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ON $\mathscr{K}$-EXTENDING MODULES

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Abstract. Let M be a right R-module and $S = End_R(M)$. We call M a $\mathscr{K}$-extending module if for every element $\phi \in S$, Kerϕ is essential in a direct summand of M. In this paper we investigate these modules. We give a characterization of $\mathscr{K}$-extending modules. We prove that if M is a projective self-generator module, then M is a $\mathscr{K}$-extending module and every finitely generated projective right ideal of S is a summand if and only if S is semiregular and $\Delta(M) = Jac(S)$, where $\Delta(M) = \{f \in S \mid Ker f \leq^e M\}$ if and only if M is $Z(M)$-$\mathscr{I}$-lifting.

1. Introduction

Throughout this paper R will denote an associative ring with identity, M a unitary right R-module and $S = End_R(M)$ the ring of all R-endomorphisms of M. We will use the notation $N \leq^e M$ to indicate that N is an essential submodule of M (i.e. $\forall 0 \neq L \leq M, L \cap N \neq 0$); $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). The notation $N \leq^\oplus M$ denotes that N is a direct summand of M. We also denote $r_M(I) = \{x \in M \mid Ix = 0\}$, for $I \subseteq S$; $\Delta(M) = \{f \in S \mid Ker f \leq^e M\}$ and $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of $R\}$.

Extending modules, continuous modules and lifting modules play important roles in rings and categories of modules, and have been studied extensively by many authors in recent years (see, [4], [5], [7], [9], [11]). A module M is called *extending* (or *CS*) if every submodule of M is essential in a direct summand of M. Dually, a module M is called *lifting* if for every $A \leq M$, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$ [9]. In [1], we introduced $\mathscr{I}$-lifting modules as a generalization of lifting modules. A module M is called *$\mathscr{I}$-lifting* if for every $\phi \in S$ there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq$ Imϕ and $M_2 \cap$ Im$\phi \ll M_2$. It is obvious that every lifting module is $\mathscr{I}$-lifting while the converse in not true (the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathscr{I}$-lifting but it is not lifting). A ring R is called a *semiregular* ring if for each $a \in R$, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$ [10]. It is easily checked that R_R is an $\mathscr{I}$-lifting module if and only if R is a semiregular ring. In [11], Nicholson and Yousif introduced right I-semiregular rings for an ideal I of a ring R. A ring R is called a *right*

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I-semiregular ring if for each $a \in R$, there exists $e^2 = e \in aR$ such that $(1-e)a \in I$. In this note, motivated by [11], we introduce F-$\mathscr{I}$-lifting modules for a submodule F of a module M as a generalization of the right I-semiregular ring. A module M is called *F-$\mathscr{I}$-lifting* if for every $\phi \in S$ there exists a decomposition $M = A \oplus B$ such that $A \subseteq \phi M$ and $\phi M \cap B \leq F$. Let $F = I$ be an ideal of R. It is clear that R_R is an I-$\mathscr{I}$-lifting module if and only if R is a right I-semiregular ring.

In [11], a ring R is called an *ACS-ring* if for every element $a \in R$, $r_R(a) \leq^e fR$ for some $f^2 = f \in R$. Inspired by this definition we introduce and investigate $\mathscr{K}$-extending modules as a generalization of the ACS-ring. We call M a *$\mathscr{K}$-extending* module if for every element $\phi \in S$, Kerϕ is essential in a direct summand of M. These modules are also a generalization of extending modules and dual of $\mathscr{I}$-lifting modules. It is clear that R_R is a $\mathscr{K}$-extending module if and only if R is an ACS-ring. In this paper our aim is to generalize the some results of [11] from the ring case to the module case.

In Section 2, we characterize semi-projective F-$\mathscr{I}$-lifting modules. We show that the following are equivalent for a semi-projective retractable module M:

(1) M is $Z(M)$-$\mathscr{I}$-lifting.

(2) S is $\Delta(M)$-semiregular.

(3) S is $Z_r(S)$-semiregular.

(4) M is $\Delta(M)M$-$\mathscr{I}$-lifting.

In Section 3, we give a characterization of $\mathscr{K}$-extending modules. We prove that if M is a projective self-generator module, then M is a $\mathscr{K}$-extending module and every finitely generated projective right ideal of S is a summand if and only if S is semiregular and $\Delta(M) = Jac(S)$, where $\Delta(M) = \{f \in S \mid Ker f \leq^e M\}$ if and only if M is $Z(M)$-$\mathscr{I}$-lifting.

