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EXTENSION OF AN INEQUALITY WITH POWER EXPONENTIAL FUNCTIONS

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Abstract. V. Cîrtoaje et al. [2] conjectured and proved [3, 4] that the inequality $a^{rb} + b^{ra} \leq 2$ holds for all nonnegative numbers $r \leq 3$ and nonnegative real numbers a, b with $a + b = 2$. In this paper, we will show that $a^{rb} + b^{ra} \leq 2$ holds for all nonnegative $r \geq 3$ and all nonnegative real numbers a, b with $a + b = 2$ and some conditions. This gives an extended inequality of conjectured by V. Cîrtoaje.

1. Introduction

Inequalities appear on the various branches of mathematics. In this paper, we give a result of an inequality with power exponential functions which is studied by V. Cîrtoaje et al. [1, 2, 3, 4, 5, 6, 7, 8]. The formula of inequalities with power exponential functions are very simple, but their proof is not as simple as it seems. V. Cîrtoaje et al. [3, 4] proved that the inequality

$$a^{rb} + b^{ra} \leq 2 \tag{1.1}$$

holds for all nonnegative real number $r \leq 3$ and all nonnegative real numbers a, b with $a + b = 2$. Miyagi et al. [7] proved that the stronger inequality

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \leq 2 \tag{1.2}$$

holds for the same conditions. These inequalities (1.1) and (1.2) are conjectures by V. Cîrtoaje [2]. The following is our main theorem.

Theorem 1.1. *The inequality*

$$a^{rb} + b^{ra} \leq 2 \tag{1.3}$$

holds for all numbers $r \geq 3$ *and all real numbers* $a, b \in [0, 1 - ((r-3)/(r-2))^{1/3}] \cup [1 + ((r-3)/(r-2))^{1/3}, 2]$ *with* $a + b = 2$.

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The above inequality (1.3) is an extension of the inequality (1.1).

2. Preliminaries

In this section, we will show some lemmas to prove our main theorem.

Lemma 2.1. *If* $0 < b < 1$, *then*

$$(2-b)^{3b-1}\ln(2-b) + b^{5-3b}\ln b > 0.$$

Proof.

$$(2-b)^{3b-1}\ln(2-b) + b^{5-3b}\ln b > 0$$

is equivalent to

$$\frac{-(b(2-b))^{3b}\ln(2-b)}{(2-b)b^5\ln b} > 1.$$

We set

$$f(b) = b^{3b-4} - (2-b)$$

then we have derivatives

$$f'(b) = 1 + b^{3b-4}\left(\frac{3b-4}{b} + 3\ln b\right)$$

and

$$f''(b) = b^{3b-4}\left(\frac{3}{b} + \frac{4}{b^2}\right) + b^{3b-4}\left(\frac{3b-4}{b} + 3\ln b\right)^2.$$

Since $f''(b) > 0$, $f'(b)$ is strictly increasing for b. Since $f'(1) = 0$, we have $f'(b) < 0$ for all $0 < b < 1$. Therefore, $f(b)$ is strictly decreasing for $0 < b < 1$. Since $f(1) = 0$, we have $f(b) > 0$ for all $0 < b < 1$. Therefore, we get

$$\frac{b^{3b-4}}{2-b} > 1$$

for all $0 < b < 1$. We set

$$g(b) = \ln(2-b) + b\ln b$$

then we have derivatives

$$g'(b) = 1 - \frac{1}{2-b} + \ln b$$

and

$$g''(b) = \frac{(b-4)(b-1)}{(b-2)^2 b}.$$

Since $g''(b) > 0$, $g'(b)$ is strictly increasing for b. Since $g'(1) = 0$, we have $g'(b) < 0$ for all $0 < b < 1$. Therefore, $g(b)$ is strictly decreasing for $0 < b < 1$. Since $g(1) = 0$, we have $g(b) > 0$ for all $0 < b < 1$. Therefore, we get

$$-\frac{\ln(2-b)}{b\ln b} > 1$$

for all $0 < b < 1$. Since $2-b > 1$ and $(2-b)^{3b} > 1^{3b} = 1$, we have $(2-b)^{3b} > 1$. Thus, we can get

$$\frac{-(b(2-b))^{3b}\ln(2-b)}{(2-b)b^5\ln b} > 1.$$

for all $0 \le b \le 1$. □

Lemma 2.2. *If $0 < t < 1$, then $G_1(t) > 0$, where*

$$G_1(t) = 18 + 54t + 45t^2 + 12t^3 - 63t^4 - 60t^5 - 22t^6 + 36t^7 + 30t^8 + 16t^9 > 0.$$

