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ON THE FIRST PROLONGATIONAL LIMIT SET OF FLOWS OF FREE MAPPINGS

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Abstract. We consider an equivalence relation defined for a given flow of the plane which has no fixed points. We prove that the first prolongational limit set of every point from the boundary of an equivalence class is contained in the boundary of this class. Moreover, if two points lying on two different boundary orbits of a class are contained in the same component of the complement of an orbit of this class, then each of these two points is contained in the first prolongational limit set of the other one.

1. Introduction

We assume that f is a *free mapping*, i.e. a homeomorphism of the plane onto itself without fixed points which preserves orientation. We consider a relation in $\mathbb{R}^2$ defined in the following way:

$p \sim q$ if $p = q$ or p and q are endpoints of some arc K for which $f^n(K) \to \infty$ as $n \to \pm\infty$.

By an *arc* K with endpoints p and q we mean the image of a homeomorphism $c : [0,1] \to c([0,1])$ satisfying conditions $c(0) = p$, $c(1) = q$, where the topology on $c([0,1])$ is induced by the topology of $\mathbb{R}^2$. It turns out that the relation defined above is an equivalence relation (see [5]) and has the same equivalence classes as the relation defined by S. Andrea in [1].

From now on we assume that f is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. It follows from the Jordan theorem that each orbit C of $\{f^t : t \in \mathbb{R}\}$ divides the plane into two simply connected regions. Note that each of them is invariant under f^t for $t \in \mathbb{R}$. Thus two different orbits C_p and C_q of points p and q, respectively, divide the plane into three simply connected invariant regions, one of which contains both C_p and C_q in its boundary. We will call this region by the *strip* between C_p and C_q and denote by D_{pq}.

For any distinct orbits C_{p_1}, C_{p_2}, C_{p_3} of $\{f^t : t \in \mathbb{R}\}$ one of the following two possibilities must be satisfied: exactly one of the orbits C_{p_1}, C_{p_2}, C_{p_3} is contained in the strip between the other two or each of the orbits C_{p_1}, C_{p_2}, C_{p_3} is contained in the strip between the other two. In the first case if C_{p_j} is the orbit which lies in the strip between C_{p_i} and C_{p_k} we will write $C_{p_i}|C_{p_j}|C_{p_k}$ ($i, j, k \in \{1,2,3\}$ and i, j, k are different). In the second case we will write $|C_{p_i}, C_{p_j}, C_{p_k}|$ (see [3]).

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Put

$$J^+(q) := \{p \in \mathbb{R}^2 : \text{ there exist a sequence} (q_n)_{n\in\mathbb{N}} \text{ and a sequence } (t_n)_{n\in\mathbb{N}} \text{ such that } q_n \to q, t_n \to +\infty, f^{t_n}(q_n) \to p \text{ as } n \to +\infty\},$$

$$J^-(q) := \{p \in \mathbb{R}^2 : \text{ there exist a sequence} (q_n)_{n\in\mathbb{N}} \text{ and a sequence } (t_n)_{n\in\mathbb{N}} \text{ such that } q_n \to q, t_n \to -\infty, f^{t_n}(q_n) \to p \text{ as } n \to +\infty\}.$$

The set $J(q) := J^+(q) \cup J^-(q)$ is called the *first prolongational limit set* of q. Let us observe that $p \in J(q)$ if and only if $q \in J(p)$ for any $p, q \in \mathbb{R}^2$. For a subset $H \subset \mathbb{R}^2$ we define

$$J(H) := \bigcup_{q\in H} J(q).$$

One can observe that for each $p \in \mathbb{R}^2$ the set $J(p)$ is invariant.

In this paper we study the structure of the first prolongational limit set $J(\mathbb{R}^2)$ of a flow of free mappings. To this end we use equivalence classes of the equivalence relation defined above. These results can be useful in finding maximal parallelizable domains of such flows, because the boundary of each maximal parallelizable domain is contained in $J(\mathbb{R}^2)$ (see [9]).

Now we collect the results from [5], [6], [7] and [8] which are needed in our considerations.

Proposition 1.1 (see [5]) *Each equivalence class is invariant under* f^t *for* $t \in \mathbb{R}$. *In particular, each orbit of* $\{f^t : t \in \mathbb{R}\}$ *is contained in exactly one equivalence class of the relation* $\sim$.

