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UNIQUENESS OF DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF ENTIRE FUNCTIONS SHARING ONE VALUE

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Abstract. In this paper, we study the uniqueness of difference-differential polynomials of entire functions f and g sharing one value with counting multiplicity. In this paper we extend and generalize the results of X. Y. Zhang, J. F. Chen and W. C. Lin [17], L. Kai, L. Xin-ling and C. Ting-bin [7] and many others [2, 16].

1. Introduction and main results

In this paper, the term 'meromorphic' will always mean meromorphic in the whole complex plane $\overline{\mathbb{C}}$. It is assumed that the reader is familiar with standard notations and fundamental results of Nevanlinna theory [6], [13] and [15]. We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \longrightarrow +\infty$, possibly outside of a set of finite linear measure.

For $a \in \overline{\mathbb{C}}$ and k be a positive integer, we denote by $N_{(k}(r,a,f)$ be the counting function for the zeros of $f(z)-a$ with multiplicity $\geq k$, and $\overline{N}_{(k}(r,a,f)$ be the corresponding one for which the multiplicity is not counted. In this paper, we denote by

$$N_k(r,a,f) = \overline{N}_{(1}(r,a,f) + \overline{N}_{(2}(r,a,f) + \ldots + \overline{N}_{(k}(r,a,f)$$

Let $f(z)$ and $g(z)$ be two meromorphic functions. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value $'a'$ CM, where $'a'$ is a complex number.

In 1993, Wang and Fang [11, 12] proved the following theorem for transcendental entire functions.

Theorem A. *Let $f(z)$ be a transcendental entire function. n and k be two positive integers with $n \geq k+1$, then $[f^n]^{(k)} - 1$ has infinitely many zeros.*

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In 2002, M. L. Fang [3] proved the unicity theorem corresponding to the above result.

Theorem B. *Let f and g be two non-constant entire functions, and let $n \geq 11$ be a positive integer with $n > 2k+4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

In 2008, X. Y. Zhang, J. F. Chen and W. C. Lin [17] proved the following results on uniqueness of two polynomials sharing a common value.

Theorem C. *Let f be a transcendental entire function, let n, k and m be positive integers with $n \geq k+2$, and $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m$, where $a_0, a_1, a_2, \ldots, a_m$ are complex constants. Then $[f^n P(f)]^{(k)} = 1$ has infinitely many solutions.*

Theorem D. *Let f and g be two non-constant entire functions. Let n, k and m be three positive integers with $n \geq 3m + 2k + 5$, and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$ or $P(z) \equiv c_0$, where $a_0(\neq 0), a_1, a_2, a_3, \ldots a_{m-1}, a_m(\neq 0)$, $c_0(\neq 0)$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share 1 CM, then*

(1) *when $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$, either $f(z) \equiv t g(z)$ for a constant t such that $t^d = 1$, where $d = (n+m, \ldots. n+m-i, \ldots, n)$, $a_{m-i}(\neq 0)$ for some $i = 0, 1, 2, \ldots m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \ldots + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \ldots + a_0)$*
(2) *when $P(z) \equiv c_0$, either $f(z) = \frac{c_1}{\sqrt[n]{c_0} e^{cz}}$, $g(z) = \frac{c_2}{\sqrt[n]{c_0} e^{-cz}}$, where c_1, c_2 and c are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

In 2012, L. Kai, L. Xin-ling, C. Ting-bin [7] considered Theorem B for difference-differential polynomials and proved the following results.

Theorem E. *Let $f(z)$ be a transcendental entire function of finite order. If $n \geq k+2$, then the difference-differential polynomial $[f^n(z) f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem F. *Let f and g be transcendental entire functions of finite order, $n \geq 2k+6$ and c is a non-zero complex constant. If $[f^n(z) f(z+c)]^{(k)}$ and $[g^n(z) g(z+c)]^{(k)}$ share the value 1CM, then either $f(z) = c_1 e^{Cz}$, $g(z) = c_2 e^{-Cz}$, where c_1, c_2 and C are constants satisfying $(-1)^k(c_1c_2)^{n+1}((n+1)C)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In the same direction J. Zhang [16] investigated the value distribution and uniqueness of difference polynomials of entire functions and obtained the following results.

