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NEW PROOFS OF MONOTONICITIES OF GENERALIZED WEIGHTED MEAN VALUES

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Abstract. Two new proofs of monotonicities with either r, s or x, y of the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are given.

1. Introduction

Let x, y, r, $s \in \mathbb{R}$, let $p \neq 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval between x and y. Then the generalized weighted mean values $M_{p,f}(r,s;x,y)$ of function f with nonnegative weight p and two parameters r and s are defined in [4, 6] as follows

$$M_{p,f}(r,s;x,y) = \left(\frac{\int_x^y p(u)f^s(u)du}{\int_x^y p(u)f^r(u)du}\right)^{1/(s-r)}, \quad (r-s)(x-y) \neq 0; \tag{1}$$

$$M_{p,f}(r,r;x,y) = \exp\left(\frac{\int_x^y p(u)f^r(u)\ln f(u)du}{\int_x^y p(u)f^r(u)du}\right), \quad x-y \neq 0; \tag{2}$$

$$M(r,s;x,x) = f(x).$$

In [4, 12], using Tchebycheff's integral inequality, it was proved that, if $p \not\equiv 0$ is a nonnegative and continuous function, f a positive, monotonic and continuous function, then $M_{p,f}(r,s;x,y)$ increases with both r and s strictly.

In [13], from Cauchy-Schwarz-Buniakowski inequality, it follows that, the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are increasing strictly with both r and s for any given continuous nonnegative weight p and continuous positive function f.

In [3], it was verified that the generalized weighted mean values $M_{p,f}(r,s;x,y)$ have the same monotonicity with x and y as the continuous positive function f for any continuous nonnegative weight p.

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In [11], another two proofs of monotonicity of variables x and y for the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are given.

Recently, in [1], a new proof of monotonicity of the generalized weighted mean values was given.

The concepts of the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are further generalized to the generalized abstract mean values in [3].

Applications of the generalized weighted mean values $M_{p,f}(r,s;x,y)$ to theory of convex function, to constructing Steffensen pairs, and to theory of gamma functions are researched in [2, 5, 8, 9, 10]. For further information, please see the expository article [7].

In this short note, we will give by double integral method two new proofs of monotonicities with either x, y or r, s of the generalized weighted mean values $M_{p,f}(r,s;x,y)$ stated as follows.

Theorem 1. *Let p and f be nonconstant continuous positive functions defined on the interval between x and y, then the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are increasing strictly with respect to both r and s.*

Theorem 2. *Let p and f be continuous positive functions defined on $\mathbb{R}$. If f is increasing (decreasing), then the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are also increasing (decreasing) with respect to both x and y.*

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Since the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are symmetric on r and s, it suffices to prove its monotonicity of $M_{p,f}(r,s;x,y)$ with respect to s for fixed r. Without loss of generality, assume $x<y$. Taking logarithm on the both sides of (1), we have

$$\ln M_{p,f}(r,s;x,y)=\frac{1}{s-r}\ln\left(\frac{\int_x^y p(u)f^s(u)du}{\int_x^y p(u)f^r(u)du}\right). \tag{3}$$

Differentiating with respect to s on both sides of (3) and rearranging leads to

$$\begin{aligned}\phi(r,s;x,y)&\triangleq\frac{(s-r)^2}{M_{p,f}(r,s;x,y)}\frac{\partial M_{p,f}(r,s;x,y)}{\partial s}\\&=\frac{(s-r)\int_x^y p(u)f^s(u)\ln f(u)du}{\int_x^y p(u)f^s(u)du}-\ln\left(\frac{\int_x^y p(u)f^s(u)du}{\int_x^y p(u)f^r(u)du}\right).\end{aligned} \tag{4}$$

The function $\phi(r,s;x,y)$ has the same sign as $\frac{\partial M_{p,f}(r,s;x,y)}{\partial s}$. Furthermore, direct computing and rearranging yields

$$\frac{\partial\phi(r,s;x,y)}{(s-r)\partial s}=\frac{\int_x^y p(u)f^s(u)(\ln f(u))^2du\int_x^y p(u)f^s(u)du-(\int_x^y p(u)f^s(u)\ln f(u)du)^2}{(\int_x^y p(u)f^s(u)du)^2} \tag{5}$$

$$\triangleq \frac{I}{(\int_x^y p(u)f^s(u)du)^2}.$$

The term I can be expressed by double integral as follows

$$\begin{aligned}I &= \int_x^y p(u)f^s(u)(\ln f(u))^2du \int_x^y p(v)f^s(v)dv \\ &\quad - \int_x^y p(u)f^s(u)\ln f(u)du \int_x^y p(v)f^s(v)\ln f(v)dv \\ &= \int_x^y \int_x^y p(u)p(v)f^s(u)f^s(v)[(\ln f(u))^2 - \ln f(u)\ln f(v)]dudv.\end{aligned} \tag{6}$$

Commuting between u and v, we obtain

$$I = \int_x^y \int_x^y p(v)p(u)f^s(v)f^s(u)[(\ln f(v))^2 - \ln f(v)\ln f(u)]dudv. \tag{7}$$

Adding (6) and (7), we have

$$I = \frac{1}{2}\int_x^y \int_x^y p(v)p(u)f^s(v)f^s(u)[(\ln f(u))^2 + (\ln f(v))^2 - 2\ln f(u)\ln f(v)]dudv > 0. \tag{8}$$

Combination of (5) and (8) impleis that the function $\frac{\partial\phi(r,s;x,y)}{\partial s}$ has a zero $s_0 = r$ uniquely, and then $\frac{\partial\phi(r,s;x,y)}{\partial s}$ is negative for $s < r$ and positive for $s > r$, therefore $\phi(r, s; x, y)$ takes its minimum $\phi(r, r; x, y) = 0$ at $s = r$, this is equivalent to $\frac{\partial M_{p,f}(r,s;x,y)}{\partial s} > 0$ for $s \neq r$. Hence the generalized weighted mean values $M_{p,f}(r, s; x, y)$ increases with respect to s strictly. So does $M_{p,f}(r, s; x, y)$ with respect to r, since $M_{p,f}(r, s; x, y) = M_{p,f}(s, r; x, y)$.

