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ON SOME CLASSES OF INVARIANT SUBMANIFOLDS OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study invariant submanifolds of Lorenzian Para-Sasakian manifolds. We consider the recurrent and bi-recurrent invariant submanifolds of Lorentzian para-Sasakian manifolds and pseudo-parallel and generalized Ricci pseudo-parallel invariant submanifolds of Lorentzian para-Sasakian manifolds. Also we search for the conditions $\mathcal{Z}(X,Y)\cdot\alpha = fQ(g,\alpha)$ and $\mathcal{Z}(X,Y)\cdot\alpha = fQ(S,\alpha)$ on invariant submanifolds of Lorentzian para-Sasakian manifolds, where $\mathcal{Z}$ is the concircular curvature tensor. Finally, we construct an example of an invariant submanifold of Lorentzian para Sasakian manifold.

1. Introduction

In 1989, Matsumoto [14] introduced the notion of Lorentzian para-Sasakian manifolds. An example of a five dimensional Lorentzian para-Sasakian manifold was given by Matsumoto, Mihai and Rosaca [16]. Lorentzian para-Sasakian manifold is called LP-Sasakian manifold

The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghuic. The geometry of submanifolds have become an interesting subject in applied mathematics and theoretical physics. Lorentzian para-Sasakian manifolds have been studied by De and Shaikh [25], $\ddot{O}zg\ddot{u}r$ [6], Shaikh and De [1] and also by several authors, such as ([15], [16], [21]) and many others.

Let M and $\tilde{M}$ be two Riemannian or semi-Riemannian manifolds, $f : M \to \tilde{M}$ an isometric immersion , α the second fundamental form and $\bar{\nabla}$ the van der Waerden-Bortolotti connection of M. An immersion is said to be semiparallel if

$$\bar{R}(X,Y)\cdot\alpha = (\bar{\nabla}_X\bar{\nabla}_Y - \bar{\nabla}_Y(\bar{\nabla}_X - \bar{\nabla}_{[X,Y]})\alpha = 0 \tag{1.1}$$

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holds for all vector fields tangent to M [11], where $\bar{R}$ denotes the curvature tensor of the connection $\bar{\nabla}$. Semiparallel immersion has been studied by many authors, see, for example, ([12], [10], [13] and [20]).

An immersion is said to be pseudo-parallel ([2], [3]) if

$$\bar{R}(X,Y)\cdot\alpha = fQ(g,\alpha) \tag{1.2}$$

holds for all vector fields X, Y tangent to M, where f denotes real valued function on M and for a $(0,k)$, $k \geq 1$ tensor T and a $(0,2)$ tensor E, $Q(E,T)$ is defined by [19]

$$\begin{aligned} Q(E,T)(X_1,X_2,\dots,X_k;X,Y) &= -T((X\wedge_E Y)X_1,\dots,X_k) \\ &\quad -T(X_1,(X\wedge_E Y)X_2,\dots,X_k)-\cdots \\ &\quad -T(X_1,X_2,\dots,X_{k-1},(X\wedge_E Y)X_k), \end{aligned} \tag{1.3}$$

where $X\wedge_E Y$ is defined by

$$(X\wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y. \tag{1.4}$$

Therefore we can take the following equation instead of (1.2)

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V)-\alpha(R(X,Y)U,V)-\alpha(U,R(X,Y)V) \\ &\quad = -f[\alpha((X\wedge_g Y)U,V)+\alpha(U,(X\wedge_g Y)V)]. \end{aligned} \tag{1.5}$$

Also Murathan, Arsalan and Ezentas [5] defined submanifolds satisfying the condition

$$\bar{R}(X,Y)\cdot\alpha = fQ(S,\alpha). \tag{1.6}$$

The equation (1.7) can be written as

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V)-\alpha(R(X,Y)U,V)-\alpha(U,R(X,Y)V) \\ &\quad = -f[\alpha((X\wedge_S Y)U,V)+\alpha(U,(X\wedge_S Y)V)]. \end{aligned} \tag{1.7}$$

This kind of submanifolds are called generalized Ricci-pseudo parallel.

