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A REMARK ON THE NUMBER OF DISTINCT PRIME DIVISORS OF INTEGERS

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Abstract. We study the asymptotic formula for the sum $\sum_{n\leqslant x}\omega(n)$ where $\omega(n)$ denotes the number of distinct prime divisors of n, and we perform some computations which detect curve patterns in the distribution of a related sequence.

1. Introduction

Let $\omega(n)=\sum_{p|n}1$ be the number of distinct prime divisors of the positive integer n. In 1917, Hardy and Ramanujan [2] proved the following average result

$$\sum_{n\leqslant x}\omega(n)=x\log\log x+Mx+O\Big(\frac{x}{\log x}\Big), \tag{1}$$

where M is known as the Meissel–Mertens constant [1] and defined by

$$M=\gamma+\sum_{p}\left(\log\left(1-p^{-1}\right)+p^{-1}\right)\approxeq 0.2614972128476427837554268386 09,$$

and γ refers to Euler's constant. In the present note we improve the error term in (1), by evaluating the exact value of O-term. Indeed we prove the following.

Theorem 1. *As $x\to\infty$ one has*

$$\sum_{n\leqslant x}\omega(n)=x\log\log x+Mx-(1-\gamma)\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right). \tag{2}$$

Remark 2. To clear more details of the asymptotic relation (2) we consider the function

$$\mathscr{R}(x)=\frac{\log x}{x}\left(\sum_{n\leqslant x}\omega(n)-x\log\log x-Mx\right). \tag{3}$$

While the truth of Theorem 1 implies that $\mathscr{R}(x)\to\gamma-1$ as $x\to\infty$, we perform computations which detect curve patterns in the distribution of the points $(n,\mathscr{R}(n))$. For integers n with $2\leqslant n\leqslant 82$ we have $\mathscr{R}(n)>1-\gamma$. There are some partial curve patters, as pictured in Figures 1 and 2, waiting for a mathematical justification. It seems that for $n\geqslant 343$ one has $\mathscr{R}(n)<1-\gamma$, and the minimum value of $\mathscr{R}(n)$ occurs at $n=1879$.

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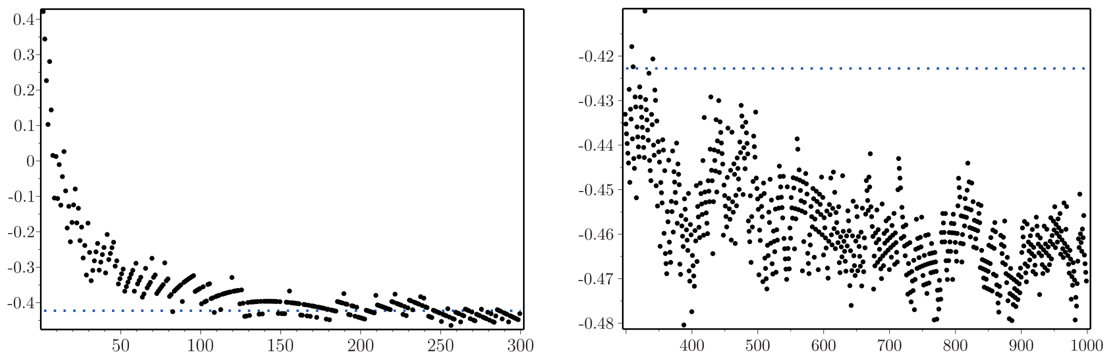

Figure 1: Graphs of the points $(n, \mathscr{R}(n))$ for $2 \leqslant n \leqslant 1000$, and the dotted line $y = \gamma - 1$.

Proof of Theorem 1. Let us denote integer part and fractional part of the real number x by $[x]$ and $\{x\}$, respectively. Also, let

$$\mathscr{A}(x) = \sum_{p \leqslant x} \frac{1}{p}, \quad \text{and} \quad \mathscr{F}(x) = \sum_{p \leqslant x} \left\{ \frac{x}{p} \right\}.$$

We have

$$\sum_{n \leqslant x} \omega(n) = \sum_{n \leqslant x} \sum_{p \mid n} 1 = \sum_{p \leqslant x} \sum_{\substack{n \leqslant x \\ p \mid n}} 1 = \sum_{p \leqslant x} \left[\frac{x}{p} \right] = \sum_{p \leqslant x} \left(\frac{x}{p} - \left\{ \frac{x}{p} \right\} \right),$$

and hence

$$\sum_{n \leqslant x} \omega(n) = x \mathscr{A}(x) - \mathscr{F}(x). \tag{4}$$

The sum $\mathscr{F}(x)$ on the fractional parts has been studied by de la Vallée Poussin [5], where he showed by elementary methods that $\mathscr{F}(x) \sim (1-\gamma)\frac{x}{\log x}$ as $x \to \infty$. More precisely, by using Perron's formula, Lee [3] proved that

$$\sum_{p^\alpha \leqslant x} \left\{ \frac{x}{p^\alpha} \right\} = (1-\gamma) \frac{x}{\log x} + O\left(\frac{x}{\log^2 x} \right).$$

