EXTENSION OF AN INEQUALITY WITH POWER EXPONENTIAL FUNCTIONS

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Abstract. V. Cîrtoaje et al. [2] conjectured and proved [3, 4] that the inequality $a^r b + b^r a \leq 2$ holds for all nonnegative numbers $r \leq 3$ and nonnegative real numbers $a, b$ with $a + b = 2$. In this paper, we will show that $a^r b + b^r a \leq 2$ holds for all nonnegative $r \geq 3$ and all nonnegative real numbers $a, b$ with $a + b = 2$ and some conditions. This gives an extended inequality of conjectured by V. Cîrtoaje.

1. Introduction

Inequalities appear on the various branches of mathematics. In this paper, we give a result of an inequality with power exponential functions which is studied by V. Cîrtoaje et al. [1, 2, 3, 4, 5, 6, 7, 8]. The formula of inequalities with power exponential functions are very simple, but their proof is not as simple as it seems. V. Cîrtoaje et al. [3, 4] proved that the inequality

$$a^r b + b^r a \leq 2 \quad (1.1)$$

holds for all nonnegative real number $r \leq 3$ and all nonnegative real numbers $a, b$ with $a + b = 2$. Miyagi et al. [7] proved that the stronger inequality

$$a^3 b + b^3 a + \left(\frac{a - b}{2}\right)^4 \leq 2 \quad (1.2)$$

holds for the same conditions. These inequalities (1.1) and (1.2) are conjectures by V. Cîrtoaje [2]. The following is our main theorem.

Theorem 1.1. The inequality

$$a^r b + b^r a \leq 2 \quad (1.3)$$

holds for all numbers $r \geq 3$ and all real numbers $a, b \in [0, 1 - ((r - 3) / (r - 2))^{1/3}] \cup [1 + ((r - 3) / (r - 2))^{1/3}, 2]$ with $a + b = 2$.

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The above inequality (1.3) is an extension of the inequality (1.1).

2. Preliminaries

In this section, we will show some lemmas to prove our main theorem.

**Lemma 2.1.** If $0 < b < 1$, then

$$(2 - b)^{3b-1} \ln (2 - b) + b^{5-3b} \ln b > 0.$$ 

**Proof.**

$$(2 - b)^{3b-1} \ln (2 - b) + b^{5-3b} \ln b > 0$$

is equivalent to

$$\frac{-(b(2-b))^{3b} \ln (2 - b)}{(2 - b)b^5 \ln b} > 1.$$ 

We set

$$f(b) = b^{3b-4} - (2 - b)$$

then we have derivatives

$$f'(b) = 1 + b^{3b-4} \left( \frac{3b-4}{b} + 3 \ln b \right)$$

and

$$f''(b) = b^{3b-4} \left( \frac{3}{b} + \frac{4}{b^2} \right) + b^{3b-4} \left( \frac{3b-4}{b} + 3 \ln b \right)^2.$$ 

Since $f''(b) > 0$, $f'(b)$ is strictly increasing for $b$. Since $f'(1) = 0$, we have $f'(b) < 0$ for all $0 < b < 1$. Therefore, $f(b)$ is strictly decreasing for $0 < b < 1$. Since $f(1) = 0$, we have $f(b) > 0$ for all $0 < b < 1$. Therefore, we get

$$\frac{b^{3b-4}}{2 - b} > 1$$

for all $0 < b < 1$. We set

$$g(b) = \ln (2 - b) + b \ln b$$

then we have derivatives

$$g'(b) = 1 - \frac{1}{2 - b} + \ln b$$

and

$$g''(b) = \frac{(b-4)(b-1)}{(b-2)^2 b}.$$ 

Since $g''(b) > 0$, $g'(b)$ is strictly increasing for $b$. Since $g'(1) = 0$, we have $g'(b) < 0$ for all $0 < b < 1$. Therefore, $g(b)$ is strictly decreasing for $0 < b < 1$. Since $g(1) = 0$, we have $g(b) > 0$ for all $0 < b < 1$. Therefore, we get

$$-\frac{\ln (2 - b)}{b \ln b} > 1.$$
for all $0 < b < 1$. Since $2 - b > 1$ and $(2 - b)^{3b} > 1^{3b} = 1$, we have $(2 - b)^{3b} > 1$. Thus, we can get
\[ \frac{-(b(2-b))^{3b} \ln(2-b)}{(2-b)b^5 \ln b} > 1. \]
for all $0 \leq b \leq 1$.