In a recent paper [22] , Pradip Majhi studied invariant submanifolds of Kenmotsu manifolds. Also in [26], V. Mangione studied invariant submanifolds of Kenmotsu manifolds and some conditions that these submanifolds are totally geodesic are given. Besides these, invariant submanifolds of Lorentzian para-Sasakian manifolds were studied by Özgür and Murathan [7]. Submanifolds of Lorentzian para-Sasakian manifolds have also been studied by several authors, such as ([21], [24], [25]) and many others.

Motivated by these studies, in this paper we consider invariant submanifolds of Lorentzian para-Sasakian manifolds and search some conditions that these submanifolds are totally geodesic.

The paper is organized as follows: In section 2, we give necessary details about submanifolds and concircular curvature tensor. In section 3, we discuss about Lorentzian para-Sasakian manifolds and its submanifolds. In section 4, we study recurrent and bi-recurrent invariant submanifolds of Lorentzian para-Sasakian manifolds. In section 5 and 6, pseudo-parallel and generalized Ricci pseudo-parallel invariant submanifolds of Lorentzian para-Sasakian manifolds have been studied. Section 7 and 8 are devoted to study invariant submanifolds satisfying the conditions $\mathcal{Z}(X,Y)\cdot\alpha = fQ(g,\alpha)$ and $\mathcal{Z}(X,Y)\cdot\alpha = fQ(S,\alpha)$ respectively. Finally, we construct an example of an invariant submanifold of Lorentzian para-Sasakian manifold.

2. Basic concepts

Let (M,g) be an n-dimensional Riemannian submanifold of an $(n+d)$-dimensional Riemannian manifold $(\tilde{M},\tilde{g})$. We denote by $\tilde{\nabla}$ and ∇ the Levi-civita connections of $\tilde{M}$ and M, respectively. Then we have the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y),$$

and

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.1}$$

where X,Y are vector fields tangent to M and N is normal vector field on M, respectively. $\nabla^{\perp}$ is called the normal connection of M. We call α the second fundamental form of the submanifold M. If $\alpha=0$, then the submanifold is said to be totally geodesic. For the second fundamental form α, the covariant derivative of α is defined by

$$(\bar{\nabla}_X\alpha)(Y,Z) = \nabla^{\perp}_X N - \alpha(\nabla_X Y, Z) - \alpha(Y,\nabla_X Z) \tag{2.2}$$

for any vector fields X,Y,Z tangent to M. Then $\bar{\nabla}\alpha$ is a normal bundle valued tensor of type $(1,3)$ and is called the third fundamental form of M. $\bar{\nabla}$ is called the van der waerden-Bortolotti connection M, i.e, $\bar{\nabla}$ is the connection in $TM \oplus T^{\perp}M$ built with ∇ and $\nabla^{\perp}$. If $\bar{\nabla}=0$, then M is said to have parallel second fundamental form [4]. From the Gauss and Weingarten formulas we obtain

$$(\tilde{R}(X,Y)Z)^T = R(X,Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X. \tag{2.3}$$

Therefore from (1.1) we have

$$\begin{aligned}(\bar{R}(X,Y)\cdot\alpha)(U,V) &= R^{\perp}(X,Y)\alpha(U,V) \\ &\quad -\alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V)\end{aligned} \tag{2.4}$$

for all vector fields $X,Y,U \; and \; V$ tangent to M, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X,\nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}, \tag{2.5}$$

and $\bar{R}$ denotes the curvature tensor of $\bar{\nabla}$.

For an n-dimensional $(n \geq 3)$ Riemannian manifold (M, g), the concircular curvature tensor $\mathcal{Z}$ is defined by [17]

$$\mathcal{Z}(X,Y)V = R(X,Y)V - \frac{r}{n(n-1)}\{g(Y,V)X - g(X,V)Y\} \tag{2.6}$$

for all vector fields X, Y and V on M^n, where r is the scalar curvature and R denotes the curvature tensor of M.

Concircular curvature tensor appeared in the study of concircular mappings, i.e., conformal mappings preserving geodesic circles [20]. Concircular curvature tensors play an important role in the theory of projective and conformal transformations.