We also prove the following which generalizes [11, Corollary 2.7]:

Let M be a projective self-generator module. Then the following are equivalent:

(1) M is quasi-injective.

(2) M has (C_2) and $M \oplus M$ is extending.

(3) M is $Z(M)$-$\mathscr{I}$-lifting and $M \oplus M$ is extending.

(4) M is weakly continuous and $M \oplus M$ is extending.

2. F-$\mathscr{I}$-lifting modules

Definition 2.1. Let F be a submodule of an R-module M. A module M is called *F-$\mathscr{I}$-lifting* if for every $\phi \in S$ there exists a decomposition $M = A \oplus B$ such that $A \subseteq \phi M$ and $\phi M \cap B \leq F$.

It is clear that every $\mathcal{I}$-lifting module is $Rad(M)$-$\mathcal{I}$-lifting.

A module M is called *semi-projective* if for any epimorphism $f: M \to N$, where N is a submodule of M, and for any homomorphism $g: M \to N$, there exists $h: M \to M$ such that $fh = g$.

Lemma 2.2. *Let M be a semi-projective module and F be a fully invariant submodule of M. Then the following are equivalent for $\phi \in S$:*

(1) *There exists $e^2 = e \in \phi S$ with $(\phi - e\phi)M \subseteq F$.*

(2) *There exists $e^2 = e \in \phi S$ with $\phi M \cap (1-e)M \subseteq F$.*

(3) *$\phi M = eM \oplus N$ where $e^2 = e \in S$ and $N \subseteq F$.*

Proof. (1) $\Rightarrow$ (2) If $x \in \phi M \cap (1-e)M$, then $x = \phi m = (1-e)m = (1-e)m'$ for some $m, m' \in M$. Thus $x = (1-e)\phi m \in F$.

(2) $\Rightarrow$ (3) It is clear that $\phi M = eM \oplus [\phi M \cap (1-e)M]$. Set $N = \phi M \cap (1-e)M$.

(3) $\Rightarrow$ (1) First we show that $e^2 = e \in \phi S$. Consider the epimorphisms $\phi: M \to \phi M$ and $e: M \to eM$. Since M is semi-projective, there exists a homomorphism $g \in S$ such that $\phi g = ie = e$, where $i: eM \to \phi M$ is the inclusion map. Hence $e \in \phi S$. Since $\phi M = eM \oplus N$, for every $m \in M$, we have $\phi m = em' + n$ for some $m' \in M$ and $n \in N$. Then $\phi m - e\phi m = n - en \in F$ because $N \subseteq F$. Hence $(\phi - e\phi)M \subseteq F$. □

Theorem 2.3. *Let F be a fully invariant submodule of a semi-projective module M. Then the following conditions are equivalent:*

(1) *M is F-$\mathcal{I}$-lifting.*

(2) *For any finitely generated right ideal $I \subseteq S$, there exists a homomorphism γ from M to IM such that $\gamma^2 = \gamma$ and $(1-\gamma)IM \subseteq F$.*

(3) *For any finitely generated right ideal $I \subseteq S$, there exists a decomposition $M = L \oplus H$ such that L is a submodule of IM and $IM \cap H \subseteq F$.*

(4) *For any finitely generated right ideal $I \subseteq S$, IM can be written as $IM = L \oplus N$ where L is a direct summand of M and $N \subseteq F$.*

When these conditions are satisfied we have:

(i) *For every right ideal I of S such that $IM \nsubseteq F$ there exists an idempotent $e^2 = e \in I$ such that $eM \nsubseteq F$.*