Proof. We set

$$f(t) = 6 + 45t^2 + 12t^3 - 63t^4$$

and

$$g(t) = 12 + 54t - 60t^5 - 22t^6 + 36t^7.$$

Since $f'(t) = 18t(5+2t-14t^2)$ and $f'(0) = f'((1+\sqrt{71})/14) = 0$, $f(t)$ is strictly increasing for $0 < t < (1+\sqrt{71})/14$ and $f(t)$ is strictly decreasing for $(1+\sqrt{71})/14 < t < 1$. From $f(0) = 6$ and $f(1) = 0$, $f(t) > 0$ for all $0 < t < 1$. Since $g'(t) = 6(9 - 50t^4 - 22t^5 + 42t^6)$ and $g''(t) = 12t^3(-100 - 55t + 126t^2) < 0$, $g'(t)$ is strictly decreasing for $0 < t < 1$. From $g'(0) = 54$ and $g'(1) = -126$, there exists uniquely a number t_0 with $0 < t_0 < 1$ such that $g'(t_0) = 0$. Since $g'(t) > 0$ for $0 < t < t_0$ and $g'(t) < 0$ for $t_0 < t < 1$, $g(t)$ is strictly increasing for $0 < t < t_0$ and $g(t)$ is strictly decreasing for $t_0 < t < 1$. From $g(0) = 12$ and $g(1) = 20$, $g(t) > 0$ for all $0 < t < 1$. Since $G_1(t) > f(t) + g(t)$ and $f(t) + g(t) > 0$, we have $G_1(t) > 0$ for $0 < t < 1$. □

Lemma 2.3. *There exists uniquely a number t_1 with $0 < t_1 < 1$ such that $G_2(t_1) = 0$, $G_2(t) < 0$ for $0 < t < t_1$ and $G_2(t) > 0$ for $t_1 < t < 1$, where*

$$G_2(t) = -18 + 18t + 9t^2 + 36t^3 + 24t^5 + 2t^6 + 16t^7.$$

Proof. From $G_2'(t) > 0$, $G_2(t)$ is strictly increasing for $0 < t < 1$. Since $G_2(0) = -18$ and $G_2(1) = 87$, there exists uniquely a number t_1 with $0 < t_1 < 1$ such that $G_2(t_1) = 0$. Therefore, we have $G_2(t) < 0$ for $0 < t < t_1$ and $G_2(t) > 0$ for $t_1 < t < 1$. □

Lemma 2.4. *If $0 < t < 1$, then*

$$H_1(t) > 0, \quad H_2(t) > 0, \quad H_3(t) > 0,$$

where

$$\begin{aligned} H_1(t) &= 648 - 5184t + 13986t^2, \\ H_2(t) &= 31320t^3 + 73143t^4 + 14742t^5 - 35433t^6 - 137844t^7 \\ &\quad -53988t^8 - 2000t^9 - 3828t^9 + 121410t^{10}, \\ H_3(t) &= 50100\, t^{11} + 44862t^{12} - 36280\, t^{13} + 7156t^{14} \\ &\quad -20384t^{15} + 1408t^{16} - 22064t^{17} - 840t^{18}. \end{aligned}$$

Proof. We have following inequalities

$$H_1(t) > H_1\left(\frac{48}{259}\right) = \frac{43416}{259}$$

and

$$\begin{aligned} H_2(t) > {} & 30000t^3 + 70000t^4 + 14000t^5 - 36000t^6 - 138000t^7 \\ & -54000t^8 - 2000t^9 - 4000t^9 + 120000t^{10} \\ = {} & 2000(-1+t)t^3(-15-50t-57t^2-39t^3+30t^4+57t^5+60t^6). \end{aligned}$$

