OSCILLATION THEOREMS FOR SECOND ORDER DIFFERENCE EQUATIONS WITH NEGATIVE NEUTRAL TERM

DEVARAJULU SEGHAR, ETHIRAJU THANDAPANI AND SANDRA PINELAS

Abstract. In this paper we obtain some new oscillation criteria for the neutral difference equation

$$\Delta\Big(a_n(\Delta(x_n - p_n x_{n-k}))\Big) + q_n f(x_{n-l}) = 0$$

where $0 \le p_n \le p < 1$, $q_n > 0$ and l and k are positive integers. Examples are presented to illustrate the main results. The results obtained in this paper improve and complement to the existing results.

1. Introduction

Consider the second order neutral difference equation of the form

$$\Delta\Big(a_n(\Delta(x_n - p_n x_{n-k}))\Big) + q_n f(x_{n-l}) = 0, \, n \in \mathbb{N}(n_0) \tag{1.1}$$

where $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \ldots\}$, n_0 is a nonnegative integer, subject to the following conditions:

(H_1) $\{a_n\}$ is a positive real sequence with $\sum\limits_{n=n_0}^{\infty} \dfrac{1}{a_n} < \infty$;

(H_2) $\{p_n\}$ is a real sequence with $0 \le p_n \le p < 1$ for all $n \in \mathbb{N}(n_0)$;

(H_3) $\{q_n\}$ is a positive real sequence for all $n \in \mathbb{N}(n_0)$;

(H_4) l and k are positive integers;

(H_5) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(u) > 0$ for $u \ne 0$, and there exists a constant $M > 0$ such that $\dfrac{f(u)}{u^\alpha} > M$ for all $u \ne 0$, where α is a ratio of odd positive integers.

Let $\theta = \max\{k, l\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for $n \geq n_0 - \theta$ and satisfying equation (1.1) for all $n \in \mathbb{N}(n_0)$. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and it is nonoscillatory otherwise.

From a review of literature, it is known that there are many results available on the oscillatory and asymptotic behavior of solutions of equation (1.1) when the neutral term is non-negative, i.e., $p_n \leq 0$; see for example [1, 2, 3, 9, 15] and the references cited therein. However, there are few results available on the oscillatory behavior of solutions of equation (1.1) when the neutral term is negative; see, for example [4, 6, 7, 8, 11, 12, 13, 14, 16, 17] and the references therein.

In [1], we see that the oscillatory behavior of the equation

$$\Delta^2\left(x_n - p x_{n-k}\right) + q_n x_{n-l} = 0, \; n \in \mathbb{N}(n_0), \tag{1.2}$$

is discussed and in [14], the authors studied the oscillatory and asymptotic behavior of equation

$$\Delta\left(a_n(\Delta(x_n - p_n x_{n-k}))\right) + q_n x_{n-l}^{\alpha} = 0, \; n \in \mathbb{N}(n_0), \tag{1.3}$$

with $\sum\limits_{n=n_0}^{\infty} \dfrac{1}{a_n} = \infty$. The results obtained for equations (1.2) and (1.3) has been improved and generalized by other authors. We mention Thandapani et al.[12] studied the oscillation of

$$\Delta\left(a_n(\Delta(x_n - p_n x_{n-k}))^{\alpha}\right) + q_n f(x_{n-l}) = 0, \tag{1.4}$$

under the conditions

$$\frac{f(u)}{u^{\alpha}} \geq M, \, and \, \sum\limits_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty.$$

In all the results, the authors assumed that either $a_n = 1$ or $\sum\limits_{n=n_0}^{\infty} \dfrac{1}{a_n} = \infty$, and as far as the authors knowledge there are no worthwhile results available in the literature when $\sum\limits_{n=n_0}^{\infty} \dfrac{1}{a_n} < \infty$ for the equation (1.1). This observation motivated us to study the oscillatory behavior of equation (1.1) when condition (H_1) is satisfied. In Section 2, we obtain some new sufficient conditions for the oscillation of all solutions of equation (1.1), and in Section 3, we provide some examples to illustrate the main results. Thus the results presented in this paper improve and complement to those established in [4, 11, 12, 13, 14, 16, 17].