\[ \text{Lemma 2.2. If } 0 < t < 1, \text{ then } G_1(t) > 0, \text{ where} \]
\[ G_1(t) = 18 + 54t + 45t^2 + 12t^3 - 63t^4 - 60t^5 - 22t^6 + 36t^7 + 30t^8 + 16t^9 > 0. \]

**Proof.** We set
\[ f(t) = 6 + 45t^2 + 12t^3 - 63t^4 \]
and
\[ g(t) = 12 + 54t - 60t^5 - 22t^6 + 36t^7. \]
Since $f'(t) = 18t(5 + 2t - 14t^2)$ and $f'(0) = f'((1 + \sqrt{7})/14) = 0$, $f(t)$ is strictly increasing for $0 < t < (1 + \sqrt{7})/14$ and $f(t)$ is strictly decreasing for $(1 + \sqrt{7})/14 < t < 1$. From $f(0) = 6$ and $f(1) = 0$, $f(t) > 0$ for all $0 < t < 1$. Since $g''(t) = 12t^3(-100 - 55t + 126t^2) < 0$, $g'(t)$ is strictly decreasing for $0 < t < 1$. From $g'(0) = 54$ and $g'(1) = -126$, there exists uniquely a number $t_0$ with $0 < t_0 < 1$ such that $g'(t_0) = 0$. Since $g'(t) > 0$ for $0 < t < t_0$ and $g'(t) < 0$ for $t_0 < t < 1$, $g(t)$ is strictly increasing for $0 < t < t_0$ and $g(t)$ is strictly decreasing for $t_0 < t < 1$. From $g(0) = 12$ and $g(1) = 20$, $g(t) > 0$ for all $0 < t < 1$. Since $G_1(t) > f(t) + g(t)$ and $f(t) + g(t) > 0$, we have $G_1(t) > 0$ for $0 < t < 1$.

**Lemma 2.3. There exists uniquely a number $t_1$ with $0 < t_1 < 1$ such that $G_2(t_1) = 0$, $G_2(t) > 0$ for $0 < t < t_1$ and $G_2(t) > 0$ for $t_1 < t < 1$, where**
\[ G_2(t) = -18 + 18t + 9t^2 + 36t^3 + 24t^5 + 2t^6 + 16t^7. \]

**Proof.** From $G_2'(t) > 0$, $G_2(t)$ is strictly increasing for $0 < t < 1$. Since $G_2(0) = -18$ and $G_2(1) = 87$, there exists uniquely a number $t_1$ with $0 < t_1 < 1$ such that $G_2(t_1) = 0$. Therefore, we have $G_2(t) < 0$ for $0 < t < t_1$ and $G_2(t) > 0$ for $t_1 < t < 1$.

**Lemma 2.4. If** $0 < t < 1$, **then**
\[ H_1(t) > 0, \quad H_2(t) > 0, \quad H_3(t) > 0, \]
**where**
\[ H_1(t) = 648 - 5184t + 13986t^2, \]
\[ H_2(t) = 31320t^3 + 73143t^4 + 14742t^5 - 35433t^6 - 137844t^7 - 53988t^8 - 2000t^9 - 3828t^9 + 121410t^{10}, \]
\[ H_3(t) = 50100t^{11} + 44862t^{12} - 36280t^{13} + 7156t^{14} - 20384t^{15} + 1408t^{16} - 22064t^{17} - 840t^{18}. \]
Proof. We have following inequalities

\[ H_1(t) > H_1\left(\frac{48}{259}\right) = \frac{43416}{259} \]

and

\[ H_2(t) > 30000t^3 + 70000t^4 + 14000t^5 - 36000t^6 - 138000t^7 - 54000t^8 - 2000t^9 - 4000t^9 + 120000t^{10} \]
\[ = 2000(-1 + t)t^3(-15 - 50t - 57t^2 - 39t^3 + 30t^4 + 57t^5 + 60t^6). \]

Here, we set

\[ f(t) = -39t^3 + 30t^4, \]
\[ g(t) = -57t^2 + 57t^5 \]

and

\[ h(t) = -15 - 50t + 60t^6. \]