Here, we set

$$\begin{aligned} f(t) &= -39t^3 + 30t^4, \\ g(t) &= -57t^2 + 57t^5 \end{aligned}$$

and

$$h(t) = -15 - 50t + 60t^6.$$

Then we have following inequalities

$$\begin{aligned} f(t) &= 3t^3(-13+10t) < 3t^3(-13+10) = -9t^3 < 0, \\ g(t) &= 57(-1+t)t^2(1+t+t^2) < 0 \end{aligned}$$

and

$$h(t) = -15 + 10t(-5+6t^5) < -15 + 10t(-5+6) < -5.$$

Since $f(t) + g(t) + h(t) < 0$, we have $H_2(t) > 0$. We have

$$\begin{aligned} H_3(t) > {} & 50000t^{11} + 44000t^{12} - 37000t^{13} + 7000t^{14} - 21000t^{15} + 1400t^{16} \\ & - 23000t^{17} - 1000t^{18} \\ = {} & -200t^{11}(-250 - 220t + 185t^2 - 35t^3 + 105t^4 - 7t^5 + 115t^6 + 5t^7). \end{aligned}$$

Since

$$-250 + 105t^4 + 115t^6 + 5t^7 < -250 + 105 + 115 + 5 < 0$$

and

$$\begin{aligned} -220t + 185t^2 - 35t^3 - 7t^5 &= -t(220 - 185t + 35t^2 + 7t^4) \\ &< -t(220 - 185t) \\ &< 0, \end{aligned}$$

we have $H_3(t) > 0$. Therefore, we have $H_1(t) > 0, H_2(t) > 0$ and $H_3(t) > 0$ for $0 < t < 1$. □

Lemma 2.5. *If* $0 \le t \le 1$, *then* $G(t) \le 0$, *where*

$$G(t) = e^{(1+t)\left(\frac{1}{1-t^3}+2\right)\left(-t-\frac{t^2}{2}-\frac{t^3}{3}\right)} + e^{(1-t)\left(\frac{1}{1-t^3}+2\right)\left(t-\frac{t^2}{2}+\frac{t^3}{3}\right)} - 2.$$

Proof. We have

$$G'(t) = e^{(1+t)\left(\frac{1}{1-t^3}+2\right)\left(-t-\frac{t^2}{2}-\frac{t^3}{3}\right)} \frac{-G_1(t)}{6(-1+t)^2(1+t+t^2)^2} + e^{(1-t)\left(\frac{1}{1-t^3}+2\right)\left(t-\frac{t^2}{2}+\frac{t^3}{3}\right)} \frac{-G_2(t)}{6(1+t+t^2)^2}$$

where

$$G_1(t) = 18+54t+45t^2+12t^3-63t^4-60t^5-22t^6+36t^7+30t^8+16t^9$$

and

$$G_2(t) = -18+18t+9t^2+36t^3+24t^5+2t^6+16t^7.$$

According to Lemmas 2.2 and 2.3, we have $G_1(t) > 0$ and $G_2(t) \geq 0$ for $t_1 \leq t < 1$, therefore $G'(t) < 0$ for $t_1 \leq t < 1$. We will show further that $G'(t)$ is also negative for $0 < t < t_1$, which involves $G'(t) < 0$ for $0 < t < 1$. The inequality $G'(t) < 0$ for $0 < t < t_1$ is equivalent to $H(t) > 0$, where

$$\begin{aligned} H(t) =&(1+t)\left(-t-\frac{t^2}{2}-\frac{t^3}{3}\right)\left(\frac{1}{1-t^3}+2\right)+\ln G_1(t)-\ln((-1+t)^2) \\ &-(1-t)\left(t-\frac{t^2}{2}+\frac{t^3}{3}\right)\left(\frac{1}{1-t^3}+2\right)-\ln(-G_2(t))>0. \end{aligned}$$

The derivative of $H(t)$ is

$$H'(t) = \frac{-t^2(H_1(t)+H_2(t)+H_3(t))}{(-1+t)^2(1+t+t^2)^2 G_1(t)G_2(t)},$$

where

$$H_1(t) = 648-5184t+13986t^2,$$

$$\begin{aligned} H_2(t) =& 31320t^3+73143t^4+14742t^5-35433t^6-137844t^7 \\ &-53988t^8-2000t^9-3828t^9+121410t^{10}, \end{aligned}$$

and

$$\begin{aligned} H_3(t) =& 50100\,t^{11}+44862t^{12}-36280\,t^{13}+7156t^{14} \\ &-20384t^{15}+1408t^{16}-22064t^{17}-840t^{18}. \end{aligned}$$