Theorem G. *Let $f(z)$ be a transcendental entire function of finite order, and $\alpha(z)$ be a small function with respect to $f(z)$. Suppose that c is a non-zero complex constant. If $n \geq 2$, then $f^n(z)(f(z)-1) f(z+c) - \alpha(z)$ has infinitely many zeros.*

Theorem H. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero constant and n is an integer. If $n \geq 7$, then $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.*

Recently, R. S. Dyavanal and R. V. Desai [2] extended the results of J. Zhang[16] and proved the following results.

Theorem I. *Let $f(z)$ be a transcendental entire function of finite order, and $\alpha(z)$ be a small function with respect to $f(z)$. Suppose that c is a non-zero complex constant and n is an integer. If $n \geq 2$, $k_1 \geq 1$ then $f^n(z)(f(z)-1)^{k_1} f(z+c)-\alpha(z)$ has infinitely many zeros.*

Theorem J. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant, $k_1 \geq 1$, $n \geq k_1+6$. If $f^n(z)(f(z)-1)^{k_1} f(z+c)$ and $g^n(z)(g(z)-1)^{k_1} g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv t\, g(z)$, where $t^{k_1}=1$.*

In this paper, we consider Theorem C and Theorem D to difference-differential polynomials and extends the above theorems as follows.

Theorem 1.1. *Let f be a transcendental entire function. n, k and m be positive integers with $n \geq k+2$ and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$, where $a_0, a_1, a_2, a_3, \ldots a_{m-1}, a_m$ are complex constants and $\alpha(z)$ be a small function with respect to $f(z)$. Then $[f^n(z)P(f)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Remark 1.1. If $P(f)=1$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem E.

Remark 1.2. If $P(f)=(f-1)$ and $k=0$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem G.

Remark 1.3. If $P(f)=(f-1)^{k_1}$ and $k=0$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem I.

The unicity theorem corresponding to Theorem 1.1 is as follows.

Theorem 1.2. *Let f and g be two non-constant entire functions of finite order. Let n, k and m be three positive integers with $n \geq m+2k+6$, $'c'$ is a non-zero complex constant and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$ or $P(z) \equiv c_0$, where $a_0(\neq 0), a_1, a_2, a_3, \ldots, a_{m-1}$, $a_m(\neq 0), c_0(\neq 0)$ are complex constants. If $[f^n(z)P(f)f(z+c)]^{(k)}$ and $[g^n(z)P(g)g(z+c)]^{(k)}$ share 1 CM, then*

(1) *when $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$, we get $f(z) \equiv t g(z)$ for a constant t such that $t^d = 1$, where $d = GCD\{n+m+1, n+m, \ldots, n+m+1-i, \ldots, n+1\}$ and $i = 0, 1, 2, \ldots, m$.*

(2) *when* $P(z)\equiv c_0$ *either* $f(z)=\frac{c_1e^{Cz}}{\sqrt[n]{c_0}}$, $g(z)=\frac{c_2e^{-Cz}}{\sqrt[n]{c_0}}$, *where* c_1,c_2,c_0 *and* C *are constants satisfying* $(-1)^k(c_1c_2)^{n+1}((n+1)C)^{2k}=(\sqrt[n]{c_0})^2$, *or* $f\equiv tg$ *for a constant* t *such that* $t^{n+1}=1$.

Remark 1.4. If $P(f)=1$ in Theorem 1.2, then Theorem 1.2 reduces to Theorem F.

Remark 1.5. If $P(f)=(f-1)$ and $k=0$ in Theorem 1.2, then Theorem 1.2 reduces to Theorem H, when $\alpha(z)=1$.

Remark 1.6. If $P(f)=(f-1)^{k_1}$ and $k=0$ in Theorem 1.2, then Theorem 1.2 reduces to Theorem J, when $\alpha(z)=1$.

2. Some lemmas

For the proof of our main results, we need the following lemmas.

Lemma 2.1 ([1]). *Let* $f(z)$ *be a trancendental meromorphic function of finite order, then*

$$T(r,f(z+c))=T(r,f)+S(r,f)$$

Lemma 2.2 ([15]). *Let* $f(z)$ *be a non-constant meromorphic function, and* $a_n(\neq 0), a_{n-1}, \ldots, a_0$ *be small functions with respect to* f. *Then*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0)=nT(r,f)+S(r,f)$$

Lemma 2.3 ([5]). *Let* f *be a transcendental meromorphic function of finite order. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=S(r,f)$$