2. Oscillation results

Throughout this paper, we use the following notation without further mention:

$$z_n = x_n - p_n x_{n-k} \tag{2.1}$$

$$A_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}, \qquad \text{and} \qquad B_n = \sum_{s=n}^{\infty} \frac{1}{a_s}.$$

Remark 2.1. Without loss of generality, we can deal only with positive solutions of equation (1.1), since the proof for the other case is similar.

We begin with the following lemma.

Lemma 2.1. *Let* $\{x_n\}$ *be an eventually positive solution of equation* (1.1). *Then one of the following three cases holds for all sufficiently large* n:

(I) $z_n > 0,\ a_n \Delta z_n > 0,\ \Delta(a_n \Delta z_n) \le 0$;

(II) $z_n > 0,\ a_n \Delta z_n < 0,\ \Delta(a_n \Delta z_n) \le 0$;

(III) $z_n < 0,\ a_n \Delta z_n > 0,\ \Delta(a_n \Delta z_n) \le 0$.

Proof. Assume that $x_{n-\theta} > 0$ for $n \ge N \in \mathbb{N}(n_0)$. Then by the condition (H_3), we have from equation (1.1) that $\Delta(a_n \Delta z_n) \le 0$ for all $n \ge N$. Hence $\{z_n\}$ and $\{a_n \Delta z_n\}$ are eventually of one sign for all $n \ge N$. Thus $\{z_n\}$ satisfying one of the following four cases for all $n \ge N$:

(I) $z_n > 0,\ a_n \Delta z_n > 0,\ \Delta(a_n \Delta z_n) \le 0$;

(II) $z_n > 0,\ a_n \Delta z_n < 0,\ \Delta(a_n \Delta z_n) \le 0$;

(III) $z_n < 0,\ a_n \Delta z_n > 0,\ \Delta(a_n \Delta z_n) \le 0$;

(IV) $z_n < 0,\ a_n \Delta z_n < 0,\ \Delta(a_n \Delta z_n) \le 0$.

Now, we shall show that case (IV) cannot happen. If so, then we have $\lim\limits_{n\to\infty} z_n = -\infty$. From the definition of z_n, we obtain $x_n > \left(\dfrac{-z_{n+k}}{p} \right)$, and therefore $\limsup\limits_{n\to\infty} x_n = \infty$. Thus there exists a subsequence $\{n_j\}$ of positive integers such that $\lim\limits_{j\to\infty} n_j = \infty$ and $x_{n_j} = \max\limits_{n_0 \le n \le n_j} x_n \to \infty$ as $j \to \infty$. Then

$$z_{n_j} = x_{n_j} - p_{n_j} x_{n_j - k} \ge x_{n_j} - p x_{n_j} = x_{n_j}(1 - p) \to \infty$$

as $j \to \infty$, a contradiction. This completes the proof. □

Lemma 2.2. *If* $\{x_n\}$ *is an eventually positive solution of equation* (1.1) *such that case* (I) *holds, then*

$$x_n \ge z_n \ge A_n a_n \Delta z_n, \quad n \ge N \in \mathbb{N}(n_0), \tag{2.2}$$

and $\left\{ \dfrac{z_n}{A_n} \right\}$ *is eventually strictly decreasing.*

Proof. The proof is similar to that of Lemma 2 in [12], and hence the details are omitted. $\square$

Lemma 2.3. *If* $\{x_n\}$ *is an eventually positive solution of equation* (1.1) *such that case* (II) *holds, then*

$$x_n \geq z_n \geq -B_n a_n \Delta z_n, \ n \geq N \in \mathbb{N}(n_0). \tag{2.3}$$

Proof. From the definition of z_n, it is clear that $x_n \geq z_n$ for all $n \geq N$. Since $a_n \Delta z_n$ is nonincreasing, we have

$$a_s \Delta z_s \leq a_n \Delta z_n, \ s \geq n \geq N.$$

Dividing the last inequality by a_s and then summing it from n to j, we obtain

$$z_{j+1} \leq z_n + a_n \Delta z_n \sum_{s=n}^{j} \frac{1}{a_s}, \ j \geq n \geq N.$$

