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ON GENERALIZED (σ,τ)-DERIVATIONS IN 3-PRIME NEAR-RINGS

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Abstract. Let N be a 2-torsion free 3-prime left near-ring with multiplicative center Z, I be a nonzero semigroup ideal of N and f be a right generalized (σ,τ)-derivation on N associated with a (σ,τ)-derivation d. Assume $d\sigma = \sigma d, d\tau = \tau d$, $f\sigma = \sigma f$, $f\tau = \tau f$. We prove that N is a commutative ring or $d = 0$ if any one of the following holds: i) $f(N) \subseteq Z$ ii) $f(I) \subseteq Z$. Moreover, if f is a generalized (σ,τ) derivation on N associated with d, then $d = 0$ if any one of the following is satisfied : iii) f acts as a homomorphism on I iv) f acts as an anti-homomorphism on I.

1. Introduction

An additively written group $(N,+)$ equipped with a binary operation $\cdot : N\times N \to N$, $(x, y) \mapsto xy$ such that $(xy)z = x(yz)$ and $z(x+y) = zx+zy$ for all $x, y, z \in N$ is called a left near-ring. A near-ring N is called 3-prime if for any $x, y \in N$, $xNy = 0$ implies that $x = 0$ or $y = 0$ and N is called zero-symmetric if $0x = 0$ for all $x \in N$. A nonempty subset I of N is called a semigroup left ideal (resp. semigroup right ideal) if $NI \subseteq I$ (resp. $IN \subseteq I$) and if I is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal. For $x, y \in N$, the symbol $[x, y]$ will denote $xy - yx$. Z is the multiplicative center of N. An additive mapping $d : N \to N$ is said a derivation if $d(xy) = xd(y)+d(x)y$ for all $x, y \in N$, or equivalently, as noted in [11], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. Recently, in [7], Bresar defined the following concept. An additive mapping $f : N \to N$ is called a generalized derivation if there exists a derivation $d : N \to N$ such that

$$f(xy) = f(x)y + xd(y), \text{ for all } x, y \in N.$$

Inspired by the definition of derivation, we define the notion of (σ,τ)-derivation as follows: Let σ,τ be two near-ring automorphisms of N. An additive mapping $d : N \to N$ is called a (σ,τ)-derivation if $d(xy) = \tau(x)d(y)+d(x)\sigma(y)$ holds for all $x, y \in N$. It is noted that $d(xy) = d(x)\sigma(y)+\tau(x)d(y)$, for all $x, y \in N$ in [9, Lemma 1].

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Definition 1 ([10], Definition 1)**.** Let N be a near-ring and d be a (σ,τ)-derivation of N. An additive mapping $f: N \to N$ is called a right generalized (σ,τ)-derivation associated with d if

$$f(xy) = f(x)\sigma(y) + \tau(x)d(y), \text{ for all } x, y \in N,$$

and $f: N \to N$ is called a left generalized (σ,τ)-derivation associated with d if

$$f(xy) = d(x)\sigma(y) + \tau(x)f(y), \text{ for all } x, y \in N.$$

f is called a generalized (σ,τ)-derivation associated with d if it is both left and right generalized (σ,τ)-derivation associated with d.

Of course a $(1,1)$-derivation (resp. generalized $(1,1)$-derivation) is a derivation (resp. generalized derivation) on N, where 1 is the identity on N.

Several authors have obtained commutativity results for prime or semiprime rings admitting derivations or generalized derivations. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [4] and [6]. Some recent results on rings deal with commutativity on prime and semiprime rings admitting suitably constrained derivations. It is natural to look for comparable results on near-rings and this has been done in [9], [8], [1], [10], [2] and [3] .

Throughout this paper, N will denote a zero-symmetric left near-ring and $d\sigma = \sigma d$, $d\tau = \tau d$, $f\sigma = \sigma f$ and $f\tau = \tau f$. It is our purpose to extend some of these results on prime near-rings admitting suitably constrained generalized (σ,τ)-derivation.