(ii) *$Jac(S)M \subseteq F$, and $\Delta(M)M \subseteq F$.*

Proof. (1) $\Rightarrow$ (2) We induct on n where $I = f_0S + \cdots + f_nS$. If $n = 1$ there is nothing to prove by (1). If $n \geq 2$, then (1) and Lemma 2.2 give $\beta^2 = \beta \in f_nS$ with $(1-\beta)f_nM \subseteq F$. Set $J = (1-\beta)f_0S + \cdots + (1-\beta)f_{n-1}S$. By induction, choose $\alpha : M \to JM$ such that $\alpha^2 = \alpha \in J$, and $(1-\alpha)JM \subseteq F$. Define $\gamma = \beta + \alpha - \alpha\beta$. Then $\gamma = \gamma^2$ and $\gamma M = \beta M \oplus \alpha M$ since $\beta\alpha = 0$. It remains to verify that $(1-\gamma)IM \subseteq F$. Since $\alpha \in J$ and $\beta \in f_nS$, $\gamma \in I$. Hence $\gamma M \subseteq IM$. But $(1-\gamma) = (1-\gamma)(1-\beta)$ and $(1-\beta)J = J$, so $(1-\gamma)I \subseteq (1-\alpha)(1-\beta)J + (1-\alpha)(1-\beta)f_nS$. Hence $(1-\gamma)IM \subseteq (1-\alpha)JM + (1-\alpha)(1-\beta)f_nM \subseteq F$.

(2) $\Rightarrow$ (3) Let I and γ be as in (2). Then $(1-\gamma)IM = (1-\gamma)M \cap IM$. Hence $M = \gamma M \oplus (1-\gamma)M$ and $IM \cap (1-\gamma)M = (1-\gamma)IM \subseteq F$.

(3) $\Rightarrow$ (4) $\Rightarrow$ (1) They are clear.

Suppose these conditions hold. Then (i) follows from Lemma 2.2, and (ii) follows (i). $\square$

An R-module M is called *retractable* if $Hom_R(M,N) \neq 0$ for all nonzero submodules N of M.

Lemma 2.4. *Let M be a semi-projective module. Consider the following conditions for $\phi \in S$:*

(1) *$\phi M = eM \oplus N$ where $e^2 = e \in S$ and N is a singular submodule of M.*

(2) *$\phi S = eS \oplus B$ where $e^2 = e \in S$ and $B \subseteq \Delta(M)$ is a right ideal of S.*

Then (1) $\Rightarrow$ (2) *holds and if moreover M is a retractable module, then* (2) $\Rightarrow$ (1) *holds.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\phi M = eM \oplus N$ as in (1). First we show that $N = \phi hM$ for some $h \in S$. Consider the homomorphism $\phi : M \to \phi M$. Since M is semi-projective, there exists a homomorphism $h : M \to M$ such that $\phi h = i\pi\phi$, where $i : N \to \phi M$ and $\pi : \phi M \to N$ are injection and projection maps respectively. Hence $\phi hM = \pi(\phi M) = N$. Now, by [13, 18.4], we have $Hom_R(M, \phi M) = Hom(M, eM) + Hom(M, \phi hM)$. Since M is semi-projective, $\phi S = eS + \phi hS$. As $eM \cap N = 0$, $eS \cap \phi hS = Hom_R(M, eM) \cap Hom_R(M, \phi hM) = Hom_R(M, eM \cap \phi hM) = 0$. Thus $\phi S = eS \oplus \phi hS$. Finally, since $N = \phi hM$ is singular and $\phi hM \cong \frac{M}{\mathrm{Ker}\phi h}$, $\mathrm{Ker}\phi h \leq^e M$ by [11, Lemma 2.1]. So $\phi h \in \Delta(M)$.

(2) $\Rightarrow$ (1) Let $\phi S = eS \oplus B$ as in (2). Clearly, $\phi M = eM + BM$. Since $eS \cap B = 0$ and M is semi-projective, we have $Hom_R(M, eM) \cap Hom_R(M, BM) = 0$. Therefore $Hom_R(M, eM \cap B) = 0$. Hence $eM \cap BM = 0$ by retractability. It follows that $\phi M = eM \oplus BM$ and $BM \subseteq \Delta(M)M \subseteq Z(M)$. $\square$

Corollary 2.5. *Let M be a semi-projective retractable module. Then the following are equivalent:*

(1) *M is $Z(M)$-$\mathscr{I}$-lifting.*

(2) *S is $\Delta(M)$-semiregular.*

(3) *S is $Z_r(S)$-semiregular.*

(4) *M is $\Delta(M)M$-$\mathscr{I}$-lifting.*

Proof. (1) $\Leftrightarrow$ (2) By Lemma 2.4. (2) $\Leftrightarrow$ (3) By [6, Proposition 2.4]. (2) $\Leftrightarrow$ (4) Similar to the proof of Lemma 2.4. $\square$

3. $\mathscr{K}$-extending modules

Definition 3.1. We call a module M a $\mathscr{K}$-*extending module* if for every element $\phi \in S$, $\mathrm{Ker}\phi \leq^e eM$ for some $e^2 = e \in S$.