Lemma 2.4 ([6, 13]). *Let* $f(z)$ *be a non-constant meromorphic function and* $a_1(z),a_2(z)$ *be two meromorphic functions such that* $T(r,a_i)=S(r,f),\ i=1,2$. *Then*

$$T(r,f)\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)+S(r,f)$$

Lemma 2.5 ([15]). *Let* $f(z)$ *and* $g(z)$ *be two trancendental entire functions, and* k *be a positive integer. Then*

$$T(r,f^{(k)})\le T(r,f)+k\overline{N}(r,f)+S(r,f)$$

Lemma 2.6 ([1],[4]). *Let* $f(z)$ *be a meromorphic function of finite order and* c *is a non-zero complex constant. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=S(r,f)$$

Lemma 2.7 (Lemma 3 in [14]). *Let F and G be non-constant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds*

(1) $max\{T(r,F),T(r,G)\} \le N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)+S(r,F)+S(r,G),$

(2) $F \equiv G,$

(3) $FG \equiv 1.$

Lemma 2.8 ([8], Lemma 2.3). *Let $f(z)$ be a non-constant meromorphic function and p, k be positive integers. Then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \le T(r,f^{(k)})-T(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f) \tag{1}$$

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \le k\overline{N}(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f) \tag{2}$$

Lemma 2.9. *Let $f(z)$ be a transcendental entire function of finite order and let $F^* = f(z)^n P(f) f(z+c)$. Then*

$$T(r,F^*) = (n+m+1)T(r,f)+S(r,f)$$

Proof. Since f is a transcendental entire function and also from Lemma 2.2, Lemma 2.3, Lemma 2.6, we obtain

$$\begin{aligned}(n+m+1)T(r,f)+S(r,f) &= T(r,f(z)^{n+1}P(f)) \le m(r,f(z)^{n+1}P(f))+S(r,f)\\ &\le m\left(r,\frac{f(z)F^*}{f(z+c)}\right)+S(r,f)\\ &\le m(r,F^*)+S(r,f)\\ &\le T(r,F^*)+S(r,f)\end{aligned}$$

On the other hand, using Lemma 2.1 and f is a transcendental entire function of finite order, we have

$$\begin{aligned}T(r,F^*) &\le nT(r,f)+mT(r,f)+T(r,f(z+c))+S(r,f)\\ &\le (n+m+1)T(r,f)+S(r,f)\end{aligned}$$

Hence we get Lemma 2.9.

Lemma 2.10. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, let n, k be two positive integers with $n>k$, $'c'$ is a non-zero complex constant and let $P(z) = a_m z^m + a_{m-1}z^{m-1}+\ldots+a_1 z + a_0$ be a non-zero polynomial, where $a_0, a_1, a_2, \ldots a_{m-1}, a_m$ are complex constants. If $[f^n P(f) f(z+c)]^{(k)}[g^n P(g) g(z+c)]^{(k)} \equiv 1$, then $P(z)$ is reduced to a non-zero monomial, that is $P(z) = a_i z^i \ne 0$ for some $i = 0,1,2,\ldots,m$.*

Proof. If $P(z)$ is not reduced to a non-zero monomial, then without loss of generality, we may assume that

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$$

where $a_0(\neq 0), a_1, a_2, \ldots a_{m-1}, a_m(\neq 0)$ are complex constants. Since

$$\left[f^n(a_m f^m + \ldots + a_0) f(z+c)\right]^{(k)} \left[g^n(a_m g^m + \ldots + a_0) g(z+c)\right]^{(k)} \equiv 1 \tag{2.1}$$

From $n > k$ and the assumption that $f(z)$ and $g(z)$ are two non-constant entire functions we deduce by (2.1) that

$$f(z) \neq 0 \ , \ g(z) \neq 0. \tag{2.2}$$

Let $f(z) = e^{\alpha(z)}$, where $\alpha(z)$ is a non-constant entire function. Thus, by induction we get

$$\left[a_i f^{i+n} f(z+c)\right]^{(k)} = P_i\left(\alpha', \ldots \alpha^{(k)}, \alpha'(z+c), \ldots \alpha^{(k)}(z+c)\right) e^{(i+n)\alpha} e^{\alpha(z+c)}. \tag{2.3}$$

Where $P_i\left(\alpha', \alpha'', \ldots \alpha^{(k)}, \alpha'(z+c), \alpha''(z+c), \ldots \alpha^{(k)}(z+c)\right)$ $(i = 0, 1, \ldots m)$ are difference-differential polynomials.
Obviously,

$$P_m\left(\alpha', \ldots \alpha^{(k)}, \alpha'(z+c), \ldots \alpha^{(k)}(z+c)\right) \not\equiv 0$$
$$\vdots$$
$$P_0\left(\alpha', \ldots \alpha^{(k)}, \alpha'(z+c), \ldots \alpha^{(k)}(z+c)\right) \not\equiv 0$$