2. Results

Lemma 1 ([4], Lemma 3)**.** *Let N be a 3-prime near-ring.*

i) *If $z \in Z-(0)$, then z is not a zero divisor.*
ii) *If $Z-(0)$ contains an element z for which $z+z \in Z$, then $(N,+)$ is abelian.*
iii) *If $z \in Z-(0)$ and x is an element of N such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.*

Lemma 2 ([1], Lemma 3.1)**.** *Let N be a 3-prime near-ring, d a non trivial (σ,τ)-derivation and $a \in N$. If $ad(N) = (0)$ or $d(N)a = (0)$, then $a = 0$.*

Lemma 3 ([10], Lemma 2)**.** *Let N be a left near-ring.*

i) *Let d be a (σ,τ)-derivation of N. Then*

$$(d(x)\sigma(y) + \tau(x)d(y))z = d(x)\sigma(y)z + \tau(x)d(y)z, \text{ for all } x, y, z \in N.$$

ii) *Let* (f,d) *be a nonzero right generalized* (σ,τ)*-derivation of* N. *Then*

$$(f(x)\sigma(y)+\tau(x)d(y))z = f(x)\sigma(y)z+\tau(x)d(y)z, \text{ for all } x,y,z\in N.$$

iii) *Let* (f,d) *be a nonzero left generalized* (σ,τ)*-derivation of* N. *Then*

$$(d(x)\sigma(y)+\tau(x)f(y))z = d(x)\sigma(y)z+\tau(x)f(y)z, \text{ for all } x,y,z\in N.$$

Lemma 4 ([5], Lemma 1.3)**.** *Let* N *be a* 3*-prime near-ring,* d *a non trivial* (σ,τ)*-derivation and* I *a nonzero semigroup ideal of* N.

i) *If* $x,y\in N$ *and* $xIy=(0)$, *then* $x=0$ *or* $y=0$.

ii) *If* $x,y\in N$ *and* $xI=(0)$ *or* $Ix=(0)$, *then* $x=0$.

Lemma 5 ([8], Lemma 4)**.** *Let* N *be a* 3*-prime near-ring,* d *a* (σ,τ)*-derivation and* I *a nonzero right (or left) semigroup ideal of* N. *If* $d(I)=(0)$, *then* $d=0$.

Lemma 6. *Let* N *be a* 3*-prime near-ring and* I *a nonzero semigroup right ideal of* N. *If* $[I,I]=(0)$, *then* N *is commutative.*

Proof. By the hypothesis, we have

$$uv=vu, \text{ for all } u,v\in I.$$

Replacing u by ur, $r\in N$, we get

$$I[v,r]=(0), \text{ for all } v\in I,\ r\in N.$$

By Lemma 4, we obtain that $[v,r]=(0)$, for all $v\in I, r\in N$. Again, replacing v by vx, $x\in N$ this implies that $I[x,r]=(0)$, for all $r,x\in N$. Using Lemma 4, we conclude that N is commutative. □

Theorem 1. *Let* N *be a* 3*-prime near-ring,* (f,d) *a nonzero right generalized* (σ,τ)*- derivation of* N. *If* $f(N)\subseteq Z$, *then* $(N,+)$ *is abelian. Moreover,* N *is a commutative ring or* $d=0$.

Proof. As $f(N)\subseteq Z$ and f is nonzero, there exists a nonzero element x in N such that $f(x)\in Z-(0)$ and $f(x+x)=f(x)+f(x)\in Z$. Hence, $(N,+)$ is abelian by Lemma 1 (ii).

Suppose that $d=0$. We have $f(xy)=f(x)\sigma(y)\in Z$, for all $x,y\in N$. Thus,

$$f(x)\sigma(y)\sigma(z)=\sigma(z)f(x)\sigma(y), \text{ for all } x,y,z\in N.$$

Using the hypothesis, we get

$$\begin{aligned} 0 &= f(x)\sigma(y)\sigma(z)-\sigma(z)f(x)\sigma(y) \\ &= f(x)\sigma(y)\sigma(z)-f(x)\sigma(z)\sigma(y) \\ &= f(x)\left(\sigma(y)\sigma(z)-\sigma(z)\sigma(y)\right). \end{aligned}$$

Using Lemma 1 (iii), $f(x) \neq 0$ and $f(x) \in Z$, we have

$$\sigma([y,z]) = 0, \text{ for all } y,z \in N.$$

As σ is an automorphism, we obtain that

$$[y,z] = 0, \text{ for all } y,z \in N.$$

Hence, N is commutative ring.