Letting $j \to \infty$, we have

$$0 \leq z_n + B_n a_n \Delta z_n, \ n \geq N.$$

This completes the proof. $\square$

Theorem 2.1. *Assume that* $\alpha = 1$ *and* $l > k$. *If*

$$\limsup_{n \to \infty} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t > \frac{p}{M}, \tag{2.4}$$

$$\liminf_{n \to \infty} \sum_{s=n-l}^{n-1} q_s(A_{s-l} + p_{s-l} A_{s-l-k}) > \frac{1}{M} \left(\frac{l}{l+1} \right)^{l+1}, \tag{2.5}$$

and

$$\sum_{n=n_0}^{\infty} \left[M q_n B_{n+1} - \frac{1}{4 a_n B_{n+1}} \right] = \infty, \tag{2.6}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume that there exists a nonoscillatory solution $\{x_n\}$ of equation (1.1), say, $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq N \in \mathbb{N}(n_0)$, where N is chosen so that all three cases of Lemma 2.1 hold for all $n \geq N$.

Case I: From (2.1), we have

$$x_n \geq z_n + p_n z_{n-k} \geq \left(1 + p_n \frac{A_{n-k}}{A_n} \right) z_n, \ n \geq N, \tag{2.7}$$

where we have used $\{z_n/A_n\}$ is decreasing. Using (2.7) and (H_5) in equation (1.1), we obtain

$$\Delta(a_n \Delta z_n) + M q_n \left(1 + p_{n-l} \frac{A_{n-l-k}}{A_{n-l}} \right) z_{n-l} \leq 0, \ n \geq N. \tag{2.8}$$

From (2.2) and (2.8), we have

$$\Delta(a_n\Delta z_n) + Mq_n(A_{n-l} + p_{n-l}A_{n-l-k})a_{n-l}\Delta z_{n-l} \le 0, \, n \ge N. \tag{2.9}$$

Let $w_n = a_n\Delta z_n$. Then $w_n > 0$ and $\{w_n\}$ is an eventually positive solution of the inequality

$$\Delta w_n + Mq_n(A_{n-l} + p_{n-l}A_{n-l-k})w_{n-l} \le 0. \tag{2.10}$$

But by Theorem 7.6.1 of [5], and (2.5), the inequality (2.10) has no eventually positive solution, a contradiction.

Case II: Define

$$w_n = \frac{a_n\Delta z_n}{z_n}, \, n \ge N. \tag{2.11}$$

Then $w_n < 0$ for all $n \ge N$. From (2.3) and (2.11), we have

$$-1 \le B_n w_n \le 0, \, n \ge N. \tag{2.12}$$

From the equations (1.1), (2.1) and (H_5), we have

$$\Delta(a_n\Delta z_n) + Mq_n z_{n-l} \le 0, \, n \ge N. \tag{2.13}$$

From (2.11) and (2.13), we obtain

$$\begin{aligned}\Delta w_n &\le -Mq_n\frac{z_{n-l}}{z_{n+1}} - \frac{a_n(\Delta z_n)^2}{z_n z_{n+1}} \\ &\le -Mq_n - \frac{w_n^2}{a_n}, \, n \ge N,\end{aligned} \tag{2.14}$$

where we have used $\{z_n\}$ is positive decreasing and l is a positive integer. Multiplying (2.14) by B_{n+1} and then summing it from N to $n-1$, we have

$$\sum_{s=N}^{n-1} B_{s+1}\Delta w_s + \sum_{s=N}^{n-1} MB_{s+1}q_s + \sum_{s=N}^{n-1} B_{s+1}\frac{w_s^2}{a_s} \le 0. \tag{2.15}$$

Using summation by parts formula in the first term of (2.15), and then rearranging we obtain

$$B_n w_n - B_N w_N + \sum_{s=N}^{n-1} MB_{s+1}q_s + \sum_{s=N}^{n-1}\left(\frac{w_s}{a_s} + \frac{w_s^2}{a_s}B_{s+1}\right) \le 0.$$

Using completing the square in the fourth term of the last inequality and then using (2.12), we obtain

$$\sum_{s=N}^{n-1}\left(MB_{s+1}q_s - \frac{1}{4a_s B_{s+1}}\right) \le B_N w_N - B_n w_n \le B_N w_N + 1.$$

Letting $n \to \infty$ in the last inequality , we obtain a contradiction with (2.6).