Similar to (2.4) and (2.6) the tensor $\mathcal{Z}(X,Y)\cdot\alpha$ is defined by

$$(\mathcal{Z}(X,Y)\cdot\alpha)(U,V) = R^{\perp}(X,Y)\alpha(U,V) - \alpha(\mathcal{Z}(X,Y)U,V) - \alpha(U,\mathcal{Z}(X,Y)V). \tag{2.7}$$

3. Lorentzian Para-Sasakian manifolds

Let $\tilde{M}$ be an n-dimensional differential manifold endowed with a $(1,1)$ tensor field φ, a vector field ξ, a 1-form and a Lorentzian metric $\tilde{g}$ of type $(0,2)$ such that for each point $p \in \tilde{M}$, the tensor $\tilde{g}_p : T_p\tilde{M} \times T_p\tilde{M} \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,\ldots,+)$, where $T_p\tilde{M}$ denotes the tangent space of $\tilde{M}$ at p and $\mathbb{R}$ is the real number space which satisfies

$$\varphi^2(X) = X + \eta(X)\xi, \ \eta(\xi) = -1, \tag{3.1}$$

$$\tilde{g}(X,\xi) = \eta(X), \ \tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X,Y) + \eta(X)\eta(Y) \tag{3.2}$$

for all vector fields X, Y. Then such a structure $(\varphi, \xi, \eta, \tilde{g})$ is termed as Lorentzian almost paracontact structure and the manifold with the structure $(\varphi, \xi, \eta, \tilde{g})$ is called a Lorentzian almost paracontact manifold [14]. In the Lorentzian almost paracontact manifold $\tilde{M}$, the following relations hold [14]:

$$\varphi\xi = 0, \qquad \eta(\varphi X) = 0, \tag{3.3}$$

$$\Phi(X,Y) = \Phi(Y,X), \tag{3.4}$$

where $\Phi(X,Y) = \tilde{g}(X, \varphi Y)$.

A Lorentzian almost paracontact manifold $\tilde{M}$ equipped with the structure $(\varphi, \xi, \eta, \tilde{g})$ is called an LP-Sasakian manifold [14] if

$$(\tilde{\nabla}_X \varphi)Y = \tilde{g}(\varphi X, \varphi Y)\xi + \eta(Y)\varphi^2 X, \tag{3.5}$$

where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $\tilde{g}$. In an LP-Sasakian manifold $\tilde{M}$ with structure $(\varphi, \xi, \eta, \tilde{g})$, it is easily seen that

$$\tilde{\nabla}_X \xi = \varphi X, \tag{3.6}$$
$$\tilde{R}(\xi, X)Y = \tilde{g}(X,Y)\xi - \eta(Y)X, \tag{3.7}$$
$$\tilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{3.8}$$
$$\tilde{S}(X,\xi) = (n-1)\eta(X) \tag{3.9}$$

for all vector fields X, Y on $\tilde{M}$ [14], where $\tilde{S}$ denotes the Ricci tensor of $\tilde{M}$ and $\tilde{R}$ is the curvature tensor of $\tilde{M}$.

A submanifold M of an LP-Sasakian manifold $\tilde{M}$ is called an invariant submanifold of $\tilde{M}$ if $\varphi(TM) \subset TM$. In an invariant submanifold of an LP-Sasakian manifold

$$\alpha(X,\xi) = 0, \tag{3.10}$$

for any vector field X tangent to M ([24], [21]).

In [7] Özgür and Murathan proved the following Lemma:

Lemma 3.1. *Let M be an n-dimensional invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then the following equations hold on M:*

$$\nabla_X \xi = \varphi X, \tag{3.11}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{3.12}$$
$$R(\xi,Y)\xi = \eta(Y)\xi - Y, \tag{3.13}$$
$$S(X,\xi) = (n-1)\eta(X), \tag{3.14}$$
$$S(\xi,\xi) = (n-1)\xi, Q\xi = (n-1)\xi, \tag{3.15}$$
$$(\nabla_X \varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{3.16}$$
$$\mathcal{Z}(\xi,X)\xi = \Big(1 - \frac{r}{n(n-1)}\Big)(\eta(X)\xi + X), \tag{3.17}$$
$$\alpha(X,\varphi Y) = \varphi\alpha(X,Y), \tag{3.18}$$

where Q denotes the Ricci operator of M defined by $S(X,Y) = g(QX,Y)$.

4. Recurrent and bi-recurrent invariant submanifolds of LP-Sasakian manifolds

Let us consider that M be an invariant n-dimensional submanifold of an $(n+d)$-dimensional LP-Sasakian manifold $\tilde{M}$.