Now, we suppose that $d \neq 0$. Let distinguish this into two situations. Firstly, we have $d(Z) \neq (0)$. Thus, there exists a nonzero element c in $Z-(0)$ such that $d(c) \neq 0$. By the hypothesis, we get

$$f(xc) = f(x)\sigma(c) + \tau(x)d(c) \in Z, \text{ for all } x \in N.$$

We have

$$\left(f(x)\sigma(c) + \tau(x)d(c)\right)\tau(y) = \tau(y)\left(f(x)\sigma(c) + \tau(x)d(c)\right), \text{ for all } x,y \in N.$$

By Lemma 3(ii), we obtain that

$$f(x)\sigma(c)\tau(y) + \tau(x)d(c)\tau(y) = \tau(y)f(x)\sigma(c) + \tau(y)\tau(x)d(c), \text{ for all } x,y \in N.$$

Using the hypothesis and $\sigma(c) \in Z$ in the last equation, we get

$$f(x)\sigma(c)\tau(y) + \tau(x)d(c)\tau(y) = f(x)\sigma(c)\tau(y) + \tau(y)\tau(x)d(c),$$

and so

$$\tau(x)d(c)\tau(y) = \tau(y)\tau(x)d(c), \text{ for all } x,y \in N. \tag{2.1}$$

Replacing x by xz, $z \in N$ in this equation, we have

$$\tau(x)\tau(z)d(c)\tau(y) = \tau(y)\tau(x)\tau(z)d(c), \text{ for all } x,y,z \in N.$$

Appliying (2.1), we obtain that

$$\tau(x)\tau(z)d(c)\tau(y) = \tau(y)\tau(z)d(c)\tau(x), \text{ for all } x,y,z \in N,$$

and so

$$\tau(z)d(c)\tau(x)\tau(y) = \tau(z)d(c)\tau(y)\tau(x), \text{ for all } x,y,z \in N.$$

That is,

$$\tau(z)d(c)\tau(xy-yx) = 0, \text{ for all } x,y,z \in N.$$

As τ is an automorphism, we obtain that

$$Nd(c)(xy-yx) = 0, \text{ for all } x,y \in N.$$

As N is a 3-prime near ring, we get

$$d(c)xy = d(c)yx, \text{ for all } x, y \in N.$$

Taking x by xz in this equation and using this equation, we find that

$$d(c)xzy = d(c)yxz = d(c)xyz, \text{ for all } x, y, z \in N,$$

and so,

$$d(c)N[z, y] = 0, \text{ for all } z, y \in N.$$

Again, as N is a 3-prime near ring and $d(c) \neq 0$, we get

$$[z, y] = 0, \text{ for all } z, y \in N.$$

Thus, N is commutative ring. Secondly, $d(Z) = (0)$. Using $f(x) \in Z$, we have $d(f(x)) = 0$, for all $x \in N$. Replacing x by xy in the last equation, we have

$$\begin{aligned} 0 &= d(f(xy)) = d(f(x)\sigma(y) + \tau(x)d(y)) \\ &= d(f(x))\sigma^2(y) + \tau(f(x))d(\sigma(y)) + d(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y) \\ &= \tau(f(x))d(\sigma(y)) + d(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y). \end{aligned}$$

That is,

$$\tau(f(x))d(\sigma(y)) + d(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y) = 0, \text{ for all } x, y \in N. \tag{2.2}$$

If d is applied in the last equation, we have

$$\begin{aligned} &d(\tau(f(x)))\sigma(d(\sigma(y))) + \tau^2(f(x))d^2(\sigma(y)) + d^2(\tau(x))\sigma^2(d(y)) \\ &\quad + \tau(d(\tau(x)))d(\sigma(d(y))) + d(\tau^2(x))\sigma(d^2(y)) + \tau^3(x)d^3(y) = 0. \end{aligned}$$

Using $d\tau = \tau d$, $d\sigma = \sigma d$ and $d(f(x)) = 0$, for all $x \in N$, we have

$$\tau^2(f(x))d^2(\sigma(y)) + d^2(\tau(x))\sigma^2(d(y)) + d(\tau^2(x))(\sigma(d^2(y)) + d(\tau^2(x))\sigma(d^2(y)) + \tau^3(x)d^3(y) = 0.$$