Case III: From (2.1) and (H_2), we have

$$x_{n-k} > \left(\frac{-z_n}{p}\right). \tag{2.16}$$

Using (H_5) and (2.16) in equation (1.1), we obtain

$$\Delta(a_n \Delta z_n) - \frac{M}{p} q_n z_{n-l+k} \le 0, \, n \ge N. \tag{2.17}$$

Summing (2.17) from s to $n-1$ for $n > s+1$, we have

$$a_n \Delta z_n - a_s \Delta z_s - \frac{M}{p} \sum_{t=s}^{n-1} q_t z_{t-l+k} \le 0.$$

Again summing the last inequality from $n-l+k$ to $n-1$ for s, we have

$$z_{n-l+k} - z_n \le \frac{M}{p} z_{n-l+k} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t$$

or

$$\frac{p}{M} \ge \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t$$

which contradicts (2.4). This completes the proof of the theorem. $\qquad \square$

Theorem 2.2. *Assume that* $0 < \alpha < 1$ *and* $l > k$. *If*

$$\limsup_{n \to \infty} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t > 0, \tag{2.18}$$

$$\sum_{n=n_0}^{\infty} q_n (A_{n-l} + p_{n-l} a_{n-l-k})^{\alpha} = \infty, \tag{2.19}$$

and for any constant $M_1 > 0$

$$\sum_{n=n_0}^{\infty} \left[M_1 B_{n+1} q_n - \frac{1}{4 a_n B_{n+1}} \right] = \infty, \tag{2.20}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume that there exists a nonoscillatory solution $\{x_n\}$ of equation (1.1), say, $x_n > 0$ and $x_{n-\theta} > 0$ for $n \ge N \in \mathbb{N}(n_0)$, where N is chosen so that all three cases of Lemma 2.1 are hold for all $n \ge N$.

Case I: Proceeding as in Case (I) of Theorem 2.1, we obtain $\{w_n\}$ is an eventually positive solution of the inequality

$$\Delta w_n + M q_n (A_{n-l} + p_{n-l} A_{n-l-k})^{\alpha} w_{n-l}^{\alpha} \le 0. \tag{2.21}$$

But by Theorem 1 of [10], and (2.19), the inequality (2.21) has no eventually positive solution, a contradiction.

Case II: Define

$$w_n = \frac{a_n \Delta z_n}{z_n}, n \geq N.$$

Proceeding as in Case (II) of Theorem 2.1, we obtain (2.12) and

$$\Delta w_n \leq -M q_n \frac{z_{n-l}^\alpha}{z_{n+1}} - \frac{w_n^2}{a_n}$$

$$\leq -M q_n z_{n-l}^{\alpha-1} - \frac{w_n^2}{a_n}$$

$$\leq -M_1 q_n - \frac{w_n^2}{a_n}, n \geq N,$$

where we have used $\{z_n\}$ is a positive decreasing, $\alpha < 1$, and $M_1 = M z_{N-l}^{\alpha-1}$. The remaining part of the proof is similar to that of Case (II) of Theorem 2.1 and hence the details are omitted.