In [7], Özgür and Murathan proved the following lemma:

Lemma 4.1. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then M^n has parallel second fundamental form if and only if M^n is totally geodesic.*

Now we prove the following Lemma:

Lemma 4.2. *If a non-flat Riemannian manifold has a recurrent second fundamental form, then it is semiparallel.*

Proof. The second fundamental form α is said to be recurrent if

$$\nabla\alpha = A \otimes \alpha,$$

where α is non-zero and A is an everywhere nonzero 1-form. We now define a function f on M by

$$f^2 = g(\alpha,\alpha),$$

where the metric g is extended to the inner product between the tensor fields in the standard fashion. Then we know that $f(Yf) = f^2 A(Y)$. So from this we have $Yf = fA(Y)$, since f is nonzero. This implies that

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (XA(Y))f.$$

Hence, $X(Yf) - Y(Xf) = \{XA(Y) - YA(X)\}f$.
Therefore, we get

$$(\tilde{\nabla}_X\tilde{\nabla}_Y - \tilde{\nabla}_Y\tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]})f = \{XA(Y) - YA(X) - A([X,Y])\}f.$$

Since the left hand side of the above equation is identically zero and f is nonzero on M by our assumption, we obtain

$$dA(X,Y) = 0, \tag{4.1}$$

i.e., the 1-form A is closed.

Now from

$$(\nabla_X\alpha)(U,V) = A(X)\alpha(U,V),$$

we get

$$(\tilde{\nabla}_U\tilde{\nabla}_V\alpha)(X,Y) - (\tilde{\nabla}_{\tilde{\nabla}_U V}\alpha)(X,Y) = [(\tilde{\nabla}_U A)V + A(U)A(V)]\alpha(X,Y).$$

Hence using (4.1), we get

$$(\tilde{R}(X,Y)\cdot\alpha)(U,V) = [2dA(X,Y)]\alpha(X,Y) = 0.$$

Therefore, we have

$$\tilde{R}(X,Y)\cdot\alpha = 0.$$

for any vector field X, Y. If $f = 0$, then from $f^2 = g(\alpha,\alpha)$ we get $\alpha = 0$ and hence obviously $R(X,Y)\cdot\alpha = 0$. Hence the result. $\square$

From Lemma 4.1 and Lemma 4.2, we have the following:

Corollary 4.1. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then M^n has recurrent second fundamental form if and only if M^n is totally geodesic.*

The second fundamental form α is bi-recurrent if there exists a non-zero covariant tensor field B such that $(\nabla_X\nabla_W\alpha - \nabla_{\nabla_X W}\alpha)(Y,Z) = B(X,W)\alpha(Y,Z)$. In a recent paper [23] Aikawa and Matsuyama proved that if a tensor field T is bi-recurrent, then $R(X,Y)\cdot T = 0$. Therefore by virtue of the lemma 4.1 we can state the following:

Corollary 4.2. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then M^n has bi-recurrent second fundamental form if and only if M^n is totally geodesic.*

5. Pseudo-parallel invariant submanifolds of LP-Saskian manifolds

In this section we study pseudo-parallel invariant submanifolds of LP-Sasakian manifolds. Therefore we have

$$\bar{R}(X,Y)\cdot\alpha = fQ(g,\alpha) \tag{5.1}$$

holds for all vector fields X, Y tangent to M, where f denotes the real valued function on M^n. The equation (5.1) can be written as

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V)\\ &-\alpha(U,R(X,Y)V) = -f\big[\alpha((X\wedge_g Y)U,V) +\\ &\alpha(U,(X\wedge_g Y)V)\big], \end{aligned} \tag{5.2}$$

where $(X\wedge_g Y)$ is defined by

$$(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{5.3}$$

Substituting (5.3) in (5.2), we have

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V)\\ &-\alpha(U,R(X,Y)V) = -f[g(Y,U)\alpha(X,V) - g(X,U)\alpha(Y,V) \end{aligned} \tag{5.4}$$

$$+g(Y,V)\alpha(U,X)-g(X,V)\alpha(U,Y)].$$

Putting $X=V=\xi$ in (5.4), we obtain

$$\begin{aligned}&R^{\perp}(\xi,Y)\alpha(U,\xi)-\alpha(R(\xi,Y)U,\xi)-\alpha(U,R(\xi,Y)\xi)\\&\quad=-f[g(Y,U)\alpha(\xi,\xi)-g(\xi,U)\alpha(Y,\xi)+g(Y,\xi)\alpha(U,\xi)-g(\xi,\xi)\alpha(U,Y)].\end{aligned}\tag{5.5}$$