Ir it clear that for $M = R_R$, the notion of a $\mathscr{K}$-extending module coincides with that of an ACS-ring.

Example 3.2.

(1) Every extending module is a $\mathscr{K}$-extending module.

(2) A module M is said to be *Rickart* if, for every $\phi \in End_R(M)$, $\mathrm{Ker}\phi \leq^{\oplus} M$ [8]. Rickart modules are precisely nonsingular $\mathscr{K}$-extending modules.

(3) $\mathbb{Z}^{(\mathbb{N})}$ is a Rickart $\mathbb{Z}$-module by [8, Example 2.3]. Hence it is a $\mathscr{K}$-extending module. But $\mathbb{Z}^{(\mathbb{N})}$ is not extending, since if it were, then we would have an epimorphism $f : \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Q}$ with nonessential kernel. Then by the extending property, $\mathrm{Ker}(f)$ is essential in some direct summand K of $\mathbb{Z}^{(\mathbb{N})}$. Hence $\mathbb{Q} \cong K/\mathrm{Ker}(f) \oplus T$ for some direct summand T of $\mathbb{Z}^{(\mathbb{N})}$. Since $\mathbb{Q}$ is nonsingular, $K = \mathrm{Ker}(f)$. It follows that $\mathbb{Q}$ embeds in $\mathbb{Z}$, which is a contradiction.

The following proposition generalizes [11, Proposition 2.2].

Proposition 3.3. *Let M be a projective module. Consider the following conditions for an element $\phi \in S = End_R(M)$.*

(1) *M is a $\mathscr{K}$-extending module.*

(2) *$\phi M = P \oplus N$ where P_R is a projective module and N_R is a singular module.*

(3) *$\phi S = A \oplus B$ where A_S is a projective right ideal of S and B_S is a right ideal of S with $B \subseteq \Delta(M)$.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Moreover, if M generates $r_M(I)$ for every $I \leq S_S$, then* (3) $\Rightarrow$ (1) *holds.*

Proof. (1) $\Rightarrow$ (2) Let $r_M(\phi) \leq^e (1-e)M$ where $e^2 = e \in S$. First we show that $\phi M = \phi eM \oplus \phi(1-e)M$. Clearly $\phi M = \phi eM + \phi(1-e)M$. If $x \in \phi eM \cap \phi(1-e)M$, then $x = \phi em = \phi(1-e)m'$ where $m, m' \in M$. Hence $em - (1-e)m' \in r_M(\phi) \subseteq (1-e)M$, so $em \in (1-e)M \cap eM = 0$, thus

$em = 0$. Hence $x = \phi em = 0$ and so $\phi M = \phi eM \oplus \phi(1-e)M$. Now $\phi eM \cong eM$ because the multiplication map $\overline{\phi} : eM \to \phi eM$ has kernel $\{em \mid \phi em = 0\} = eM \cap r_M(\phi) = 0$. Since M is projective, eM is projective. Hence ϕeM is projective. Finally, $\overline{\phi} : (1-e)M \to \phi(1-e)M$ has kernel $(1-e)M \cap r_M(\phi) = r_M(\phi)$. Hence $\phi(1-e)M \cong \frac{(1-e)M}{r_M(\phi)}$, and so $\phi(1-e)M$ is singular by [11, Lemma 2.1] because $r_M(\phi) \leq^e (1-e)M$.

(2) $\Rightarrow$ (1) Suppose that $\phi M = P \oplus N$ as in (2), and let $\pi : \phi M \to P$ be the projection with $\mathrm{Ker}(\pi) = N$. Then define $\gamma : M \to P$ by $\gamma(m) = \pi(\phi m)$, and write $K = \mathrm{Ker}(\gamma)$. Then γ is onto so, as P is projective, $K = fM$ for some $f^2 = f \in S$. Clearly $r_M(\phi) \subseteq fM$; it remains to verify that $r_M(\phi) \leq^e fM$. If $k \in K$, then $\phi(k) \in N$ because $\pi(\phi(k)) = \gamma(k) = 0$. Hence we have a map $\theta : K \to N$ defined by $\theta(k) = \phi(k)$. Then $\mathrm{Ker}(\theta) = K \cap r_M(\phi) = r_M(\phi)$. So $\frac{K}{r_M(\phi)} \cong \mathrm{Im}(\theta) \subseteq N$. Thus $\frac{K}{r_M(\phi)}$ is singular. Since K is projective, it follows that $r_M(\phi) \leq^e K$ by [11, Lemma 2.1].