Replacing y by $d(y)$ and x by $\tau(x)$ in (2.2) and using $f\tau = \tau f$, $d\sigma = \sigma d$, we have

$$\tau^2(f(x))d^2(\sigma(y)) + d(\tau^2(x))\sigma(d^2(y)) + \tau^3(x)d^3(y) = 0, \text{ for all } x, y \in N.$$

Using this equation in the above equation, we obtain that

$$d^2(\tau(x))\sigma^2(d(y)) + d(\tau^2(x))\sigma(d^2(y)) = 0, \text{ for all } x, y \in N. \tag{2.3}$$

Writing x by $d(x)$ and y by $\sigma(y)$ in (2.2), we have

$$\tau(f(d(x))d(\sigma^2(y)) + d^2(\tau(x))\sigma^2(d(y)) + \tau^2(d(x))d^2(\sigma(y)) = 0, \text{ for all } x, y \in N.$$

Using (2.3) in the above equation and $f\tau = \tau f$, $d\sigma = \sigma d$, we get

$$f(d(\tau(x))d(\sigma^2(y)) = 0, \text{ for all } x, y \in N.$$

As σ, τ are automorphisms, we have $f(d(x))d(y) = 0$, for all $x, y \in N$. Replacing y by yz in this equation, we obtain that $f(d(x))\tau(y)d(z) = 0$, for all $x, y, z \in N$. As N is a 3-prime near ring, we have $f(d(x)) = 0$, for all $x \in N$ or $d = 0$. Suppose that, $f(d(x)) = 0$, for all $x \in N$. By the hypothesis, we get $f(d(x)\tau(y)) \in Z$. That is, $f(d(x))\sigma(\tau(y)) + \tau(d(x))d(\tau(y)) \in Z$, and so $\tau(d(x))d(\tau(y)) \in Z$, for all $x, y \in N$. Using $d\tau = \tau d$, we get $\tau(d(x)d(y)) \in Z$, for all $x, y \in N$. Since τ is an automorphism, we get $d(x)d(y) \in Z$, for all $x, y \in N$. Assume that, $\{d(x)d(y) = 0\}$or $\{d(y)d(x) = 0\}$ or $\{d(x)d(y) \neq 0 \text{ and } d(y)d(x) \neq 0\}$ for all $x, y \in N$. In the first two cases, we have $d = 0$. In the last case, $d(x)d(y) \in Z - (0)$ and $d(y)d(x) \in Z - (0)$, for all $x, y \in N$. That is, $d(x)d(y) - d(y)d(x) \in Z - (0)$. That is, $d(x)d(x)d(y) = d(x)d(y)d(x)$, for all $x, y \in N$. We conclude that, $d(x)\left(d(x)d(y) - d(y)d(x)\right) = 0$, for all $x, y \in N$. Using $d(x)d(y) - d(y)d(x) \in Z - (0)$ in the last equation, $d(x) = 0$, for all $x \in N$ by Lemma 1 (iii). Thus, $d = 0$. □

Theorem 2. *Let N be a 3-prime near-ring, (f, d) a nonzero right generalized (σ, τ)- derivation of N and I a nonzero semigroup ideal of N. If $f(I) \subseteq Z$, then $(N, +)$ is abelian. Moreover, N is a commutative ring or $d = 0$.*

Proof. Suppose that $f(I) = (0)$. Then, $f(ux) = 0$, for all $u \in I$, $x \in N$. That is, $f(u)\sigma(x) + \tau(u)d(x) = 0$. Using $f(I) = (0)$, we have $\tau(u)d(x) = 0$, for all $u \in I$, $x \in N$. Using Lemma 2, we have $d = 0$. Therefore, $f(xu) = 0 = f(x)\sigma(u)$, for all $u \in I$, $x \in N$. As σ is an automorphism of N, we get $f(x)u = 0$, for all $u \in I$, $x \in N$. By Lemma 4 (ii), we conclude that $f = 0$. This is a contradiction. Thus, $f(I) \neq (0)$. There exists a nonzero element a in I such that $f(a) \neq 0$.

As I is a semigroup ideal of N, we get $ax \in I$, for all $x \in N$. Thus, $ax + ax = a(x + x) \in I$. Using $f(I) \subseteq Z$, we have $f(ax+ax) = f(ax)+f(ax) \in Z$. Firstly, suppose that there exists $x \in N$ such that $f(ax) \neq 0$. This implies that $f(ax) \in Z - (0)$ and $f(ax) + f(ax) \in Z$. We obtain that $(N, +)$ is abelian by Lemma 1 (ii).