Using (3.10) and (3.13) in (5.5), we have

$$-\alpha(U,\eta(Y)\xi-Y)=-f[-\alpha(U,Y)],\tag{5.6}$$

which implies

$$(1-f)\alpha(U,Y)=0,\tag{5.7}$$

i.e, $\alpha(U,Y)=0$ which gives M^n is totally geodesic, provided $f\neq 1$.
Conversely, let M^n be totally geodesic. Then, from (5.4) we get M^n is pseudo-parallel. Thus we can state the following:

Theorem 5.1. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then M^n is pseudo-parallel if and only if M^n is totally geodesic, provided $f\neq 1$.*

6. Generalized Ricci pseudo-parallel invariant submanifolds of LP-Sasakian manifolds

Let us consider M^n be a generalized Ricci pseudo-parallel invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Therefore we have

$$\bar{R}(X,Y)\cdot\alpha=fQ(S,\alpha)\tag{6.1}$$

for all vector fields X,Y tangent to M, where f denotes the real valued function on M^n. The equation (6.1) can be written as

$$\begin{aligned}&R^{\perp}(X,Y)\alpha(U,V)-\alpha(R(X,Y)U,V)-\alpha(U,R(X,Y)V)\\&\quad=-f\big[\alpha((X\wedge_S Y)U,V)+\alpha(U,(X\wedge_S Y)V)\big],\end{aligned}\tag{6.2}$$

where $(X\wedge_S Y)$ is defined by

$$(X\wedge_S Y)Z=S(Y,Z)X-S(X,Z)Y.\tag{6.3}$$

Substituting (6.3) in (6.2), we have

$$\begin{aligned}&R^{\perp}(X,Y)\alpha(U,V)-\alpha(R(X,Y)U,V)-\alpha(U,R(X,Y)V)\\&\quad=-f[S(Y,U)\alpha(X,V)-S(X,U)\alpha(Y,V)+S(Y,V)\alpha(U,X)-S(X,V)\alpha(U,Y)].\end{aligned}\tag{6.4}$$

Putting $X = V = \xi$ in (6.4), we obtain

$$\begin{aligned} &R^{\perp}(\xi, Y)\alpha(U,\xi) - \alpha(R(\xi, Y)U,\xi) - \alpha(U, R(\xi, Y)\xi) \\ &\quad = -f[S(Y,U)\alpha(\xi,\xi) - S(\xi,U)\alpha(Y,\xi) + S(Y,\xi)\alpha(U,\xi) - S(\xi,\xi)\alpha(U,Y)]. \end{aligned} \tag{6.5}$$

Using (3.10), (3.13) and (3.15) in (6.5), we get

$$-\alpha(U,\eta(Y)\xi - Y) = -f[-(n-1)\alpha(U,Y)], \tag{6.6}$$

which implies

$$(1-(n-1)f)\alpha(U,Y) = 0, \tag{6.7}$$

i.e, $\alpha(U,Y) = 0$ which gives M^n is totally geodesic, provided $f \neq \frac{1}{n-1}$.
Conversely, let M^n be totally geodesic. Then, from (6.4) we get M^n is pseudo-parallel. Thus we can state the following:

Theorem 6.1. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then M^n is generalized Ricci pseudo-parallel if and only if M^n is totally geodesic, provided $f \neq \frac{1}{n-1}$.*

7. Invariant submanifolds of LP-Sasakian manifolds satisfying $\mathcal{Z}(X,Y)\cdot\alpha = fQ(g,\alpha)$

Let us consider M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$ satisfying

$$\mathcal{Z}(X,Y)\cdot\alpha = fQ(g,\alpha) \tag{7.1}$$

for all vector fields X, Y tangent to M, where f denotes the real valued function on M^n. The equation (7.1) can be written as

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V) - \alpha(\mathcal{Z}(X,Y)U,V) - \alpha(U,\mathcal{Z}(X,Y)V) \\ &\quad = -f\big[\alpha((X\wedge_g Y)U,V) + \alpha(U,(X\wedge_g Y)V)\big]. \end{aligned} \tag{7.2}$$

