8]. In the real case, the eigenvalues $_m$ of $G_1{}^{(m)}$ are bounded by a

closed interval of the real axis such that $\alpha \le \mu_m \le< 1$, for$\alpha \ne \beta$. Moreover, the complex case is that when the eigenvalues μ_m of ${G_1}^{(m)}$ are bounded by the elliptic region R in the complex plane with center on the real axis such that $1 not \in Re, i.e., \mu_m \subseteq \Re$ [6].

Let us see two cases when the eigenvalues are complex:

Case-1. $\Rightarrow \mu_m$ is purely imaginary, i.e. $\mu_m = i\frac{1}{b}(v - \frac{a_1}{v})\sin\theta$.
If the endpoints are purely imaginary, say αi and βi, then the ellipse will be line segment on the imaginary axis.Hence it is similar to real case.

Case-2. $\Rightarrow \mu_m + \frac{1+a_1-b_1}{b_1} - \frac{v\cos\theta}{b_1} - i\frac{v\sin\theta}{b_1}$.
As given by [6] and in section 2, an optimal second degree of generalized Jacobi iterative method for the complex case is given by

$$x^{(1)} = \omega_0(G_m x^{(0)} + k_1(m)) + (1-\omega_0)x^{(0)}$$
$$x^{(n+1)} = \omega_b[\omega_0({G_1}^m x^{(n)} + {k_1}^m) + (1-\omega_0)x^{(n)}] + (1-\omega_b)x^{(n-1)}, n \ge 1,$$

where ,$\omega_0 = \frac{2}{2-(\alpha+\beta)}, \omega_b = \frac{2}{1+\sqrt{1-\overline{\sigma}^2}}, \overline{\sigma} = \frac{\beta-\alpha}{2-(\alpha+\beta)}$.

Let R be an elliptical region with foci α and β which are real.

- If $\beta = -\alpha = \overline{\mu}_m$, where $\overline{\mu}_m = \sigma {G_1}^m$, then $\omega_0 = 1$ and $\omega_b = \frac{2}{1+\sqrt{1-\overline{\mu}_m^2}}$, and
- If $\alpha = 0$ and $\beta = {\mu_m}^2$, where $\overline{\mu}_m = \sigma(G_1^{(m)})$, then $\omega_0 = \frac{2}{2-{\overline{\mu}_m}^2}$ and $\omega_b = \frac{2}{1+\sqrt{1-\overline{\delta}^2}}$, where $\overline{\delta} = \frac{\overline{\mu_m}^2}{2-{\overline{\mu}_m}^2}$ by [6].

Let R be a line segement with, and $\zeta = t\beta + (1-t)\alpha$ for $0 \le t \le 1$ [6].

- If the endpoints are purely imaginary, say αi and βi, then $\omega_b = \frac{2}{1+\sqrt{1-\overline{\mu}_m^2}}$.
- $\beta = -\alpha = \overline{\mu}_m$, where $\overline{\mu}_m = \sigma({G_1}^m)$, then $\omega_0 = 1$ and $\omega_b = \frac{2}{1+\sqrt{1-{\overline{\mu}_m}^2}}$.

3. Relationship between spectral radius

As we have seen above the spectral radius of
✓First degree Jacobi method(FDJ) is $\overline{\mu}$.
✓Second degree Jacobi method(SDJ) is $\sqrt{a_1} = \frac{\overline{\mu}}{1+\sqrt{1-\overline{\mu}^2}}$.
✓Second degree generalized Jacobi method(SDGJ) is $\sqrt{a_1} = \frac{\overline{\mu}_m}{1+\sqrt{1-\overline{\mu_m}^2}}$.

That is one can see $\frac{\overline{\mu}_m}{1+\sqrt{1-{\overline{\mu}_m}^2}} \le \frac{\overline{\mu}}{1+\sqrt{1-\overline{\mu}^2}} \le \overline{\mu}$ since $1+\sqrt{1-\overline{\mu}^2} > 0$ and also $\overline{\mu}_m \le \overline{\mu}, \forall m = 0,1,2,3,...,n$.
If $m = 0$, then $\overline{\mu}_0 = \overline{\mu}$, i.e. SDJ=SDGJ.

4. Numerical examples

a. Solve the following PD matrix using FDJ, FDGJ, SDJ and SDGJ iterative methods.

$$\begin{aligned} 6x_1 + 2x_2 + 2x_3 &= 5 \\ 2x_1 + 8x_2 + 2x_3 &= 6 \\ 2x_1 + 2x_2 + 210x_3 &= 7 \end{aligned} \quad .$$

Solution :- For FDJ of PD

i. $\sigma(G_1) = \overline{\mu} = 0.51456716,\ a_1 = 0.076744954,\ b_1 = 1.076744954.$ The spectral radius of SDJ is ${a_1}^{1/2} = \frac{\overline{\mu}}{1+\sqrt{1-\overline{\mu}^2}} = 0.2770287863.$

ii. For SDGJ when $m = 1$.