Now, finally assume that $f(ax) = 0$, for all $x \in N$. We get

$$0 = f(a(xa)) = f((ax)a) = f(ax)\sigma(a) + \tau(ax)d(a).$$

Application of $f(ax) = 0$, we find that

$$\tau(ax)d(a) = 0, \text{ for all } x \in N.$$

As τ is an automorphism of N, we get $\tau(a)Nd(a) = 0$. By the primeness of N, we have $\tau(a) = 0$ or $d(a) = 0$ and so, $a = 0$ or $d(a) = 0$. Let be $d(a) = 0$, so that

$$f(xa) = f(x)\sigma(a) + \tau(x)d(a) = f(x)\sigma(a) \in Z$$

and so

$$f(x)\sigma(a) \in Z, \text{ for all } x \in N.$$

Therefore,

$$0 = [f(u)\sigma(a), y] = f(u)\sigma(a)y - yf(u)\sigma(a), \text{ for all } u \in I.$$

Using $f(I) \subseteq Z$, we have

$$0 = f(u)\sigma(a)y - f(u)y\sigma(a) = f(u)(\sigma(a)y - y\sigma(a)) = f(u)[\sigma(a), y], \text{ for all } u \in I.$$

As $f(I) \neq (0)$ and $f(I) \subseteq Z$, we have $f(u) \in Z - \{0\}$. Thus, $[\sigma(a), y] = 0$, for all $y \in N$ by Lemma 1 (i). As σ is an automorphism, we get $a \in Z$. Using $f(ax) = 0$, for all $x \in N$ and $d(a) = 0$, we get

$$0 = f(ax) = f(xa) = f(x)\sigma(a) + \tau(x)d(a) = f(x)\sigma(a).$$

That is

$$f(x)\sigma(a) = 0, \text{ for all } x \in N.$$

Thus, $f(I)\sigma(a) = (0)$. As $f(I) \neq (0)$ and $f(I) \subseteq Z$, we have $\sigma(a) = 0$ by Lemma 1 (i). Using σ is an automorphism, we have $a = 0$. This is contradiction with $f(a) \neq 0$. Therefore, $(N,+)$ is abelian.

To complete the proof, we prove that N is a commutative ring. First case, consider $d = 0$. We obtain that

$$f(ux) = f(u)\sigma(x) + \tau(u)d(x) = f(u)\sigma(x) \in Z,$$

and so

$$f(u)\sigma(x) \in Z, \text{ for all } u \in I,\ x \in N.$$

As $f(I) \neq (0)$ and $f(I) \subseteq Z$, we have $f(u) \in Z - \{0\}$ for some $u \in I$. Using Lemma 1 (iii) in the last equation, we have $\sigma(x) \in Z$, for all $x \in N$. As σ is an automorphism, we obtain that $x \in Z$, for all $x \in N$. Therefore, N is commutative.

Now, assume that $d \neq 0$. Let $c \in Z - \{0\}$. This implies that $f(uc) = f(u)\sigma(c) + \tau(u)d(c) \in Z$, for all $u \in I$. Commuting $\tau(v)$, $v \in I$ in the last equation, we have

$$\left(f(u)\sigma(c) + \tau(u)d(c)\right)\tau(v) = \tau(v)\left(f(u)\sigma(c) + \tau(u)d(c)\right), \text{ for all } u, v \in I.$$

As N is a left near-ring and Lemma 3 (ii), we have

$$f(u)\sigma(c)\tau(v) + \tau(u)d(c)\tau(v) = \tau(v)f(u)\sigma(c) + \tau(v)\tau(u)d(c), \text{ for all } u, v \in I.$$

Using $f(u), \sigma(c) \in Z$, we get

$$f(u)\sigma(c)\tau(v) + \tau(u)d(c)\tau(v) = f(u)\sigma(c)\tau(v) + \tau(v)\tau(u)d(c),$$

and so

$$\tau(u)d(c)\tau(v) = \tau(v)\tau(u)d(c), \text{ for all } u, v \in I. \tag{2.4}$$