Substituting (5.3) in (7.2), we have

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V) - \alpha(\mathcal{Z}(X,Y)U,V) - \alpha(U,\mathcal{Z}(X,Y)V) \\ &\quad = -f[g(Y,U)\alpha(X,V) - g(X,U)\alpha(Y,V) + g(Y,V)\alpha(U,X) - g(X,V)\alpha(U,Y)]. \end{aligned} \tag{7.3}$$

Putting $X = V = \xi$ in (7.3), we obtain

$$\begin{aligned} &R^{\perp}(\xi,Y)\alpha(U,\xi) - \alpha(\mathcal{Z}(\xi,Y)U,\xi) - \alpha(U,\mathcal{Z}(\xi,Y)\xi) \\ &\quad = -f[g(Y,U)\alpha(\xi,\xi) - g(\xi,U)\alpha(Y,\xi) + g(Y,\xi)\alpha(U,\xi) - g(\xi,\xi)\alpha(U,Y)]. \end{aligned} \tag{7.4}$$

Using (3.10) and (3.17) in (7.4), we get

$$-\alpha\big(U,\big(1-\frac{r}{n(n-1)}\big)\big(\eta(Y)\xi+Y\big)\big)=-f[-\alpha(U,Y)], \tag{7.5}$$

which implies

$$\big(f+\big(1-\frac{r}{n(n-1)}\big)\big)\alpha(U,Y)=0, \tag{7.6}$$

i.e., $\alpha(U,Y)=0$ which gives M^n is totally geodesic, provided $f\neq\big(\frac{r}{n(n-1)}-1\big)$. Conversely, let M^n be totally geodesic. Then, from (7.3) we get M^n satisfies $\mathcal{Z}(X,Y)\cdot\alpha = fQ(g,\alpha)$. Thus we can state the following:

Theorem 7.1. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then M^n satisfies $\mathcal{Z}(X,Y)\cdot\alpha = fQ(g,\alpha)$ if and only if M^n is totally geodesic, provided $f\neq\big(\frac{r}{n(n-1)}-1\big)$.*

8. Invariant submanifolds of LP-Sasakian manifolds satisfying $\mathcal{Z}(X,Y)\cdot\alpha = fQ(S,\alpha)$

Let us consider M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$ satisfying

$$\mathcal{Z}(X,Y)\cdot\alpha = fQ(S,\alpha) \tag{8.1}$$

for all vector fields X,Y tangent to M, where f denotes the real valued function on M^n. The equation (8.1) can be written as

$$\begin{aligned}&R^{\perp}(X,Y)\alpha(U,V)-\alpha(\mathcal{Z}(X,Y)U,V)-\alpha(U,\mathcal{Z}(X,Y)V)\\&\quad=-f\big[\alpha((X\wedge_S Y)U,V)+\alpha(U,(X\wedge_S Y)V)\big].\end{aligned} \tag{8.2}$$

Substituting (6.3) in (8.2), we have

$$\begin{aligned}&R^{\perp}(X,Y)\alpha(U,V)-\alpha(\mathcal{Z}(X,Y)U,V)\\&-\alpha(U,\mathcal{Z}(X,Y)V)=-f[S(Y,U)\alpha(X,V)-S(X,U)\alpha(Y,V)\\&+S(Y,V)\alpha(U,X)-S(X,V)\alpha(U,Y)].\end{aligned} \tag{8.3}$$

Putting $X=V=\xi$ in (8.3), we obtain

$$\begin{aligned}&R^{\perp}(\xi,Y)\alpha(U,\xi)-\alpha(\mathcal{Z}(\xi,Y)U,\xi)-\alpha(U,\mathcal{Z}(\xi,Y)\xi)\\&\quad=-f[S(Y,U)\alpha(\xi,\xi)-S(\xi,U)\alpha(Y,\xi)+S(Y,\xi)\alpha(U,\xi)-S(\xi,\xi)\alpha(U,Y)].\end{aligned} \tag{8.4}$$