Where if $a_i \neq 0$ for some $i = 1, 2 \ldots m-1$, then $P_i\left(\alpha' \ldots \alpha^{(k)}, \alpha'(z+c), \ldots \alpha^{(k)}(z+c)\right) \not\equiv 0$
Since $g(z)$ is an entire function, we get from (2.1) that $\left[f^n(a_m f^m + \ldots + a_0) f(z+c)\right]^{(k)} \neq 0$.
Thus, by (2.3) we have

$$\begin{aligned} &P_m\left(\alpha', \alpha'', \ldots \alpha^{(k)}, \alpha'(z+c), \alpha''(z+c), \ldots \alpha^{(k)}(z+c)\right) e^{m\alpha} + \ldots \\ &\quad + \ldots P_0\left(\alpha', \alpha'', \ldots \alpha^{(k)}, \alpha'(z+c), \alpha''(z+c), \ldots \alpha^{(k)}(z+c)\right) \neq 0 \end{aligned} \tag{2.4}$$

Since $\alpha(z)$ and $\alpha(z+c)$ is an entire function, we obtain

$$\begin{aligned} T(r, \alpha^{(j)}) \leq T(r, \alpha') + S(r, f) &= m(r, \alpha') + S(r, f) \\ &= m\left(r, \frac{(e^{\alpha})'}{e^{\alpha}}\right) + S(r, f) \end{aligned}$$

Similarly, we obtain

$$T(r, \alpha^{(j)}(z+c)) \leq T(r, \alpha'(z+c)) + S(r, f) = m(r, \alpha'(z+c)) + S(r, f)$$

$$= m\left(r, \frac{(e^{\alpha(z+c)})'}{e^{\alpha(z+c)}}\right) + S(r,f)$$

for $j = 1,2,\ldots,k$. Hence, we deduce that

$$T(r,P_m) = S(r,f),\ldots,T(r,P_0) = S(r,f) \tag{2.5}$$

Note that $f = e^{\alpha(z)}$. Thus, by (2.4), (2.5) above, and Lemma 2.2 and Lemma 2.4, we get

$$\begin{aligned} mT(r,f) &= T\left(r, P_m e^{m\alpha} + \ldots + P_1 e^{\alpha}\right) + S(r,f) \\ &\le \overline{N}\left(r, \frac{1}{P_m e^{m\alpha} + \ldots + P_1 e^{\alpha}}\right) + \overline{N}\left(r, \frac{1}{P_m e^{m\alpha} + \ldots + P_1 e^{\alpha} + P_0}\right) + S(r,f) \\ &\le \overline{N}\left(r, \frac{1}{P_m e^{(m-1)\alpha} + \ldots + P_2 e^{\alpha} + P_1}\right) + S(r,f) \\ &\le (m-1)T(r,f) + S(r,f) \end{aligned}$$

which is a contradiction. This shows that $P(z)$ is reduced to a non-zero monomial, that is, $P(z) = a_i z^i \neq 0$ for some $i = 0,1,2,\ldots m$. This completes the proof of the Lemma 2.10.

3. Proof of Theorem 1.1

Denote $F(z) = [f(z)^n P(f) f(z+c)]^{(k)}$ and $F^* = f(z)^n P(f) f(z+c)$. From Lemma 2.9, F^* is not a constant. Assume that $F(z) - \alpha(z)$ has only finitely many zeros, then from the second fundamental theorem for three values and (1) of Lemma 2.8, we get

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}(r,0,F) + \overline{N}(r,0,F-\alpha(z)) + S(r,F) \\ &\le N_1(r,0,F) + \overline{N}(r,0,F-\alpha(z)) + S(r,F) \\ &\le T(r,F) - T(r,F^*) + N_{k+1}(r,0,F^*) + S(r,F^*) + S(r,F) \end{aligned}$$

$$T(r,F^*) \le N_{k+1}(r,0,F^*) + S(r,f). \tag{3.1}$$

From Lemma 2.9 and (3.1), it implies that

$$\begin{aligned} (n+m+1)T(r,f) + S(r,f) &= T(r,F^*) \le N_{k+1}(r,0,F^*) + S(r,f) \\ &\le (k+1)\overline{N}(r,0,f) + N(r,0,P(f)) + N(r,0,f(z+c)) + S(r,f) \\ &\le (k+m+2)T(r,f) + S(r,f). \end{aligned}$$

Which is contradiction to $n \ge k+2$. Hence $\left[f^n(z)(a_m f^m + \ldots + a_0) f(z+c)\right]^{(k)} - \alpha(z)$ has infinitely many zeros.