(1) $\Rightarrow$ (3) Let $r_M(\phi) \leq^e (1-e)M$ where $e^2 = e \in S$. First we show that $\phi S = \phi eS \oplus \phi(1-e)S$. Clearly $\phi S = \phi eS + \phi(1-e)S$. If $x \in \phi eS \cap \phi(1-e)S$, then $x = \phi ef = \phi(1-e)g$ where $f, g \in S$. So, for all $m \in M$, $(ef - (1-e)g)m \in r_M(\phi) \subseteq (1-e)M$. Hence $efm = 0$ and so $efM = 0$. Thus $x = \phi ef = 0$. Therefore $\phi S = \phi eS \oplus \phi(1-e)S$. Now $\phi eS \cong eS$ because the multiplication map $\overline{\phi} : eS \to \phi eS$ has kernel $\{ef \mid \phi ef = 0\} = 0$. Hence ϕeS is projective. Finally, $\overline{\phi} : (1-e)M \to \phi(1-e)M$ has kernel $(1-e)M \cap r_M(\phi) = r_M(\phi)$. Hence $\frac{(1-e)M}{r_M(\phi)} \cong \phi(1-e)M \cong \frac{M}{r_M(\phi(1-e))}$, and so $\frac{M}{r_M(\phi(1-e))}$ is singular by [11, Lemma 2.1]. Thus $r_M(\phi(1-e)) \leq^e M$ by [11, Lemma 2.1] again. Therefore $\phi(1-e)S \subseteq \Delta(M)$.

(3) $\Rightarrow$ (1) Suppose that $\phi S = A \oplus B$ as in (3), and let $\pi : \phi S \to A$ be the projection with $\mathrm{Ker}\pi = B$. Then define $\gamma : S \to A$ by $\gamma(f) = \pi(\phi f)$, and write $K = \mathrm{Ker}(\gamma)$. Then γ is onto so, as A is projective, $K = eS$ for some $e^2 = e \in S$. Clearly, $r_S(\phi) \subseteq eS$. Since M generates $r_M(I)$ for every $I \leq S_S$, $r_S(\phi)M = r_M(\phi)$. Thus $r_M(\phi) \subseteq eSM = eM$. It remains to show that $r_M(\phi) \leq^e eM$. Since $e \in K$, then $\phi em \in BM$ because $\pi(\phi e) = \gamma(e) = 0$. Hence we have a map $\theta : eM \to BM$ defined by $\theta(em) = \phi em$. Then $\mathrm{Ker}(\theta) = eM \cap r_M(\phi) = r_M(\phi)$. So $\frac{eM}{r_M(\phi)} \cong \mathrm{Im}(\theta) \subseteq BM$. Thus $\frac{eM}{r_M(\phi)}$ is singular so, since eM is projective, it follows that $r_M(\phi) \leq^e eM$ by [11, Lemma 2.1]. $\square$

Let M and N be R-modules. We say that M is *N-$\mathcal{K}$-extending* if for every homomorphism $\phi : M \to N$, there exists $L \leq^{\oplus} M$ such that $\mathrm{Ker}\phi \leq^e M$. It is clear that a module M is $\mathcal{K}$-extending if and only if M is M-$\mathcal{K}$-extending.

Proposition 3.4. *The following conditions are equivalent for a module M:*

(1) *M is a $\mathcal{K}$-extending module;*

(2) *For any submodule N of M, every direct summand L of M is N-$\mathcal{K}$-extending;*

(3) *For every pair of summands L and N of M and any $\phi \in Hom_R(M, N)$, the kernel of the restricted map $\phi|_L$ is essential in a direct summand of L.*