Using (3.10), (3.15) and (3.17) in (8.4), we have

$$-\alpha\big(U,\big(1-\frac{r}{n(n-1)}\big)\big(\eta(Y)\xi+Y\big)\big)=-f[-(n-1)\alpha(U,Y)], \tag{8.5}$$

which follows that

$$\big((n-1)f+\big(1-\frac{r}{n(n-1)}\big)\big)\alpha(U,Y)=0, \tag{8.6}$$

i.e, $\alpha(U,Y)=0$ which gives M^n is totally geodesic, provided $f\neq\big(\frac{r}{n(n-1)^2}-\frac{1}{(n-1)}\big)$. Conversely, let M^n be totally geodesic. Then, from (8.3) we get M^n satisfies $\mathcal{Z}(X,Y)\cdot\alpha = fQ(S,\alpha)$. Thus we can state the following:

Theorem 8.1. *Let M^n be an invariant submanifold of LP-Sasakian manifold $\tilde{M}$. Then M^n satisfies $\mathcal{Z}(X,Y)\cdot\alpha = fQ(S,\alpha)$ if and only if M^n is totally geodesic, provided $f \neq \left(\frac{r}{n(n-1)^2} - \frac{1}{(n-2)}\right)$.*

In view of the Corollary (4.1)−(4.2) and Theorem (5.1)−(8.1) we can state the following:

Theorem 8.2. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then the following statements are equivalent:*

(1) *Second fundamental form of M^n is recurrent.*
(2) *Second fundamental form of M^n is bi-recurrent.*
(3) *M^n is pseudo-parallel, provided $f \neq 1$.*
(4) *M^n is generalized Ricci pseudo parallel, provided $f \neq \frac{1}{n-1}$.*
(5) *M^n satisfies the condition $\mathcal{Z}(X,Y)\cdot\alpha = fQ(g,\alpha)$, provided $f \neq \left(\frac{r}{n(n-1)} - 1\right)$.*
(6) *M^n satisfies the condition $\mathcal{Z}(X,Y)\cdot\alpha = fQ(S,\alpha)$, provided $f \neq \left(\frac{r}{n(n-1)^2} - \frac{1}{(n-2)}\right)$.*
(7) *M^n is totally geodesic.*

9. Example

Let us consider the 5-dimensional manifold $\tilde{M} = \{(x,y,z,u,v) \in \mathbb{R}^5 : (x,y,z,u,v) \neq (0,0,0,0,0)\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. The vector fields

$$e_1 = e^z \frac{\partial}{\partial x},\ e_2 = e^{z-ax}\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z},\ e_4 = e^z\frac{\partial}{\partial u},\ e_5 = e^{z-u}\frac{\partial}{\partial v}$$

are linear independent at each point of $\tilde{M}$ where a is a scalar. Let $\tilde{g}$ be the metric defined by

$$\tilde{g}(e_i,e_j) = \begin{cases} 1, & \text{for } i = j \neq 3, \\ 0, & \text{for } i \neq j, \\ -1, & \text{for } i = j = 3. \end{cases} \tag{9.1}$$

Here i and j runs from 1 to 5. Let η be the 1-form defined by $\eta(Z) = \tilde{g}(Z,e_3)$, for any vector field Z tangent to $\tilde{M}$. Let φ be the $(1,1)$ tensor field defined by

$$\varphi e_1 = -e_1,\ \varphi e_2 = -e_2,\ \varphi e_3 = 0,\ \varphi e_4 = -e_4,\ \varphi e_5 = -e_5.$$

Then using the linearity property of φ and $\tilde{g}$ we have

$$\eta(e_3) = -1,\ \varphi^2 Z = Z + \eta(Z)e_3$$

for any vector field Z tangent to $\tilde{M}$. Thus for $e_3 = \xi$, $\tilde{M}(\varphi,\xi,\eta,\tilde{g})$ defines an almost para-contact metric manifold. Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$ with respect to the metric $\tilde{g}$. Then we have

$$[e_1,e_2]=-ae^z e_2,\quad [e_1,e_3]=-e_1,\quad [e_1,e_4]=0,\quad [e_1,e_5]=0,$$
$$[e_2,e_3]=-e_2,\quad [e_2,e_4]=0,\quad [e_2,e_5]=o,\quad [e_3,e_4]=e_4,$$
$$[e_3,e_5]=e_5,\qquad [e_4,e_5]=-e^z e_5.$$