Replacing u by uw, $w \in I$ in the last equation, we find that

$$\tau(u)\tau(w)d(c)\tau(v) = \tau(v)\tau(u)\tau(w)d(c), \text{ for all } u, v, w \in I.$$

Using equation (2.4) in the above equation, we have

$$\tau(w)d(c)\tau(u)\tau(v) = \tau(w)d(c)\tau(v)\tau(u), \text{ for all } u, v, w \in I.$$

That is,

$$\tau(w)d(c)(\tau(u)\tau(v) - \tau(v)\tau(u) = 0, \text{ for all } u, v, w \in I.$$

Thus,

$$I\tau^{-1}(d(c))(uv - vu) = 0, \text{ for all } u, v \in I.$$

By Lemma 4 (ii), we have

$$\tau^{-1}(d(c))(uv - vu) = 0, \text{ for all } u, v \in I,$$

and so

$$\tau^{-1}(d(c))uv = \tau^{-1}(d(c))vu, \text{ for all } u, v \in I, \tag{2.5}$$

Taking v by vw, $w \in I$ in (2.5) and using this equation, we see that

$$\tau^{-1}(d(c))vuw - \tau^{-1}(d(c))vwu = 0, \text{ for all } u, v \in I.$$

That is

$$\tau^{-1}(d(c))I(uw - wu) = 0, \text{ for all } u, w \in I.$$

Using Lemma 4 (i), we obtain that

$$\tau^{-1}(d(c)) = 0 \text{ or } [u, w] = 0, \text{ for all } u, w \in I.$$

Therefore,

$$d(c) = 0 \text{ or } [u, w] = 0, \text{ for all } u, w \in I.$$

If $[u, w] = 0$, for all $u, w \in I$, then $I \subseteq Z$ by Lemma 6. Thus, N is commutative. Then, $d(c) = 0$.

The last is $d \neq 0$ and $d(c) = 0$, $c \in Z - \{0\}$. For each $u \in I$, we have $u^2 \in I$. Assume that $\{\sigma(I) + \tau(I)\} \cap Z = (0)$. As σ, τ are automorphisms, there exists $x, y \in N$ such that $f(u) = \sigma(x)$ and $d(u) = \tau(y)$, we have

$$f(u^2) = f(u)\sigma(u) + \tau(u)d(u) = \sigma(x)\sigma(u) + \tau(u)\tau(y) = \sigma(xu) + \tau(uy) \in \sigma(I) + \tau(I).$$

Also, using the hypothesis, we have $f(u^2) \in Z$. Therefore $f(u^2) \in \{\sigma(I)+\tau(I)\} \cap Z = (0)$. That is $f(u^2) = 0$, for all $u \in I$. This implies that

$$f(u^2x) = f(u^2)\sigma(x) + \tau(u^2)d(x) = \tau(u^2)d(x), \text{ for all } u \in I,\ x \in N.$$

As τ is an automorphism, we have $\tau(u^2)d(x) = \tau(u)\tau(u)\tau(z)$ for some $z \in N$ and so $\tau(u^2)d(x) = \tau(u^2z) \in \tau(I) \subset \{\sigma(I)+\tau(I)\}$. Moreover $f(u^2x) = \tau(u^2)d(x) \in Z$. So $\tau(u^2)d(x) \in \{\sigma(I)+\tau(I)\} \cap Z = (0)$. That is $\tau(u^2)d(x) = 0$ for all $u \in I,\ x \in N$. Replacing x by $xy,\ y \in N$, we have $\tau(u^2)d(x)\sigma(y) + \tau(u^2)\tau(x)d(y) = 0$, and so $\tau(u^2)\tau(x)d(y) = 0$, for all $u \in I,\ x, y \in N$. By the primenessly of N, we have $\tau(u^2) = 0$ or $d = 0$, for all $u \in I$. We conclude that $\tau(u^2) = 0$ for all $u \in I$. Using $\tau \in Aut(N)$, we have $u^2 = 0$ for all $u \in I$. By the hypothesis, we have

$$f(xu) = f(x)\sigma(u) + \tau(x)d(u) \in Z \text{ , for all } u \in I,\ x \in N.$$