Proof of Theorem 2. Since the generalized weighted mean values $M_{p,f}(r, s; x, y)$ are symmetric with respect to variables x and y, it suffices to prove its monotonicity of $M_{p,f}(r, s; x, y)$ with respect to y for fixed x. Without loss of generality, assume $r < s$. A simple calculation yields

$$\frac{\partial \ln M_{p,f}(r,s;x,y)}{\partial y} = \frac{p(y)}{s-r}\left(\frac{f^s(y)}{\int_x^y p(u)f^s(u)du} - \frac{f^r(y)}{\int_x^y p(u)f^r(u)du}\right). \tag{9}$$

Define for $t \in \mathbb{R}$

$$J(t) = \frac{f^t(y)}{\int_x^y p(u)f^t(u)du}. \tag{10}$$

Then we have

$$J'(t) = \frac{f^t(y)\int_x^y p(u)f^t(u)[\ln f(y) - \ln f(u)]du}{(\int_x^y p(u)f^t(u)du)^2}. \tag{11}$$

It is easy to see that, if f is increasing, then $J'(t)$ is nonnegative, so is $\frac{\partial \ln M_{p,f}(r,s;x,y)}{\partial y}$. Thus $\ln M_{p,f}(s, r; x, y)$, and $M_{p,f}(s, r; x, y)$, increases with respect to y. If f is deceasing, then $J'(t)$ is nonpositive, so is $\frac{\partial \ln M_{p,f}(r,s;x,y)}{\partial y}$, and then $\ln M_{p,f}(s, r; x, y)$, and $M_{p,f}(s, r; x, y)$, decreases with respect to y.

The similar argument can be applied to the variable x, since $M_{p,f}(r,s;x,y) = M_{p,f}(r,s;y,x)$.

The proof is complete.

References

[1] Ch.-P. Chen and F. Qi, *A new proof for monotonicity for the generalized weighted mean values*, Adv. Stud. Contemp. Math. (Kyungshang) **6**(2003), 13-16.

[2] B.-N. Guo and F. Qi, *Inequalities for generalized weighted mean values of convex function*, Math. Inequal. Appl. **4**(2001), No.2, 195-202. RGMIA Res. Rep. Coll. **2**(1999), No.7, Art.11, 1059-1065. Available online at http://rgmia.vu.edu.au/v2n7.html.

[3] F. Qi, *Generalized abstracted mean values*, J. Inequal. Pure Appl. Math. **1**(2000), No.1, Art.4. Available online at http://jipam.vu.edu.au/vin1/013_99.html. RGMIA Res. Rep. Coll. **2**(1999), No.5, Art.4, 633-642. Available online at http://rgmia.vu.edu.au/v2n5.html.

[4] F. Qi, *Generalized weighted mean values with two parameters*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **454**(1998), 2723-2732.

[5] F. Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5**(2002), No.1, 61-67. RGMIA Res. Rep. Coll. **2**(1999), No.7, Art.7. Available online at http://rgmia.vu.edu.au/v2n7.html.

[6] F. Qi, *On a two-parameter family of nonhomogeneous mean values*, Tamkang J. Math. **29**(1998), 155-163.

[7] F. Qi, *The extended mean values: definition, properties, monotonicities, comparison, convexities, generalizations, and applications*, Cubo Math. Educ. **5**(2003), no.3, 63-69. RGMIA Res. Rep. Coll. **5**(2002), No.1, Art.5. Available online at http://rgmia.vu.edu.au/v5n1.html.

[8] F. Qi and J.-X. Cheng, *Some new Steffensen pairs*, Anal. Math. **29**(2003), No.3, 219-226. RGMIA Res. Coll. **3**(2000), No.3, Art.11, 431-436. http://rgmia.vu.edu.au/v3n3.html.

[9] F. Qi, J.-X. Cheng and G. Wang, *New Steffensen pairs*, Inequality Theory and Applications **1**(2001), 273-279. Y. J. Cho, J. K. Kim and S. S. Dragomir (Ed.), Nova Science Publishers.

[10] F. Qi and B.-N. Guo, *On Steffensen pairs*, J. Math. Anal. Appl. **271**(2002), No.2, 534-541. RGMIA Res. Rep. Coll. **3**(2000), No.3, Art.10, 425-430. Available online at http://rgmia.vu.edu.au/v3n3.html.

[11] F. Qi, J.-Q. Mei, D.-F. Xia and S.-L. Xu, *New proofs of weighted power mean inequalities and monotonicity for generalized weighted mean values*, Math. Inequal. Appl. **3**(2000), 377-383.

[12] F. Qi. J.-Q. Mei and S.-L. Xu, *Other proofs of monotonicity for generalized weighted mean values*, RGMIA Res. Rep. Coll. **2**(1999), No.4, Art.6, 469-472. Available online at http://rgmia.vu.edu.au/v2n4.html.

[13] F. Qi and Sh.-Q. Zhang, *Note on monotonicity of generalized weighted mean values*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455**(1999), 3259-3260.

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