By Lemma 2.4, it follows that $H'(t) > 0$ for $0 < t < t_1$, when $G_1(t) > 0$ and $G_2(t) < 0$. Therefore, $H(t)$ is strictly increasing for $0 < t < t_1$, hence $H(t) > H(0) = 0$. Thus, $G'(t) < 0$ for $0 < t < 1$, $G(t)$ is strictly decreasing, $G(t) < G(0) = 0$ for $t_1 < t \leq 1$. □

3. Proof of Theorem 1.1

Proof. Without loss of generically, we assume that

$$0 \leq b \leq 1-\left(\frac{r-3}{r-2}\right)^{\frac{1}{3}}$$

and

$$1+\left(\frac{r-3}{r-2}\right)^{\frac{1}{3}} \le a \le 2.$$

We set

$$F(b,r) = (2-b)^{rb} + b^{r(2-b)} - 2.$$

Then we have derivatives

$$\frac{\partial F}{\partial r}(b,r) = (2-b)^{br}\, b \ln(2-b) + (2-b)\, b^{(2-b)r} \ln b$$

and

$$\frac{\partial^2 F}{\partial r^2}(b,r) = (2-b)^{br}\, b^2 (\ln(2-b))^2 + (2-b)^2 b^{(2-b)r} (\ln b)^2.$$

Since $\partial^2 F(b,r)/\partial r^2 \ge 0$, the function $\partial F(b,r)/\partial r$ is strictly increasing for r. By Lemma 2.1, we have

$$\begin{aligned}\frac{\partial F}{\partial r}(b,r) &\ge \frac{\partial F}{\partial r}(b,3)\\ &= b(2-b)\left((2-b)^{3b-1}\ln(2-b) + b^{5-3b}\ln b\right)\\ &\ge 0.\end{aligned}$$

Thus, $F(b,r)$ is strictly increasing for $r \ge 3$. Since

$$0 \le b \le 1-\left(\frac{r-3}{r-2}\right)^{\frac{1}{3}},$$

we have

$$3 \le r \le \frac{1}{1-(1-b)^3} + 2.$$

Thus, we can get

$$\begin{aligned}F(b,r) &\le F\left(b, \frac{1}{1-(1-b)^3} + 2\right)\\ &= (2-b)^{\left(\frac{1}{1-(1-b)^3}+2\right)b} + b^{\left(\frac{1}{1-(1-b)^3}+2\right)(2-b)} - 2.\end{aligned}$$

Therefore, it suffices to show that

$$(2-b)^{\left(\frac{1}{1-(1-b)^3}+2\right)b} + b^{\left(\frac{1}{1-(1-b)^3}+2\right)(2-b)} - 2 \le 0.$$

Denoting

$$t = 1-b, \quad 0 \le t \le 1,$$

this desired inequality becomes

$$(1+t)^{(1-t)\left(\frac{1}{1-t^3}+2\right)} + (1-t)^{(1+t)\left(\frac{1}{1-t^3}+2\right)} - 2 \le 0.$$

From Lemma 6.1 in [3], we have

$$\ln(1+t) \le t - \frac{t^2}{2} + \frac{t^3}{3}$$

for all $t > -1$. Using this inequality, we get

$$\begin{aligned}(1-t)^{(1+t)\left(\frac{1}{1-t^3}+2\right)} &+ (1+t)^{(1-t)\left(\frac{1}{1-t^3}+2\right)} - 2 \\ &= e^{(1+t)\left(\frac{1}{1-t^3}+2\right)\ln(1-t)} + e^{(1-t)\left(\frac{1}{1-t^3}+2\right)\ln(1+t)} - 2 \\ &\le e^{(1+t)\left(\frac{1}{1-t^3}+2\right)\left(-t-\frac{t^2}{2}-\frac{t^3}{3}\right)} + e^{(1-t)\left(\frac{1}{1-t^3}+2\right)\left(t-\frac{t^2}{2}+\frac{t^3}{3}\right)} - 2.\end{aligned}$$

Therefore, it suffices to prove that $G(t) \le 0$ for $0 \le t \le 1$, where

$$G(t) = e^{(1+t)\left(\frac{1}{1-t^3}+2\right)\left(-t-\frac{t^2}{2}-\frac{t^3}{3}\right)} + e^{(1-t)\left(\frac{1}{1-t^3}+2\right)\left(t-\frac{t^2}{2}+\frac{t^3}{3}\right)} - 2.$$

This is true by Lemma 2.5. Thus, the proof of Theorem 1.1 is completed. □

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