Appliying $u^2 = 0$ for all $u \in I$, we have

$$\begin{aligned} 0 &= \left\{f(x)\sigma(u) + \tau(x)d(u)\right\}\sigma(u^2) \\ &= \sigma(u)\left\{f(x)\sigma(u) + \tau(x)d(u)\right\}\sigma(u) \\ &= \sigma(u)f(x)\sigma(u^2) + \sigma(u)\tau(x)d(u)\sigma(u) \\ &= \sigma(u)\tau(x)d(u)\sigma(u). \end{aligned}$$

Multipliying the last equation on the left $d(u)$ and as τ is an automorphism, we obtain that

$$d(u)\sigma(u)Nd(u)\sigma(u) = (0), \text{ for all } u \in I.$$

Using N is a 3- prime near-ring, we have

$$d(u)\sigma(u) = 0, \text{ for all } u \in I.$$

As $u^2 = 0$ for all $u \in I$ and using the above equation, we get

$$0 = d(u^2) = d(u)\sigma(u) + \tau(u)d(u) = \tau(u)d(u),$$

and so

$$\tau(u)d(u) = 0, \text{ for all } u \in I. \tag{2.6}$$

As $f(I) \neq (0)$, there exists $v \in I$ such that $f(v) \neq 0$. Using equation (2.6) and $v^2 = 0$, we have

$$0 = f(v^2) = f(v)\sigma(v) + \tau(v)d(v) = f(v)\sigma(v).$$

Using $f(v)\sigma(v) = 0$ and $0 \neq f(v) \in Z$, we have $f(v)N\sigma(v) = (0)$. By the primenessly of N, we obtain that $f(v) = 0$ or $\sigma(v) = 0$ and so $f(v) = 0$ or $v = 0$. This is a contradiction.

Now, suppose that $\{\sigma(I)+\tau(I)\}\cap Z \neq (0)$. Taking $0 \neq c \in \{\sigma(I)+\tau(I)\}\cap Z$ and $x \in N$ and using $d(Z)=(0)$, we find that

$$f(xc)=f(x)\sigma(c)+\tau(x)d(c)=f(x)\sigma(c).$$

As $c \in \{\sigma(I)+\tau(I)\}$, there exists $u, v \in I$ such that $c=\sigma(u)+\tau(v)$. We have

$$f(x)\sigma(c)=f(x)\{\sigma(u)+\tau(v)\}=f(x)\sigma(u)+f(x)\tau(v).$$

As $\sigma,\tau \in Aut(N)$, there exists $x', y' \in N$ such that $\sigma(x')=x$, $\tau(y')=x$ and using $\sigma f = f\sigma$, $\tau f = f\tau$, we get

$$\begin{aligned} f(xc) &= f(x(\sigma(u)+\tau(v)) = f(x\sigma(u)+x\tau(v)) \\ &= f(\sigma(x')\sigma(u)+\tau(y')\tau(v)) = f(\sigma(x'u)+\tau(y'v)) \\ &= f(\sigma(x'u))+f(\tau(y'v)) \\ &= \sigma(f(x'u))+\tau(f(y'v)). \end{aligned}$$

As $x'u, y'v \in I$ and $f(I)\subset Z$, we have $f(x'u), f(y'v) \in Z$. Using σ, τ are automorphisms, we get $\sigma(f(x'u)),\tau(f(y'v)) \in Z$. This implies that $f(xc)=\sigma(f(x'u))+\tau(f(y'v)) \in Z$. Therefore $f(x)\sigma(c) \in Z$, for all $x \in N$. Using $\sigma(c) \in Z-\{0\}$, we have $f(x) \in Z$, for all $x \in N$ by Lemma 1 (iii). Therefore, N is commutative ring or $d=0$ by Theorem 1. ☐

Theorem 3. *Let N be a 3-prime near-ring, (f,d) a nonzero generalized (σ,τ)-derivation of N and I a nonzero right semigroup ideal of N. If f acts as a homomorphism on I, then $d=0$.*

Proof. By the hypothesis, we get

$$f(uv)=f(u)f(v), \text{ for all } u, v \in I. \tag{2.7}$$

Replacing v by vw, $w \in I$, we get

$$f(uvw)=d(u)\sigma(vw)+\tau(u)f(vw)=d(u)\sigma(vw)+\tau(u)d(v)\sigma(w)+\tau(uv)f(w).$$

