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A SIMPLE PROOF OF THE DISCRETE STEFFENSEN'S INEQUALITY

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It was first given in [1] and then cited repeatedly in [2]-[5] the following discrete Steffensen's inequality:

Theorem. *Let* $(x_i)_{i=1}^n$ *be a nonincreasing finite sequence of nonnegative real numbers, and let* $(y_i)_{i=1}^n$ *be a finite sequence of real numbers such that for every* i, $0 \le y_i \le 1$. *Let* $k_1, k_2 \in \{1, ..., n\}$ *be such that* $k_2 \le \sum_{i=1}^n y_i \le k_1$. *Then*

$$\sum_{i=n-k_2+1}^{n} x_i \le \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{k_1} x_i. \tag{1}$$

It should be noted that the proof of this theorem in [1] *states as an application of a generalized Steffensen's inequality over a general measure space to the discret case which seemed as if it is not quite clear. The purpose of this note is to give a new proof which is very simple and clear.*

Proof. The proof of the second inequality in (1) goes as follows.

$$\begin{aligned}
\sum_{i=1}^{k_1} x_i - \sum_{i=1}^{n} x_i y_i &= \sum_{i=1}^{k_1}(1-y_i)x_i - \sum_{i=k_1+1}^{n} x_i y_i \\
&\ge x_{k_1}\sum_{i=1}^{k_1}(1-y_i) - \sum_{i=k_1+1}^{n} x_i y_i \\
&= x_{k_1}(k_1 - \sum_{i=1}^{k_1} y_i) - \sum_{i=k_1+1}^{n} x_i y_i \\
&\ge x_{k_1}(\sum_{i=1}^{n} y_i - \sum_{i=1}^{k_1} y_i) - \sum_{i=k_1+1}^{n} x_i y_i \\
&= x_{k_1}\sum_{i=k_1+1}^{n} y_i - \sum_{i=k_1+1}^{n} x_i y_i
\end{aligned}$$

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$$= \sum_{i=k_1+1}^{n} (x_{k_1} - x_i) y_i \geq 0.$$

Similarly, for the first inequality in (1):

$$\begin{aligned}
\sum_{i=n-k_2+1}^{n} x_i - \sum_{i=1}^{n} x_i y_i &= \sum_{i=n-k_2+1}^{n} (1-y_i) x_i - \sum_{i=1}^{n-k_2} x_i y_i \\
&\leq x_{n-k_2+1} \sum_{i=n-k_2+1}^{n} (1-y_i) - \sum_{i=1}^{n-k_2} x_i y_i \\
&= x_{n-k_2+1} (k_2 - \sum_{i=n-k_2+1}^{n} y_i) - \sum_{i=1}^{n-k_2} x_i y_i \\
&\leq x_{n-k_2+1} (\sum_{i=1}^{n} y_i - \sum_{i=n-k_2+1}^{n} y_i) - \sum_{i=1}^{n-k_2} x_i y_i \\
&= x_{n-k_2+1} \sum_{i=1}^{n-k_2} y_i - \sum_{i=1}^{n-k_2} x_i y_i \\
&= \sum_{i=1}^{n-k_2} (x_{n-k_2+1} - x_i) y_i \leq 0.
\end{aligned}$$

Remarks. (i) It is not difficult to find that the first inequality in (1) can also follows directly from the second one upon consideration of $(1-y_i)_{i=1}^n$. Since for every i, $0 \leq 1-y_i \leq 1$ and $n-k_1, n-k_2 \in \{1, \ldots, n\}$ be such that $n-k_1 \leq \sum_{i=1}^n (1-y_i) \leq n-k_2$ imply that $\sum_{i=1}^n x_i(1-y_i) \leq \sum_{i=1}^{n-k_2} x_i$ and consequently $\sum_{i=1}^n x_i y_i \geq \sum_{i=n-k_2+1}^n x_i$.

(ii) In fact, the Theorem can be restated in a more general version as follows: Let $(y_i)_{i=1}^n$ be a finite sequences of real numbers such that for every i, $0 \leq y_i \leq 1$. Then $k_2 \leq \sum_{i=1}^n y_i \leq k_1$ is a necessary and sufficient condition for inequalities (1) to hold for all nonincreasing finite sequences $(x_i)_{i=1}^n$ of nonnegative real numbers, where $k_1, k_2 \in \{1, \ldots, n\}$.

References

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