Proof. (1) $\Rightarrow$ (2) Let $L = eM$ where $e^2 = e \in S$. Let $\psi : L \to N$ be any homomorphism and set $\phi = \psi e \in S$. Since M is $\mathscr{K}$-extending, there exists $f^2 = f \in S$ such that $\mathrm{Ker}\phi \leq^e fM$. Thus $\mathrm{Ker}\psi = \mathrm{Ker}\phi \cap L \leq^e fM \cap L$. It is enough to show that there exists $g^2 = g \in S$ such that $fM \cap L \leq^e gM$. Since $(1-e)M \subseteq \mathrm{Ker}(1-f)e$, $\mathrm{Ker}(1-f)e = eM \cap \mathrm{Ker}(1-f)e \oplus (1-e)M = eM \cap \mathrm{Ker}(1-f) \oplus (1-e)M$. There exists $h^2 = h \in S$ such that $\mathrm{Ker}(1-f)e \leq^e hM$ as M is a $\mathscr{K}$-extending module. Since $(1-e)M \subseteq hM$, it follows that $hM = hM \cap (eM \oplus (1-e)M) = hM \cap eM \oplus (1-e)M$. Hence $eM \cap \mathrm{Ker}(1-f) \oplus (1-e)M \leq^e hM \cap eM \oplus (1-e)M$ and so $eM \cap fM = eM \cap \mathrm{Ker}(1-f) \leq^e hM \cap eM$. Since $eM \cap hM$ is a direct summand of hM, $eM \cap hM$ is a direct summand of M.

(2) $\Rightarrow$ (3) is obvious to take that N is a direct summand of M.

(3) $\Rightarrow$ (1) is clear to see by taking $L = N = M$. □

Corollary 3.5. *Every direct summand of a $\mathscr{K}$-extending module is $\mathscr{K}$-extending.*

We recall that the module M is *$\mathscr{K}$-nonsingular* if, for all $\phi \in S$, $\mathrm{Ker}\phi \leq^e M$ implies $\phi = 0$.

Proposition 3.6. *The following conditions are equivalent for a $\mathscr{K}$-nonsingular module M:*

(1) *M is an indecomposable $\mathscr{K}$-extending module;*

(2) *Every nonzero endomorphism $\phi \in S$ is a monomorphism.*

Proof. (1) $\Rightarrow$ (2) Let M is an indecomposable $\mathscr{K}$-extending module. Assume that $0 \neq \phi \in S$. Then there exists $e^2 = e \in S$ such that $\mathrm{Ker}\phi \leq^e eM$. Since M is indecomposable, $e = 0$ or $e = 1$. If $e = 1$, then $\mathrm{Ker}\phi \leq^e M$. By $\mathscr{K}$-nonsingularity, $\phi = 0$, a contradiction. Thus $e = 0$ and so ϕ is a monomorphism.

(2) $\Rightarrow$ (1) is clear. □

A ring is called *I-finite* if it contains no infinite set of orthogonal idempotents.

Proposition 3.7. *Let M be a $\mathscr{K}$-extending module.*

(1) *For every $X \subseteq M$, if $\ell_S(X) \nsubseteq \Delta(M)$, then $\ell_S(X)$ contains a nonzero idempotent, where $\ell_S(X) = \{\phi \in S \mid \phi(X) = 0\}$.*

(2) *If S is I-finite, every left annihilator $\ell_S(X)$ with $X \subseteq M$, has the form $\ell_S(X) = Se \oplus T$ where $e^2 = e \in S$ and ${}_R T \subseteq \Delta(M)$.*

Proof. (1) Choose $\phi \in \ell_S(X)$, $\phi \notin \Delta(M)$. By hypothesis, $r_M(\phi) \leq^e eM$ where $e^2 = e \in S$ and $e \neq 1$ because $\phi \notin \Delta(M)$. Hence $X \subseteq r_M(\phi) \subseteq eM$, so $0 \neq (1-e) \in \ell_S(X)$.

(2) If $\ell_S(X) \subseteq \Delta(M)$, then take $e = 0$ and $T = \ell_S(X)$. Otherwise use (1) and the I-finite hypothesis to choose e maximal in $\{e \in S \mid 0 \neq e^2 = e \in \ell_S(X)\}$, where $e \leq f$ means $e \in fSf$. Then

$\ell_S(X) = Se \oplus [\ell_S(X) \cap S(1-e)]$ so it suffices to show that $\ell_S(X) \cap S(1-e) \subseteq \Delta(M)$. If not, let $0 \neq f^2 = f \in \ell_S(X) \cap S(1-e)$ by (1). Then $fe = 0$ so $g = e + f - ef$ satisfies $g^2 = g \in \ell_S(X)$ and $e \leq g$. Thus $g = e$ by the choice of e, and so $f = ef$ and $f = f^2 = f(ef) = 0$, a contradiction. $\square$

According to [12], M is called a *Baer* module if for every left ideal I of S, $\cap_{\phi \in I} \mathrm{Ker}\phi$ is a direct summand of M.