Proof of theorem 1.2.

(1) If $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$.

Then by assumption and Theorem 1.1 we know that either both f and g are transcendental entire functions or both f and g are polynomials.
First, we consider the case when f and g are transcendental entire functions.
Considering $F=[f^n(z)P(f)f(z+c)]^{(k)}$, $G=[g^n(z)P(g)g(z+c)]^{(k)}$. Since F and G share 1 CM, let us assume (1) of Lemma 2.7 holds. That is

$$max\{T(r,F),T(r,G)\} \le N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)+S(r,F)+S(r,G) \tag{3.2}$$

Since f is an entire function and from Lemma 2.5, Lemma 2.9 we have $S(r,F)=S(r,f)$. We also have $S(r,G)=S(r,g)$.
From (1) of Lemma 2.8, we obtain

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &= N_2\left(r,\frac{1}{[f^n(z)P(f)f(z+c)]^{(k)}}\right)\\ &\le T(r,F)-T\left(r,f^n(z)P(f)f(z+c)\right)+N_{k+2}\left(r,\frac{1}{f^n(z)P(f)f(z+c)}\right)+S(r,f). \end{aligned} \tag{3.3}$$

From Lemma 2.9 and (3.3), we get

$$\begin{aligned} (n+m+1)T(r,f) &= T\left(r,f^n(z)P(f)f(z+c)\right)+S(r,f)\\ &\le T(r,F)-N_2\left(r,\frac{1}{F}\right)+N_{k+2}\left(r,\frac{1}{f^n(z)P(f)f(z+c)}\right)+S(r,f) \end{aligned} \tag{3.4}$$

From (2) of Lemma 2.8, we get

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &\le N_{k+2}\left(r,\frac{1}{f^n(z)P(f)f(z+c)}\right)\\ &\le (k+2)N\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f(z+c)}\right)\\ &\le (k+m+3)T(r,f)+S(r,f). \end{aligned} \tag{3.5}$$

Similarly as above, we obtain

$$(n+m+1)T(r,g) \le T(r,G)-N_2\left(r,\frac{1}{G}\right)+N_{k+2}\left(r,\frac{1}{g^n(z)P(g)g(z+c)}\right)+S(r,g) \tag{3.6}$$

$$N_2\left(r,\frac{1}{G}\right) \le N_{k+2}\left(r,\frac{1}{g^n(z)P(g)g(z+c)}\right) \le (k+m+3)T(r,g)+S(r,g). \tag{3.7}$$

Using equations (3.2)−(3.7), we deduce

$$\begin{aligned} (n+m+1)[T(r,f)+T(r,g)] &\le 2N_{k+2}\left(r,\frac{1}{f(z)^nP(f)f(z+c)}\right)+2N_{k+2}\left(r,\frac{1}{g(z)^nP(g)g(z+c)}\right)\\ &\quad +S(r,f)+S(r,g)\\ &\le 2(k+m+3)[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \end{aligned}$$

Which is contradiction to $n \geq m+2k+6$. Hence, by Lemma 2.7, we get either $FG \equiv 1$ or $F \equiv G$. Suppose $FG \equiv 1$ holds,

$$i.e.,\ \left[f^n(z)(a_m f^m+\ldots+a_0)f(z+c)\right]^{(k)}\left[g^n(z)(a_m g^m+\ldots+a_0)g(z+c)\right]^{(k)} \equiv 1 \tag{3.8}$$

By assumption that $a_m \neq 0, a_0 \neq 0$, we can arrive at a contradiction by Lemma 2.10. Hence, by Lemma 2.7 $F(z) \equiv G(z)$,

$$i.e.,\ \left[f^n(z)(a_m f^m+\ldots+a_0)f(z+c)\right]^{(k)} \equiv \left[g^n(z)(a_m g^m+\ldots+a_0)g(z+c)\right]^{(k)} \tag{3.9}$$