Taking $e_3=\xi$ and using Koszul's formula for $\tilde{g}$, it can be easily calculated that

$$\begin{array}{lllll}
\tilde{\nabla}_{e_1}e_1=e_3, & \tilde{\nabla}_{e_1}e_2=0, & \tilde{\nabla}_{e_1}e_3=-e_1, & \tilde{\nabla}_{e_1}e_4=0, & \tilde{\nabla}_{e_1}e_5=0,\\
\tilde{\nabla}_{e_2}e_1=ae^z e_2, & \tilde{\nabla}_{e_2}e_2=-ae^z e_1+e_3, & \tilde{\nabla}_{e_2}e_3=-e_2, & \tilde{\nabla}_{e_2}e_4=0, & \tilde{\nabla}_{e_2}e_5=0,\\
\tilde{\nabla}_{e_3}e_1=0, & \tilde{\nabla}_{e_3}e_2=0, & \tilde{\nabla}_{e_3}e_3=0, & \tilde{\nabla}_{e_3}e_4=0, & \tilde{\nabla}_{e_3}e_5=0,\\
\tilde{\nabla}_{e_4}e_1=0, & \tilde{\nabla}_{e_4}e_2=0, & \tilde{\nabla}_{e_4}e_3=-e_4, & \tilde{\nabla}_{e_4}e_4=0, & \tilde{\nabla}_{e_4}e_5=0,\\
\tilde{\nabla}_{e_5}e_1=0, & \tilde{\nabla}_{e_5}e_2=0, & \tilde{\nabla}_{e_5}e_3=-e_5, & \tilde{\nabla}_{e_5}e_4=e^z e_5, & \tilde{\nabla}_{e_5}e_5=e_3-e^z e_5.
\end{array}$$

From the above calculations, we see the manifold under consideration satisfies $\eta(\xi)=-1$ and $\tilde{\nabla}_X\xi=\varphi X$. Hence, $\tilde{M}$ is an LP-Sasakian manifold.

Let f be an isometric immersion from M to $\tilde{M}$ defined by $f(x,y,z)=(x,y,z,0,0)$.

Let $M=\{(x,y,z)\in\mathbb{R}^3:(x,y,z)\neq(0,0,0)\}$ where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1=e^z\frac{\partial}{\partial x},\ e_2=e^{z-ax}\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the metric defined by

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0$$
$$g(e_1,e_1)=g(e_2,e_2)=1,\qquad g(e_3,e_3)=-1.$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$, for any vector field Z tangent to M. Let φ be the $(1,1)$ tensor field defined by

$$\varphi e_1=-e_1,\ \varphi e_2=-e_2,\ \varphi e_3=0$$

Then using the linearity property of φ and g we have

$$\eta(e_3)=-1,\ \varphi^2 Z=Z+\eta(Z)e_3$$

for any vector field Z tangent to M. Thus for $e_3=\xi$, $M(\varphi,\xi,\eta,g)$ defines an almost para-contact metric manifold. Let ∇ be the Levi-Civita connection on M with respect to the metric g. Then we have

$$[e_1,e_2]=-ae^z e_2,\quad [e_1,e_3]=-e_1,\quad [e_2,e_3]=-e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula for g, it can be easily calculated that

$$\begin{aligned} &\nabla_{e_1} e_1 = e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = -e_1, \\ &\nabla_{e_2} e_1 = ae^z e_2, && \nabla_{e_2} e_2 = -ae^z e_1 + e_3, && \nabla_{e_2} e_3 = -e_2, \\ &\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0. \end{aligned}$$

We see that the (φ,ξ,η,g) structure satisfies the formula $\nabla_X \xi = \varphi X$ and $\eta(\xi) = -1$. Hence $M(\varphi,\xi,\eta,g)$ is a 3-dimensional LP-Sasakian manifold. It is obvious that the manifold M under consideration is a submanifold of the manifold $\tilde{M}$.

Also $\varphi X = -X \in TM$ for $X \in TM$. Hence M is invariant.

Let $U = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in TM$ and $V = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 \in TM$ where λ_i and μ_i are scalars, $i = 1,2,3$.

Then

$$\begin{aligned} \alpha(U,V) &= \sum \lambda_i \mu_j \alpha(e_i, e_j) \\ &= \sum \lambda_i \mu_j (\tilde{\nabla}_{e_i} e_j - \nabla_{e_i} e_j) \\ &= 0. \end{aligned}$$

Hence the submanifold is totally geodesic.

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