Corollary 3.8. *Let S be I-finite and $\Delta(M) = 0$, then M is a Baer module if and only if M is a $\mathcal{K}$-extending module.*

An R-module M has (C_2) if any submodule of M isomorphic to a summand of M is itself a summand. A ring R is called a *right C_2-ring* if R_R has (C_2).

Lemma 3.9. *If M has (C_2), then $\Delta(M) \subseteq Jac(S)$.*

Proof. Let $\phi \in \Delta(M)$. Since $r_M(\phi) \cap r_M(1-\phi) = 0$, we have $r_M(1-\phi) = 0$ and so $\mathrm{Im}(1-\phi) \cong M$. Hence $\mathrm{Im}(1-\phi)$ is a direct summand of M by hypothesis. $\mathrm{Im}(1-\phi)$ is also essential in M because, for every $m \in r_M(\phi)$, we have $(1-\phi)m = m$ and so $r_M(\phi) \subseteq \mathrm{Im}(1-f)$. Therefore $\mathrm{Im}(1-\phi) = M$. Since this holds for every $\phi \in \Delta(M)$, we have $\Delta(M) \subseteq Jac(S)$. $\square$

An R-module M is called *continuous* if M is extending and has $(C2)$ [9]. A ring R is called *right continuous* if R_R is a continuous module. In [11], Nicholson and Yousif introduced the notion of weakly continuous rings. A ring R is called *right weakly continuous* if R is a right ACS-ring which is also a right C_2-ring. Motivated by this concept, we define a weakly continuous module as follows:

Definition 3.10. We call a module M *weakly continuous* if M is a $\mathcal{K}$-extending module and has (C_2).

Clearly, a ring R is a right weakly continuous ring if R_R is a weakly continuous module.

Example 3.11.

(1) It is clear that every continuous module is weakly continuous.

(2) The $\mathbb{Z}$-module $\mathbb{Z}$ is a noetherian $\mathcal{K}$-extending module which is not weakly continuous.

(3) Given a field F and an isomorphism $a \mapsto \bar{a}$ from $F \to \bar{F} \subseteq F$, let $M = R$ be the left F-space on basis $\{1, t\}$ with multiplication given by $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. Then M_R is a weakly continuous module, but M_R is not a continuous module if $dim_{\bar{F}}(F) \geq 2$ (see [11, Example 2.5] or [3, Page 70]).

Theorem 3.12. *Let M be a projective self-generator module. Then the following are equivalent:*

(1) *M is $\Delta(M)M$-$\mathcal{I}$-lifting.*

(2) *M is $Z(M)$-$\mathcal{I}$-lifting.*

(3) *If I is a finitely generated right ideal of S, then $IM = eM \oplus N$ where $e^2 = e \in S$ and N is a singular submodule of M.*

(4) *M is a $\mathcal{K}$-extending module and for every finitely generated right ideal I of S such that IM is projective, we have $IM \leq^{\oplus} M$.*

(5) *If I is a finitely generated right ideal of S, then $I = eS \oplus B$ where $e^2 = e \in S$ and $B \subseteq \Delta(M)$ is a right ideal of S.*

(6) *M is a $\mathcal{K}$-extending module and every finitely generated projective right ideal of S is a summand.*

(7) *M is a $\mathcal{K}$-extending module and S is a right C_2-ring.*

(8) *S is semiregular and $\Delta(M) = Jac(S)$.*

(9) *M is weakly continuous.*

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) By Theorem 2.3.

(3) $\Rightarrow$ (4) If $\phi \in S$, taking $I = \phi S$ in (3) shows that M is a $\mathcal{K}$-extending module by Proposition 3.3. If I is a finitely generated right ideal of S with IM projective, write $IM = eM \oplus N$ as in (3). Then N_R is both singular and projective, so $N = 0$ by [11, Lemma 2.1].

(4) $\Rightarrow$ (2) Let $\phi \in S$. Then $\phi M = P \oplus N$ where P_R is projective and N_R is a singular submodule. Consider the homomorphism $\phi : M \to \phi M$. Since M is projective, there exists a homomorphism $h : M \to M$ such that $\phi h = \iota\pi\phi$, where $\iota : P \to \phi M$ and $\pi : \phi M \to P$ are injection and projection maps. Hence $\phi h(M) = \pi\phi(M) = P$. Take $I = \phi hS$ in (4), so P is a direct summand of M.