From (3.9), we get

$$f^n(z)P(f)f(z+c) \equiv g^n(z)P(g)g(z+c)+q(z)$$

where $q(z)$ is a polynomial of degree at most $k-1$. If $q(z) \not\equiv 0$, then we have

$$\frac{f^n(z)P(f)f(z+c)}{q(z)} = \frac{g^n(z)P(g)g(z+c)}{q(z)}+1 \tag{3.10}$$

Thus, from the second main theorem for three small values and (3.10), we have

$$\begin{aligned}
(n+m+1)T(r,f) &\leq T\left(r,\frac{f^n(z)P(f)f(z+c)}{q(z)}\right)+S(r,f)\\
&\leq \overline{N}\left(r,\frac{f^n(z)P(f)f(z+c)}{q(z)}\right)+\overline{N}\left(r,\frac{q(z)}{f^n(z)P(f)f(z+c)}\right)\\
&\quad+\overline{N}\left(r,\frac{q(z)}{g^n(z)P(g)g(z+c)}\right)+S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{f(z)}\right)+\overline{N}\left(r,\frac{1}{P(f)}\right)+\overline{N}\left(r,\frac{1}{f(z+c)}\right)+\overline{N}\left(r,\frac{1}{g(z)}\right)+\overline{N}\left(r,\frac{1}{P(g)}\right)\\
&\quad+\overline{N}\left(r,\frac{1}{g(z+c)}\right)+S(r,f)\\
&\leq (2+m)T(r,f)+(2+m)T(r,g)+S(r,f).
\end{aligned}$$

Similarly as above, we have

$$(n+m+1)T(r,g) \leq (2+m)T(r,g)+(2+m)T(r,f)+S(r,g)$$

Thus, we get

$$(n+m+1)[T(r,f)+T(r,g)] \leq 2(2+m)([T(r,f)+T(r,g)]+S(r,f)+S(r,g)$$

which is contradiction to $n \geq m+2k+6$. Hence, we get $q(z) \equiv 0$, which implies that

$$f^n(z)P(f)f(z+c) = g^n(z)P(g)g(z+c)$$

$$i.e.,\qquad f^n(z)(a_m f^m+a_{m-1}f^{m-1}+\ldots+a_0)f(z+c)$$

$$\equiv g^n(z)(a_m g^m + a_{m-1}g^{m-1} + \ldots + a_0)g(z+c). \tag{3.11}$$

Let $h = \frac{f}{g}$, and then substituting $f = gh$ and $f(z+c) = g(z+c)h(z+c)$ in (3.11) we deduce

$$\Rightarrow (gh)^n \left[a_m(gh)^m + a_{m-1}(gh)^{m-1} + \ldots + a_0\right] g(z+c)h(z+c) = g^n(a_m g^m + \ldots + a_0)g(z+c)$$

$$\Rightarrow g^{n+m}\left[a_m(h^{m+n}(z)h(z+c)-1)\right] + \ldots + g^n\left[a_0(h^n(z)h(z+c)-1)\right] = 0$$

$$\Rightarrow g^m = \frac{-\left[g^{m-1}\left[a_{m-1}(h^{m+n-1}(z)h(z+c)-1)\right] + \ldots + [a_0(h^n(z)h(z+c)-1)]\right]}{a_m(h^{m+n}(z)h(z+c)-1)} \tag{3.12}$$

If $h^{n+m}(z)h(z+c) \neq 1$, then since g is transcendental and from (3.12), we have $h(z)$ is a transcendental meromorphic function with finite order. By Lemma 2.1,

$$T(r, h(z+c)) = T(r,h) + S(r,h) \tag{3.13}$$

From (3.13) and using the condition $n \geq m+2k+6$, it is easy to show that $h^{n+m}(z)h(z+c)$ is not a constant.
Suppose that there exist a point z_0 such that $h^{n+m}(z_0)h(z_0+c) = 1$.
Since $g(z)$ is an entire function and from (3.12), we deduce $h^d(z_0) = 1$, where $d = GCD\{n+m+1, n+m, \ldots, n+m+1-i, \ldots, n+1\}$ and $i = 0,1,2,\ldots,m$.
Now denote $H = h^{n+m}(z)h(z+c)$, then

$$\overline{N}\left(r, \frac{1}{H-1}\right) \leq \overline{N}\left(r, \frac{1}{h^d(z)-1}\right) \leq dT(r,h) + O(1) \leq mT(r,h) + O(1) \tag{3.14}$$