$$\overline{\mu}_m = 0.297246773, \quad a_1 = \frac{{\overline{\mu}_m}^2}{(1+\sqrt{1-{\overline{\mu}_m}^2})^2} = 0.023122209, b_1 = \frac{2}{1+\sqrt{1-{\overline{\mu}_m}^2}} = 1.023122209.$$

The spectral radius of SDGJ is ${a_1}^{1/2} = \frac{\overline{\mu}_m}{1+\sqrt{1-{\overline{\mu}_m}^2}} = 0.1520598862.$

Note:- If $m = 0$, then the spectral radius of SDJ=Spectral radius of SDGJ

Method	First degree Jacobi Method	Second degree generalized Jacobi Method when m=0	Second degree generalized Jacobi method when m=1
Spectral radius of PD	0.51456716	0.2770287863	0.1520598862

b. Solve the following SDD matrix using FDJ,FDGJ,SDJ and SDGJ iterative methods.

$$\begin{aligned} 10x_1 + 2x_2 + 4x_3 &= 8 \\ 2x_1 + 10x_2 + 3x_3 &= 7 \\ 5x_1 + 3x_2 + 10x_3 &= 9 \end{aligned}$$

Solution :- For FDJ and SDDJ of SDD

i. $\sigma(G_1) = \overline{\mu} = 0.615050468, a_1 = 0.118263184, b_1 = 1.118263184.$
The spectral radius of SDJ is ${a_1}^{1/2} = \frac{\overline{\mu}}{1+\sqrt{1-\overline{\mu}^2}} = \sqrt{0.118263184} = 0.3438941465.$

ii. For SDGJ when $m = 1$. $\overline{\mu}_m = 0.498227834,\ a_1 = \frac{{\overline{\mu}_m}^2}{(1+\sqrt{1-{\overline{\mu}_m}^2})^2} = 0.0712108,\ b_1 = \frac{2}{1+\sqrt{1-{\overline{\mu}_m}^2}} =$ 1.0712108.
The spectral radius of SDGJ is ${a_1}^{1/2} = \frac{\overline{\mu}_m}{1+\sqrt{1-{\overline{\mu}_m}^2}} = 0.2668535179.$

Note:- If $m = 0$, then the spectral radius of SDJ=Spectral radius of SDGJ

Method	First degree Jacobi Method	Second degree generalized Jacobi Method when m=0	Second degree generalized Jacobi method when m=1
Spectral radius of SDD	0.615050468	0.3438941465	0.2668535179

Table 1. All numerical result of Positive definite(PD) matrix of

First degree Jacobi(FDJ)				First degree Generalized Jacobi(FDGJ)		
n	$x_1{}^{(n)}$	$x_2{}^{(n)}$	$x_3{}^{(n)}$	$x_1{}^{(n)}$	$x_2{}^{(n)}$	$x_3{}^{(n)}$
0	0	0	0	0	0	0
1	0.833335	0.7500000	0.700000	0.692308	0.423077	0.615385
2	0.350001	0.366667	0.383333	0.454142	0.522189	0.457101
3	0.45889	0.566667	556666	0.516557	0.493229	0.510526
4	0.45889	0.465	0.47	0.495836	0.501967	0.496295
5	0.521668	0.517778	0.515222	0.501434	0.499404	0.500952
6	0.0.489001	0.490778	0.49211	0.499625	0.500174	0.499678
7	0.505705	0.504722	0.504044	0.500125	0.49947	0.500086
8	0.497079	0.497915	0.497915	0.499966	0.500015	0.499972
9	0.501508	0.501252	0.501072	0.500011	0.499995	0.500008
10	0.499226	0.499355	0.499448	0.499997	0.500001	0.499998
11	0.5004	0.500332	0.500384	0.500001	0.5	0.500001
12	0.499762	0.499804	0.499854	0.500000	0.500000	0.500000
13	0.500115	0.500096	0.500087			
14	0.49994	0.49995	0.499858			
15	0.500032	0.500026	0.500022			
16	0.499985	0.499987	0.499988			
17	0.500009	0.500007	0.500006			
18	0.499997	0.499996	0.49997			
19	0.500003	0.500002	0.500001			
20	0.5	0.499999	0.4999			
21	0.500002	0.500000	0.500000			
22	0.500000	0.500000	0.500000			
Second degree Jacobi(SDJ)				Second degree Generalized Jacobi(SDGJ)		
n	$x_1{}^{(n)}$	$x_2{}^{(n)}$	$x_3{}^{(n)}$	$x_1{}^{(n)}$	$x_2{}^{(n)}$	$x_3{}^{(n)}$
0	0	0	0	0	0	0
1	0.833333	0.75	0.7	0.692308	0.423077	0.615385
2	0.376861	0.394807	0.412752	0.464643	0.534264	0.46767
3	0.543488	0.537447	0.533822	0.508335	0.496511	0.506093
4	0.483871	0.487262	0.489267	0.498376	0.501067	0.498824
5	0.505087	0.504357	0.503621	0.500279	0.499869	0.500234
6	0.498374	0.498634	0.49879	0.499945	0.500015	0.499962
7	0.500534	0.500429	0.500366	0.500009	0.499996	0.500007
8	0.499839	0.499863	0.499885	0.499998	0.500001	0.499999
9	0.500049	0.500041	0.500036	0.500000	0.5000090	0.500000
10	0.499985	0.499988	0.499989			
11	0.500004	0.500004	0.500003			
12	0.499999	0.499999	0.499999			
13	0.500000	0.500000	0.500000			