On the other hand, using Lemma 3 (iii), we get

$$\begin{aligned} f(uvw) &= f(uv)f(w) = \left\{d(u)\sigma(v)+\tau(u)f(v)\right\}f(w) \\ &= d(u)\sigma(v)f(w)+\tau(u)f(v)f(w) = d(u)\sigma(v)f(w)+\tau(u)f(vw) \\ &= d(u)\sigma(v)f(w)+\tau(u)d(v)\sigma(w)+\tau(u)\tau(v)f(w). \end{aligned}$$

Comparing these two equations, we get

$$d(u)\sigma(vw)+\tau(u)d(v)\sigma(w)+\tau(uv)f(w)=d(u)\sigma(v)f(w)+\tau(u)d(v)\sigma(w)+\tau(u)\tau(v)f(w).$$

That is

$$d(u)\sigma(vw)=d(u)\sigma(v)f(w),$$

and so

$$d(u)\sigma(v)(\sigma(w)-f(w))=0,\ \text{for all } u,v,w\in I.$$

As σ is an automorphism, we have

$$\sigma^{-1}(d(u))\,I\sigma^{-1}(\sigma(w)-f(w))=0,\ \text{for all } u,w\in I.$$

By Lemma 4 (i), we have $d(u)=0$ or $\sigma(w)=f(w)$, for all $u,w\in I$. If $d(I)=0$, then $d=0$ by Lemma 5. In the second case, we get $\sigma(w)=f(w)$, for all $w\in I$. Replacing w by wx, $x\in N$ in the last equation and using this equation, we have

$$\sigma(wx)=f(w)\sigma(x)+\tau(w)d(x)=\sigma(w)\sigma(x)+\tau(w)d(x).$$

Therefore, $\tau(w)d(x)=0$, for all $w\in I$, $x\in N$. As τ is an automorphism and using Lemma 4 (ii), we obtain that $d=0$. Thus, in the both cases, this implies that $d=0$. □

Theorem 4. *Let N be a 3-prime near-ring, (f,d) a nonzero generalized (σ,τ)- derivation of N and I a nonzero semigroup ideal of N. If f acts as an anti-homomorphism on I, then $d=0$.*

Proof. Assume that

$$f(uv)=f(v)f(u),\ \text{for all } u,v\in I.$$

Replacing v by uv in the above equation, we have

$$f(uuv)=f(uv)f(u)$$

and so

$$f(uuv)=f(u(uv))=d(u)\sigma(uv)+\tau(u)f(uv)$$

Moreover, by Lemma 3 (iii)

$$f(uv)f(u)=d(u)\sigma(v)f(u)+\tau(u)f(v)f(u)=d(u)\sigma(v)f(u)+\tau(u)f(uv).$$

Comparing last two equation, we have

$$d(u)\sigma(uv)=d(u)\sigma(v)f(u),\ \text{for all } u,v\in I.$$

Taking vx instead of v, $x\in N$ and using the last equation, we obtain that

$$d(u)\sigma(v)f(u)\sigma(x)=d(u)\sigma(v)\sigma(x)f(u),$$

and so

$$d(u)\sigma(v)[f(u),\sigma(x)] = 0, \text{ for all } u,v \in I,\ x \in N.$$

As σ is an automorphism, we have

$$\sigma^{-1}(d(u))\, I\sigma^{-1}([f(u),x]) = 0, \text{ for all } u \in I,\ x \in N.$$

By the primeness of N, we find that

$$d(u) = 0 \text{ or } f(u) \in Z, \text{ for all } u \in I.$$

Since I is a nonzero ideal of N, there exists $u \in I-(0)$. Let $I_1 = uN$. Then I_1 is a nonzero semigroup right ideal contained in I and I_1 is an additive subgroup of N. Let $L = \{u \in I_1 \mid f(u) \in Z$ $\}$ and $K = \{u \in I_1 \mid d(u) = 0\}$. It is clear that, each of L and K is an additive subgroup of I_1 such that $I_1 = L \cup K$. But, a group can not be the set-theoretic union of two proper subgroups. Hence $I_1 = L$ or $I_1 = K$. In the first case, $f(I_1) \subset Z$, we get $f(uv) = f(v)f(u) = f(u)f(v)$. That is, f acts as a homomorphism on I_1. This implies that $d = 0$, by Theorem 3. In the second case, $d(I_1) = 0$. By Lemma 5, we get $d = 0$. This completes the proof. □

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