Case III: Proceeding as in Case (III) of Theorem 2.1, we have

$$a_n \Delta z_n - a_s \Delta z_s - \frac{M}{p^\alpha} \sum_{t=s}^{n-1} q_t z_{n-l+k}^\alpha \leq 0. \tag{2.22}$$

Let $\lim_{n\to\infty} z_n = c = 0$. Summing (2.22) from $n - l + k$ to $n - 1$ for s, we have

$$z_{n-l+k} - z_n \leq \frac{M}{p^\alpha} z_{n-l+k} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t$$

or

$$\frac{z_{n-l+k}}{z_{n-l+k}^\alpha} \geq \frac{M}{p^\alpha} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t. \tag{2.23}$$

Since

$$\frac{z_{n-l+k}}{z_{n-l+k}^\alpha} = |z_{n-l+k}|^{1-\alpha} \text{ and } 1 - \alpha > 0,$$

we have

$$\limsup_{n\to\infty} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t \leq 0,$$

which contradicts (2.18). Next assume that $\lim_{n\to\infty} z_n = c < 0$. From (2.18), we claim that

$$\limsup_{n\to\infty} \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t = \infty. \tag{2.24}$$

In fact from (2.18), there exists a subsequence $\{n_i\}$ and $n_{i+1} - n_i \geq l - k$ such that

$$\sum_{s=n_i-l+k}^{n_i-1} \frac{1}{a_s} \sum_{t=s}^{n_i-1} q_t \geq b > 0,$$

where b is some constant.

Hence

$$\lim_{n\to\infty}\sum_{s=N}^{n-1}\frac{1}{a_s}\sum_{t=s}^{n-1}q_t \geq \lim_{j\to\infty}\sum_{i=1}^{j}\sum_{s=n_i-l+k}^{n_i-1}\frac{1}{a_s}\sum_{t=s}^{n-1}q_t$$

$$\geq \lim_{j\to\infty}\sum_{i=1}^{j}\sum_{s=n_i-l+k}^{n_i-1}\frac{1}{a_s}\sum_{t=s}^{n_i-1}q_t$$

$$= \infty,$$

where $n_j = \max\{n_i : n_i \leq n\}$. From (2.22), we have

$$\Delta z_s + \frac{M z_n^\alpha}{p^\alpha a_s}\sum_{t=s}^{n-1}q_t \geq 0.$$

Summing the last inequality from N to $n-1$, we obtain

$$z_N - z_n \leq \frac{M}{p^\alpha}z_n^\alpha\sum_{s=N}^{n-1}\frac{1}{a_s}\sum_{t=s}^{n-1}q_t$$

or

$$\frac{p^\alpha z_N}{M z_n^\alpha} \geq \sum_{s=N}^{n-1}\frac{1}{a_s}\sum_{t=s}^{n-1}q_t.$$

In view of $c < 0$, $\dfrac{p^\alpha z_N}{M z_n^\alpha}$ has an upper bound, so

$$\lim_{n\to\infty}\sum_{s=N}^{n-1}\frac{1}{a_s}\sum_{t=s}^{n-1}q_t < \infty$$

which contradicts (2.24). This completes the proof of the theorem.   □

Theorem 2.3. *Let $\alpha > 1$. If*

$$\sum_{n=n_0}^{\infty}q_n\Big(1 + p_{n-l}\frac{A_{n-l-k}}{A_{n-l}}\Big)^\alpha = \infty, \tag{2.25}$$

and

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}\sum_{s=n_0}^{n-1}q_s B_{s-l}^\alpha = \infty, \tag{2.26}$$

then every solution $\{x_n\}$ of equation (1.1) is either oscillatory or $\lim\limits_{n\to\infty}x_n = 0$.

Proof. Proceeding as in the proof of Theorem 2.2, we see that Lemma 2.1 holds for all $n \geq N \in \mathbb{N}(n_0)$.