Table 2. All numerical result of Strictly diagonal dominate(SDD) matrix of

First degree Jacobi(FDJ)				First degree Generalized Jacobi(FDGJ)		
n	$x_1{}^{(n)}$	$x_2{}^{(n)}$	$x_3{}^{(n)}$	$x_1{}^{(n)}$	$x_2{}^{(n)}$	$x_3{}^{(n)}$
0	0	0	0	0	0	0
1	0.8	0.7	0.9	0.721348	0.393258	0.782022
2	0.3	0.53	0.29	0.377195	0.549981	0.374332
3	0.614	0.583	0.645	0.555536	0.473655	0.569306
4	0.4254	0.4451	0.4181	0.469783	0.512475	0.46849
5	0.54374	0.53203	0.55377	0.513906	0.493491	0.517061
6	0.472086	0.479495	0.468521	0.492554	0.5031107	0.492114
7	0.516693	0.512235	0.520109	0.503476	0.498391	0.504206
8	0.489509	0.492298	0.487983	0.498163	0.500775	0.498030
9	0.506347	0.504654	0.507556	0.500868	0.499602	0.501038
10	0.496047	0.497099	0.495430	0.499546	0.500193	0.499508
11	0.502408	0.501766	0.502847	0.500217	0.499901	0.500257
12	0.498508	0.498905	0.498266	0.499888	0.500048	0.499877
13	0.500913	0.500669	0.501075	0.500054	0.499976	0.500063
14	0.499436	0.499586	0.499343	0.499972	0.500012	0.499969
15	0.500346	0.500254	0.500406	0.500014	0.499994	0.500016
16	0.499787	0.499844	0.499751	0.499993	0.500003	0.499992
17	0.500131	0.500096	0.500153	0.500004	0.499998	0.500004
18	0.49992	0.499941	0.499906	0.499998	0.500001	0.499998
19	0.500049	0.500036	0.500058	0.500001	0.500000	0.500001
20	0.49997	0.499978	0.499965	0.500000	0.500000	0.499999
21	0.500018	0.500014	0.500022	0.500000	0.500000	0.500000
22	0.499988	0.499992	0.499987			
23	0.500007	0.500005	0.500008			
24	0.499996	0.499996	0.499995			
25	0.500003	0.500002	0.500003			
26	0.499998	0.499999	0.499998			
27	0.500001	0.500001	0.500001			
28	0.499999	0.500000	0.499999			
29	0.500000	0.500000	0.500001			
30	0.500000	0.500000	0.500000			

5. Conclusions

Several iterative techniques for the solution of linear system of equations have been proposed in different literature in the past. In my paper, new modifying technique of second degree generalized Jacobi iterative method for solving system of linear system of equations are developed using splitting the matrix A=D-L-U for FDJ and SDJ methods, and $A = D_m -$

	Second degree Jacobi(SDJ)			Second degree Generalized Jacobi(SDGJ)		
n	$x_1{}^{(n)}$	$x_2{}^{(n)}$	$x_3{}^{(n)}$	$x_1{}^{(n)}$	$x_2{}^{(n)}$	$x_3{}^{(n)}$
0	0	0	0	0	0	0
1	0.8	0.7	0.9	0.721348	0.393258	0.782022
2	0.335479	0.391392	0.324296	0.404055	0.589146	0.400988
3	0.567405	0.553690	0.581119	0.531080	0.485512	0.537932
4	0.471164	0.478093	0.465079	0.489092	0.501089	0.488173
5	0.512548	0.508590	0.513879	0.503362	0.498493	0.503903
6	0.495281	0.496531	0.494232	0.501105	0.500717	0.498803
7	0.501872	0.501447	0.502161	0.500245	0.500249	0.499088
8	0.499268	0.499476	0.499150	0.500312	0.499949	0.499954
9	0.500276	0.500196	0.500330	0.499991	0.500036	0.499882
10	0.499895	0.499920	0.499880	0.50003	0.499996	0.500010
11	0.500039	0.500029	0.500047	0.499995	0.500003	0.499991
12	0.499985	0.499989	0.499983	0.500002	0.499999	0.500003
13	0.500005	0.500004	0.500007	0.499999	0.500000	0.499999
14	0.499998	0.499998	0.499998	0.500000	0.500000	0.500000
15	0.500001	0.500000	0.500001			
16	0.500000	0.500000	0.500000			

$L_m - U_m$ for the new SDGJ method for $m = 0, 1, 2, \ldots$. I studied spectral radius and their rate of convergence properties. As consider in it from the values tabulated in different table for various examples that the new methods converge better than the Conventional methods for the solution of the problem that I consider. In general, the results of numerical examples considered clearly demonstrate the accuracy of the methods developed in this paper. It is conjectured that the rate of convergence of the method that developed in this paper can be further enhanced by using extrapolating techniques.

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