Then we have following inequalities

\[ f(t) = 3t^3(-13 + 10t) < 3t^3(-13 + 10) = -9t^3 < 0, \]
\[ g(t) = 57(-1 + t)t^2(1 + t + t^2) < 0 \]

and

\[ h(t) = -15 + 10t(-5 + 6t^5) < -15 + 10t(-5 + 6) < -5. \]

Since \( f(t) + g(t) + h(t) < 0 \), we have \( H_2(t) > 0 \). We have

\[ H_3(t) > 50000t^{11} + 44000t^{12} - 37000t^{13} + 7000t^{14} - 21000t^{15} + 1400t^{16} - 23000t^{17} - 1000t^{18} \]
\[ = -200t^{11}(-250 - 220t + 185t^2 - 35t^3 + 105t^4 - 7t^5 + 115t^6 + 5t^7). \]

Since

\[ -250 + 105t^4 + 115t^6 + 5t^7 < -250 + 105 + 115 + 5 < 0 \]

and

\[ -220t + 185t^2 - 35t^3 - 7t^5 = -t(220 - 185t + 35t^2 + 7t^4) < -t(220 - 185t) < 0, \]

we have \( H_3(t) > 0 \). Therefore, we have \( H_1(t) > 0, H_2(t) > 0 \) and \( H_3(t) > 0 \) for \( 0 < t < 1 \).

Lemma 2.5. If \( 0 \leq t \leq 1 \), then \( G(t) \leq 0 \), where

\[ G(t) = e^{(1+t)(\frac{1}{1-t^2}+2)}(-t^{\frac{1-t}{2}}+\frac{3}{z}) + e^{(1-t)(\frac{1}{1-t^2}+2)}(t^{\frac{1-t}{2}}+\frac{3}{z}) - 2. \]
The derivative of $H$ where $G$ involves $t$, we assume that

**Proof.** We have

$$G'(t) = e^{(1+t)(\frac{1}{1-t^2}+\frac{1}{2})} \left[ t^2 - \frac{G_1(t)}{6(-1 + t)^2(1 + t + t^2)^2} + e^{(1-t)(\frac{1}{1-t^2}+\frac{1}{2})} \right]$$

where

$$G_1(t) = 18 + 54t + 45t^2 + 12t^3 - 63t^4 - 60t^5 - 22t^6 + 36t^7 + 30t^8 + 16t^9$$

and

$$G_2(t) = -18 + 18t + 9t^2 + 36t^3 + 24t^5 + 2t^6 + 16t^7.$$

According to Lemmas 2.2 and 2.3, we have $G_1(t) > 0$ and $G_2(t) \geq 0$ for $t_1 \leq t < 1$, therefore $G'(t) < 0$ for $t_1 \leq t < 1$. We will show further that $G'(t)$ is also negative for $0 < t < t_1$, which involves $G'(t) < 0$ for $0 < t < 1$. The inequality $G'(t) < 0$ for $0 < t < t_1$ is equivalent to $H(t) > 0$, where

$$H(t) = (1 + t) \left( -t - \frac{t^2}{2} - \frac{t^3}{3} \right) \left( \frac{1}{1-t^2} + 2 \right) + \ln G_1(t) - \ln((-1 + t)^2)$$

$$- (1 - t) \left( t - \frac{t^2}{2} + \frac{t^3}{3} \right) \left( \frac{1}{1-t^2} + 2 \right) - \ln(-G_2(t)) > 0.$$

The derivative of $H(t)$ is

$$H'(t) = \frac{-t^2(H_1(t) + H_2(t) + H_3(t))}{(-1 + t)^2(1 + t + t^2)^2G_1(t)G_2(t)},$$

where

$$H_1(t) = 648 - 5184t + 13986t^2,$$

$$H_2(t) = 31320t^3 + 73143t^4 + 14742t^5 - 35433t^6 - 137844t^7$$

$$- 53988t^8 - 2000t^9 - 3828t^9 + 121410t^{10},$$

and

$$H_3(t) = 50100t^{11} + 44862t^{12} - 36280t^{13} + 7156t^{14}$$

$$- 20384t^{15} + 1408t^{16} - 22064t^{17} - 840t^{18}.$$

By Lemma 2.4, it follows that $H'(t) > 0$ for $0 < t < t_1$, when $G_1(t) > 0$ and $G_2(t) < 0$. Therefore, $H(t)$ is strictly increasing for $0 < t < t_1$, hence $H(t) > H(0) = 0$. Thus, $G'(t) < 0$ for $0 < t < 1$, $G(t)$ is strictly decreasing, $G(t) < G(0) = 0$ for $t_1 < t \leq 1$. □