Applying the second fundamental theorem to H, and using (3.13) and (3.14), we have

$$\begin{aligned} T(r,H) &\leq \overline{N}(r,H) + \overline{N}\left(r,\frac{1}{H}\right) + \overline{N}\left(r,\frac{1}{H-1}\right) + S(r,h) \\ &\leq \overline{N}(r,H) + \overline{N}\left(r,\frac{1}{H}\right) + mT(r,h) + S(r,h) \\ &\leq (4+m)T(r,h) + S(r,h) \end{aligned}$$

Noting this, we have

$$\begin{aligned} (n+m)T(r,h) &= T(r,h^{n+m}(z)) = T\left(r, \frac{H}{h(z+c)}\right) \\ &\leq T(r,H) + T(r,h(z+c)) + O(1) \\ &\leq (4+m)T(r,h) + T(r,h) + S(r,h) \\ &= (5+m)T(r,h) + S(r,h) \end{aligned}$$

which is contradiction to $n \geq m+2k+6$.
Hence $H = h^{n+m}(z)h(z+c) \neq 1$, then 1 is picard's exceptional value of H, then by Second fundamental theorem, we have

$$T(r,H) \leq \overline{N}(r,H) + \overline{N}\left(r,\frac{1}{H}\right) + \overline{N}\left(r,\frac{1}{H-1}\right) + S(r,H)$$

$$\leq 2\,T(r,h)+2\,T(r,h)+S(r,h)$$
$$(n+m+1)T(r,h) \leq 4\,T(r,h)+S(r,h)$$

Which is contradiction to $n \geq m+2k+6$.
$\therefore\ h^{n+m}(z)h(z+c) \equiv 1$, then, from (3.12), we get $h^n(z)h(z+c) \equiv 1 \Rightarrow h^d(z) \equiv 1$.
Hence, we get $f(z) \equiv tg(z)$, such that $t^d = 1$, where $d = GCD\{n+m+1, n+m, \ldots, n+m+1-i, \ldots, n+1\}$ and $i = 0,1,2,\ldots,m$.
Now we consider the case when f and g are two polynomials.
By $[f^n(z)P(f)f(z+c)]^{(k)}$ and $[g^n(z)P(g)g(z+c)]^{(k)}$ share 1 CM, we have

$$\left[f^n(z)(a_m f^m+\ldots+a_0)f(z+c)\right]^{(k)}-1=\beta\left[\left[g^n(z)(a_m g^m+\ldots+a_0)g(z+c)\right]^{(k)}-1\right] \tag{3.15}$$

where β is a non-zero constant. Let $deg\, f = l$, then by (3.15) we know that $deg\, g = l$. Differentiating the two sides of (3.15), we get

$$f^{n-k-1}(z)q_1(z) = g^{n-k-1}(z)q_2(z), \tag{3.16}$$

where $q_1(z)$, $q_2(z)$ are two polynomials with $deg\, q_1(z) = deg\, q_2(z) = (m+k+2)l-(k+1)$. By $n \geq m+2k+6$, we get $deg\, f^{n-k-1}(z) = (n-k-1)l > deg\, q_2(z)$.
Thus, by (3.16) we know that there exists z_0 such that $f(z_0) = g(z_0) = 0$.
Hence, by (3.15) and $f(z_0) = g(z_0) = 0$, we deduce that $\beta = 1$, that is,

$$\left[f^n(z)(a_m f^m+\ldots+a_0)f(z+c)\right]^{(k)} = \left[g^n(z)(a_m g^m+\ldots+a_0)g(z+c)\right]^{(k)} \tag{3.17}$$

Thus, we have

$$f^n(z)(a_m f^m+\ldots+a_0)f(z+c)-g^n(z)(a_m g^m+\ldots+a_0)g(z+c) = Q(z) \tag{3.18}$$

where $Q(z)$ is a polynomial of degree atmost $k-1$. Next we prove $Q(z) \equiv 0$. By rewriting (3.17) as

$$f^{n-k}(z)p_1(z) = g^{n-k}(z)p_2(z). \tag{3.19}$$

Where $p_1(z)$, $p_2(z)$ are two polynomials with $deg\, p_1(z) = deg\, p_2(z) = (m+k+1)l-k$ and $deg\, f(z) = l$.
Hence total number of common zeros of $f^{n-k}(z)$ and $g^{n-k}(z)$ is atleast k. Thus, by (3.18) we deduce that $Q(z) \equiv 0$, that is

$$f^n(z)(a_m f^m+a_{m-1}f^{m-1}+\ldots+a_0)f(z+c) = g^n(z)(a_m g^m+a_{m-1}g^{m-1}+\ldots+a_0)g(z+c). \tag{3.20}$$

Next, similar to the argument of (3.11), we get $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD\{n+m+1, n+m, \ldots, n+m+1-i, \ldots, n+1\}$ and $i = 0,1,2,\ldots,m$.