Case I: Proceeding as in the proof of Theorem 2.1(Case(I)), we have

$$\Delta(a_n\Delta z_n) + M q_n\Big(1 + p_{n-l}\frac{A_{n-l-k}}{A_{n-l}}\Big)^\alpha z_{n-l}^\alpha \leq 0,\, n \geq N.$$

Define $w_n = \dfrac{a_n \Delta z_n}{z_{n-l}^\alpha}$, then $w_n > 0$, and

$$\Delta w_n \le -Mq_n\Big(1 + p_{n-l}\frac{A_{n-l-k}}{A_{n-l}}\Big)^\alpha - \frac{\alpha a_{n+1}\Delta z_{n+1}\Delta z_{n-l}}{z_{n-l}^\alpha}$$

$$\le -Mq_n\Big(1 + p_{n-l}\frac{A_{n-l-k}}{A_{n-l}}\Big)^\alpha, \ n \ge N.$$

Summing the last inequality from N to $n-1$, we obtain

$$\sum_{s=N}^{n-1} Mq_s\Big(1 + p_{s-l}\frac{A_{s-l-k}}{A_{s-l}}\Big)^\alpha < w_N < \infty.$$

Letting $n \to \infty$, in the last inequality, we obtain a contradiction to (2.25).

Case II: From Lemma 2.3, we have

$$z_{n-l} > -B_{n-l}a_n\Delta z_n \ge -B_{n-l}a_N\Delta z_N \ge dB_{n-l} \tag{2.27}$$

where $d = -a_N\Delta z_N$.

From equation (1.1), (H_5) and (2.27), we obtain

$$\Delta(-a_n\Delta z_n) \ge Mq_n d^\alpha B_{n-l}^\alpha, \ n \ge N.$$

Summing the last inequality from N to $n-1$, we have

$$-a_n\Delta z_n \ge -a_N\Delta z_N + Md^\alpha \sum_{s=N}^{n-1} q_s B_{n-l}^\alpha.$$

Dividing the last inequality by a_n and then summing it from N to $n-1$, we obtain

$$z_N \ge z_N - z_n \ge Md^\alpha \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=N}^{s-1} q_t B_{t-l}^\alpha.$$

Letting $n \to \infty$ in the last inequality, we obtain

$$\sum_{n=N}^{\infty} \frac{1}{a_s} \sum_{s=N}^{n-1} q_s B_{s-l}^\alpha \le z_N$$

a contradiction to (2.26).

Case III. In this case $z_n < 0$ and $\Delta z_n > 0$ for all $n \ge N$. Then by Lemma 1 of [12], we see that $\lim_{n\to\infty} x_n = 0$. This completes the proof. $\square$

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the second order neutral difference equation

$$\Delta(2^n\Delta(x_n - \frac{1}{2}x_{n-2})) + 3(2^n)x_{n-3}(1 + x_{n-3}^2) = 0, n \geq 1. \tag{3.1}$$

Here $a_n = 2^n$, $p_n = \frac{1}{2}$, $q_n = 3(2^n)$, $l = 3$, $k = 2$, $\alpha = 1$, and $M = 1$. Since $A_n = 1 - \frac{1}{2^{n-1}}$ and $B_n = \frac{1}{2^{n-1}}$, it is easy to see that all conditions of Theorem 2.1 are satisfied and hence every solution of equation (3.1) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (3.1).

Example 3.2. Consider the second order neutral difference equation

$$\Delta(2^n\Delta(x_n - \frac{1}{2^n}x_{n-1})) + 2^{1-\frac{n}{3}}(15(4^n) + 3(2^n))x_{n-3}^{\frac{1}{3}} = 0, n \geq 1. \tag{3.2}$$

Here $a_n = 2^n$, $p_n = \frac{1}{2^n}$, $q_n = 2^{1-\frac{n}{3}}$, $l = 3$, $k = 1$, $\alpha = \frac{1}{3}$, and $M = 1$. Since $A_n = 1 - \frac{1}{2^{n-1}}$ and $B_n = \frac{1}{2^{n-1}}$, it is easy to see that all conditions of Theorem 2.2 are satisfied and hence every solution of equation (3.2) is oscillatory. In fact $\{x_n\} = \{(-1)^{3n}2^n\}$ is one such oscillatory solution of equation (3.2).