The difference of the later sum by $\mathscr{F}(x)$ is not large, because

$$\sum_{p^\alpha \leqslant x} \left\{ \frac{x}{p^\alpha} \right\} - \mathscr{F}(x) = \sum_{\substack{p^\alpha \leqslant x \\ \alpha \geqslant 2}} \left\{ \frac{x}{p^\alpha} \right\} \ll \sum_{\substack{p^\alpha \leqslant x \\ \alpha \geqslant 2}} 1 \ll \sqrt{x} \log^2 x.$$

Thus, we get

$$\mathscr{F}(x) = (1-\gamma) \frac{x}{\log x} + O\left(\frac{x}{\log^2 x} \right). \tag{5}$$

We recall that Theorem 5 of [4] implies

$$\mathscr{A}(x) := \sum_{p \leqslant x} \frac{1}{p} = \log\log x + M + O\left(\frac{1}{\log^2 x} \right). \tag{6}$$

Finally, we combine (4) with (5) and (6) to deduce (2). $\square$

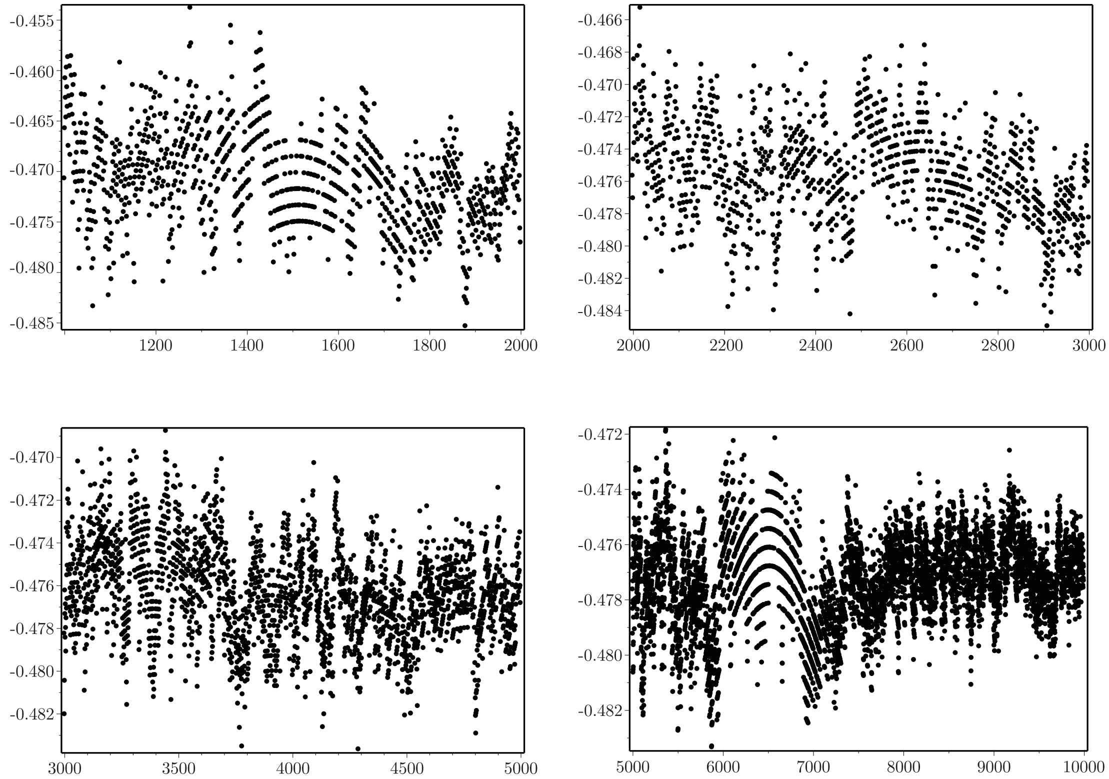

Figure 2: Graphs of the points $(n, \mathscr{R}(n))$ for $10^3 \leqslant n \leqslant 10^4$.

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