Hence proved the (1) of Theorem 1.2.

(2) If $P(z) \equiv c_0$

By the assumption and Theorem 1.1, we know that either both f and g are transcendental entire functions or both f and g are polynomials.

First, we consider the case when both f and g are transcendental entire functions. Let

$$F = f^n(z)c_0 f(z+c) \quad , G = g^n(z)c_0 g(z+c)$$

By the Theorem F and $n \geq m+2k+6$, we obtain either $f(z) = \frac{c_1 e^{Cz}}{\sqrt[n]{c_0}}$, $g(z) = \frac{c_2 e^{-Cz}}{\sqrt[n]{c_0}}$, where c_1, c_2, c_0 and C are constants satisfying $(-1)^k (c_1 c_2)^{n+1}((n+1)C)^{2k} = (\sqrt[n]{c_0})^2$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

Now we consider the case when both f and g are two polynomials.

By $[f^n(z)c_0 f(z+c)]^{(k)}$ and $[g^n(z)c_0 g(z+c)]^{(k)}$ share 1 CM, we have

$$\left[f^n(z)c_0 f(z+c)\right]^{(k)} - 1 = \gamma\left[\left[g^n(z)c_0 g(z+c)\right]^{(k)} - 1\right]. \tag{3.21}$$

Where γ is a non-zero constant. Let $deg\ f(z) = l$, then by (3.21) we know that $deg\ g(z) = l$. Differentiating the two sides of (3.21), we get

$$f^{n-k-1}(z) q_3(z) = g^{n-k-1}(z) q_4(z) \tag{3.22}$$

where $q_3(z)$, $q_4(z)$ are two polynomials with $deg\ q_3(z) = deg\ q_4(z) = (k+2)l - (k+1)$. By $n \geq 2k+6$, we get $deg\ f^{n-k-1}(z) = (n-k-1)l > deg\ q_4(z)$.

Thus, by (3.22) we know that there exists z_0 such that $f(z_0) = g(z_0) = 0$.

Hence, by (3.21) and $f(z_0) = g(z_0) = 0$, we deduce that $\gamma = 1$, that is,

$$\left[f^n(z)c_0 f(z+c)\right]^{(k)} = \left[g^n(z)c_0 g(z+c)\right]^{(k)} \tag{3.23}$$

Thus, we have

$$f^n(z) f(z+c) - g^n(z) g(z+c) = Q_1(z) \tag{3.24}$$

where $Q_1(z)$ is a polynomial of degree atmost $k-1$. Next we prove $Q_1(z) \equiv 0$. By rewriting (3.23) as

$$f^{n-k}(z) p_3(z) = g^{n-k}(z) p_4(z) \tag{3.25}$$

where $p_3(z)$, $p_4(z)$ are two polynomials with $deg\ p_3(z) = deg\ p_4(z) = (k+1)l-k$ and $deg\ f(z) = l$.

Hence total number of common zeros of $f^{n-k}(z)$ and $g^{n-k}(z)$ is atleast k.

Thus, by (3.24) we deduce that $Q_1(z) \equiv 0$, that is,

$$f^n(z) f(z+c) = g^n(z) g(z+c). \tag{3.26}$$

Let $h(z) = \frac{f(z)}{g(z)}$ and $h(z+c) = \frac{f(z+c)}{g(z+c)}$ then

$$(gh)^n g(z+c)h(z+c) = g^n g(z+c)$$

Hence $f = tg$ where $(tg)^n tg(z+c) = g^n g(z+c)$

$$\Rightarrow t^{n+1} = 1$$

Hence proved the (2) of Theorem 1.2.

4. Open questions

Question 4.1. *Can 1 point shared value in the Theorem 1.2 be replaced by fixed point?*

Question 4.2. *Can 1 point shared value with CM in the Theorem 1.2 be replaced by 1 point shared value with IM?*

Question 4.3. *Do the Theorem 1.1 and Theorem 1.2 hold for meromorphic functions f and g?*

Question 4.4. *What happens if the CM sharing is replaced by weighted sharing of small function in Theorem 1.2?*

Question 4.5. *Are the conditions $n \geq k+2$ in Theorem 1.1 and $n \geq m+2k+6$ in Theorem 1.2 sharp?*

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