(2) $\Rightarrow$ (5) By Corollary 2.5, S is $\Delta(M)$-semiregular and, by [11, Theorem 1.2], we have (5).

(5) $\Rightarrow$ (6) If $\phi \in S$, taking $I = \phi S$ in (5) shows that M is a $\mathcal{K}$-extending module by Proposition 3.3. If I is a finitely generated projective right ideal of S, write $I = eS \oplus B$, where $e^2 = e \in S$ and $B \subseteq \Delta(M)$. By [6, Proposition 2.4], $\Delta(M) = Z_r(S)$ and so $B \subseteq Z_r(S)$. Thus B_S is both singular and projective, so $B = 0$ by [11, Lemma 2.1].

(6) $\Rightarrow$ (7) To verify the right C_2-condition, let I be a right ideal of S which is isomorphic to a summand of S. Then I is projective and principal, so I is a summand by (6), as required.

(7) $\Rightarrow$ (8) By Proposition 3.3, for every $\phi \in S$, we have $\phi S = A \oplus B$ where A_S is a projective right ideal of S and B_S is a right ideal of S with $B \subseteq \Delta(M)$. By Lemma 3.9, $\Delta(M) \subseteq Jac(S)$, and so $B \subseteq Jac(S)$. Since A_S is projective, A_S is isomorphic to a summand of S. Hence $A = eS$ where

$e^2 = e \in S$ by the C_2-condition. Therefore S is semiregular. Finally, if $\phi \in Jac(S)$, then $e^2 = e \in Jac(S)$ and so $e = 0$ and $\phi \in \phi S = B \subseteq \Delta(M)$, proving that $Jac(S) \subseteq \Delta(M)$. Thus $Jac(S) = \Delta(M)$.

(8) $\Rightarrow$ (1) By Corollary 2.5.

(7) $\Rightarrow$ (9) By [11, Theorem 3.9].

(9) $\Rightarrow$ (2) Let $\phi \in S$. Since M is a $\mathcal{K}$-extending module, we have $\phi M = P \oplus N$ where P is projective and N is singular. Thus P is isomorphic to a summand of M and so the C_2-condition implies that $P = eM$ where $e^2 = e \in S$. □

Corollary 3.13 (see [11, Theorem 2.4])**.** *The following are equivalent for a ring R:*

(1) *R is semiregular and $J = Z_r$.*

(2) *R is right Z_r-semiregular.*

(3) *If T is a finitely generated right ideal, then $T = eR \oplus B$ where $e^2 = e \in R$ and B is a singular right ideal.*

(4) *R is a right ACS-ring and every finitely generated projective right ideal is a summand.*

(5) *R is a right ACS-ring which is also a right C_2-ring.*

Corollary 3.14. *Let M be a projective self-generator module. Then the following are equivalent:*

(1) *M is quasi-injective.*

(2) *M has (C_2) and $M \oplus M$ is extending.*

(3) *M is $Z(M)$-$\mathcal{I}$-lifting and $M \oplus M$ is extending.*

(4) *M is weakly continuous and $M \oplus M$ is extending.*

Proof. (1) $\Rightarrow$ (2) By [9, Proposition 1.18].

(2) $\Rightarrow$ (3) If $M \oplus M$ is extending, then M is extending. By Theorem 3.12, M is $Z(M)$-$\mathcal{I}$-lifting.

(3) $\Rightarrow$ (4) By Theorem 3.12.

(4) $\Rightarrow$ (1) If M is weakly continuous, then S is semiregular by Theorem 3.12. Since semiregularity is a Morita invariant property by [10, Corollary 2.8], the matrix ring $M_2(S) \cong End_R(M \oplus M)$ is semiregular. In particular $End_R(M \oplus M)$ has the right C_2-condition, and so $M \oplus M$ has (C_2) by [11, Theorem 3.9]. Hence $M \oplus M$ is continuous and, by [9, Theorem 3.16], M is quasi-injective. □

A module M is called *finitely lifting*, or *f-lifting* for short, if for every finitely generated submodule A of M, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$.

Corollary 3.15. *Let M be a projective self-generator module. Then M is weakly continuous if and only if S_S is f-lifting and $\Delta(M) = Jac(S)$.*

Proof. It is clear by Theorem 3.12. □

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