Example 3.3. Consider the second order neutral difference equation

$$\Delta(n(n+1)\Delta(x_n - \frac{1}{2}x_{n-1})) + \frac{(n-2)^3}{n(n-1)}x_{n-2}^3 = 0, n \geq 3. \tag{3.3}$$

Here $a_n = n(n+1)$, $p_n = \frac{1}{2}$, $q_n = \frac{(n-2)^3}{n(n-1)}$, $l = 2$, $k = 1$, $\alpha = 3$, and $M = 1$. Since $A_n = \frac{n-3}{3n}$ and $B_n = \frac{1}{n}$, it is easy to see that all conditions of Theorem 2.3 are satisfied and so any solution of equation (3.3) is either oscillatory or tends to zero as $n \to \infty$.. In fact $\{x_n\} = \{\frac{1}{n}\}$ is one such solution of equation (3.3) of the latter type.

We conclude this paper with the following remark.

Remark 3.1. It would be interesting to improve the result of Theorem 2.3 to similar that of Theorem 2.1.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Second Edition, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, M. Bohner S. R. Grace, and D. O'Regan, Discrete Oscillation Theory, Hindawi Publ. Corp., New York, 2005.

[3] R. P. Agarwal, S. R. Grace and E. Thandapani, *Oscillatory and nonoscillatory behavior of second order neutral delay difference equations*, Math. Comp. Model., **24** (1996), 5–11.

[4] W.T. Li, *Oscillation of higher order neutral nonlinear difference equations*, Appl. Math. Lett., **11** (1998), 1–8.

[5] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential equations With Applications, Clarendon press, Oxford, 1991.

[6] S. H. Saker, Oscillation Theory of Delay Differential and Difference Equations: Second and Third Orders, Verlag dr Muler, Germany, 2010.

[7] S. H. Saker and D. O'Regan, *New oscillation criteria for second order neutral functional dynamic equations via the generalized Riccati substitution*, Commu. Nonlin. Sci. Numer. Simu., **16** (2011), 423–434.

[8] S. H. Saker and D. O'Regan, *New oscillation criteria for second order nonlinear neutral functional dynamic equations on time scales via Riccati substitution*, Hirishima J. Math., **42** (2012), 77–98.

[9] Y. G. Sun and S. H. Saker, *Oscillation for second order nonlinear neutral delay difference equations*, Appl. Math. Comput., **163** (2005), 909–918.

[10] X. H. Tang and Y. Liu, *Oscillation for nonlinear delay difference equations*, Tamkang J. Math., **32** (2001), 275–280.

[11] E. Thandapani, P. Sundaram, J. R. Graef and P. W. Spikes, *Asymptotic properties of solutions of nonlinear second order neutral delay difference equations*, Dynamic Sys. Appl., **4**(1995), 125–136.

[12] E. Thandapani, V. Balasubramanian and J. R. Graef, *Oscillation criteria for second order neutral difference equations with negative neutral term*, Inter. J. Pure. Appl. Math., **87** (2013), 283–292.

[13] E. Thandapani, Z. Liu, R. Arul and P. S. Raja, *Oscillation and asymptotic behavior of second order difference equations with nonlinear neutral terms*, Appl. Math. E–Notes, **4** (2004), 59–67.

[14] E. Thandapani and K. Mahalingam, *Necessary and sufficient conditions for oscillation of second order neutral delay difference equations*, Tamkang J. Math., **34** (2003), 137–145.

[15] D. M. Wang and Z. T. Xu, *Oscillation of second order quasilinear neutral delay difference equations*, Acta. Math. Appl. Sinica., **27** (2011), 93–104.

[16] A. Zafer and R. S. Dahiya, *Oscillation of a neutral difference equations*, Appl. Math. Lett., **6**(1993), 71–74.

[17] Z. Zhou, J. Yu and G. Lei, *Oscillation for even order neutral difference equations*, Korean J. Comput. Appl. Math., **7** (2000), 601–610.

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai -600 005, India.

E-mail: ethandapani@yahoo.co.in

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai -600 005, India.

E-mail: d.seghar@gmail.com

Academia Militar, Departamento de Ciencias Exactas e Naturais, Av. Conde Castro Guimaraes, 2720-113 Amadora, Portugal.

E-mail: sandra.pinelas@gmail.com