3. **Proof of Theorem 1.1**

**Proof.** Without loss of generically, we assume that

$$0 \leq b \leq 1 - \left( \frac{r-3}{r-2} \right)^{\frac{1}{3}}.$$
and

\[ 1 + \left( \frac{r - 3}{r - 2} \right)^{\frac{1}{3}} \leq a \leq 2. \]

We set

\[ F(b, r) = (2 - b)^{rb} + b^{r(2-b)} - 2. \]

Then we have derivatives

\[ \frac{\partial F}{\partial r}(b, r) = (2 - b)^{rb} b \ln(2 - b) + (2 - b)^{r(2-b)} r \ln b \]

and

\[ \frac{\partial^2 F}{\partial r^2}(b, r) = (2 - b)^{rb} b^2 \ln(2 - b)^2 + (2 - b)^{r(2-b)} (\ln b)^2. \]

Since \( \frac{\partial^2 F(b, r)}{\partial r^2} \geq 0 \), the function \( \frac{\partial F(b, r)}{\partial r} \) is strictly increasing for \( r \). By Lemma 2.1, we have

\[ \frac{\partial F}{\partial r}(b, r) \geq \frac{\partial F}{\partial r}(b, 3) = b(2 - b) \left( (2 - b)^{3b-1} \ln(2 - b) + b^{5-3b} \ln b \right) \]

\[ \geq 0. \]

Thus, \( F(b, r) \) is strictly increasing for \( r \geq 3 \). Since

\[ 0 \leq b \leq 1 - \left( \frac{r - 3}{r - 2} \right)^{\frac{1}{3}}, \]

we have

\[ 3 \leq r \leq \frac{1}{1 - (1-b)^3} + 2. \]

Thus, we can get

\[ F(b, r) \leq F \left( b, \frac{1}{1 - (1-b)^3} + 2 \right) \]

\[ = (2 - b)^{b \left( \frac{1}{1 - (1-b)^3} + 2 \right)} b^{(1-t) \left( \frac{1}{1 - (1-b)^3} + 2 \right)^{(2-b)} - 2}. \]

Therefore, it suffices to show that

\[ (2 - b)^{b \left( \frac{1}{1 - (1-b)^3} + 2 \right)} b^{(1-t) \left( \frac{1}{1 - (1-b)^3} + 2 \right)} (2-b) - 2 \leq 0. \]

Denoting

\[ t = 1 - b, \quad 0 \leq t \leq 1, \]

this desired inequality becomes

\[ (1 + t)^{(1-t) \left( \frac{1}{1-t^3} + 2 \right)} + (1 - t)^{(1+t) \left( \frac{1}{1-t^3} + 2 \right)} - 2 \leq 0. \]

From Lemma 6.1 in [3], we have

\[ \ln(1 + t) \leq t - \frac{t^2}{2} + \frac{t^3}{3} \]
for all $t > -1$. Using this inequality, we get

\[
(1 - t)^{(1+t)\left(\frac{1}{1-t^3} + 2\right)} + (1 + t)^{(1-t)\left(\frac{1}{1-t^3} + 2\right)} - 2
\]

\[
= e^{(1+t)\left(\frac{1}{1-t^3} + 2\right)} \ln(1-t) + e^{(1-t)\left(\frac{1}{1-t^3} + 2\right)} \ln(1+t) - 2
\]

\[
\leq e^{(1+t)\left(\frac{1}{1-t^3} + 2\right)} \left(-t - \frac{c^2}{t} - \frac{c^3}{3}\right) + e^{(1-t)\left(\frac{1}{1-t^3} + 2\right)} \left(t - \frac{c^2}{t} + \frac{c^3}{3}\right) - 2.
\]

Therefore, it suffices to prove that $G(t) \leq 0$ for $0 \leq t \leq 1$, where

\[
G(t) = e^{(1+t)\left(\frac{1}{1-t^3} + 2\right)} \left(-t - \frac{c^2}{t} - \frac{c^3}{3}\right) + e^{(1-t)\left(\frac{1}{1-t^3} + 2\right)} \left(t - \frac{c^2}{t} + \frac{c^3}{3}\right) - 2.
\]

This is true by Lemma 2.5. Thus, the proof of Theorem 1.1 is completed. \qed

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**References**


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