Proposition 1.2. (see [8]) *The boundary of each equivalence class is a union of a family of orbits. In particular, the boundary of each class is invariant.*

Proposition 1.3. (see [7]) *If* $q \in \operatorname{int} G_0$ *for an equivalence class* G_0, *then* $q \notin J(\mathbf{R}^2)$, *i.e.* $J(q) = \emptyset$.

Proposition 1.4. (see [6]) *Let* $p \in \operatorname{fr} G_0$ *and* $q \in G_0 \setminus C_p$ *for an equivalence class* G_0. *Let* K *be an arc with endpoints* p *and* q *such that* $K \setminus \{p, q\} \subset D_{pq}$ *and* $C_p \mid C_r \mid C_q$ *for every* $r \in K \setminus \{p, q\}$. *Then* $r \in G_0$ *for every* $r \in K \setminus \{p\}$. *Moreover,* $C_p \mid C_s \mid C_q$ *for every* $s \in D_{pq} \cap G_0$.

Proposition 1.5. (see [6]) *Let* $p \in \operatorname{fr} G_0$ *for an equivalence class* G_0. *Then for every class* G *such that* $G \setminus C_p \neq \emptyset$ *the set* $\operatorname{cl} G \setminus C_p$ *is contained in exactly one of the components of* $\mathbb{R}^2 \setminus C_p$.

Proposition 1.6. (see [8]) *Let* C_1, C_2 *be any orbits of* $\{f^t : t \in \mathbb{R}\}$. *If* C_1, C_2 *are contained in an equivalence class, then each point of the strip between* C_1 *and* C_2 *belongs to the interior of this class.*

Proposition 1.7. (see [5]) *For any orbit* C_1, C_2, C_3 *of* $\{f^t : t \in \mathbb{R}\}$, *if* $|C_1, C_2, C_3|$, *then* C_1, C_2, C_3 *cannot be contained in the same equivalence class of* $\sim$.

2. First Prolongational Limit Set of a Boundary Point of a Class

In this section we prove that the first prolongational limit set of every point from the boundary of a class is contained in the boundary of this class.

Theomrem 2.1. *Let G_0 be an equivalence class which does not consist of just one orbit. Let H_0 the component of $\mathbb{R}^2 \setminus C_p$ which contains $\operatorname{cl} G_0 \setminus C_p$. Assume that $p \in \operatorname{fr} G_0$, $q \in H_0$ and $q \in J(p)$. Then $q \in \operatorname{cl} G_0$.*

Proof. Denote by L the straight line segment with endpoints q and p. Let p_0 be the first point of L belonging to C_p. Then by Proposition 1.2 we have $p_0 \in \operatorname{fr} G_0$. Moreover, $p_0 \in J(q)$, since $p_0 \in C_p$, $p \in J(q)$ and $J(q)$ is invariant. Let L_0 denote the subsegment of L having q and p_0 as its endpoints.

Let q_0 be the first point of L_0 belonging to $\operatorname{cl} G_0$. Such a point q_0 exists on account of the Weierstrass theorem, since $L_0 \cap \operatorname{cl} G_0$ is a compact set. Now we will prove that $q_0 \in \operatorname{fr} G_0$. If $s \in L_0 \cap \operatorname{int} G_0$, then there exists a ball $B(s,\varepsilon)$ centered at s with a radius $\varepsilon > 0$ such that $\operatorname{cl} B(s,\varepsilon) \subset \operatorname{int} G_0$. Hence s cannot be the first point of L_0 belonging to $\operatorname{cl} G_0$, and consequently $s \neq q_0$. Thus $q_0 \in \operatorname{fr} G_0$, since $q_0 \in L_0 \cap \operatorname{cl} G_0$ and $s \neq q_0$ for all $s \in L_0 \cap \operatorname{int} G_0$.

Now we will show that $q_0 \notin C_p$, i.e. $q_0 \neq p_0$. On account of the Whitney-Bebutov theorem (see [2], p. 52) there exists a neighbourhood V_1 of p which is a union of orbits such that $\{f^t|_{V_1} : t \in \mathbb{R}\}$ is conjugate with the flow of translations, i.e. there exists a homeomorphism $\psi : V_1 \to (-\infty,+\infty) \times (-1,1)$ such that $\psi(p) = (0,0)$ and for every orbit C contained in V_1 the image of C under ψ is a horizontal straight line in the open set $(-\infty,+\infty) \times (-1,1)$. Then $J(V_1) \cap V_1 = \emptyset$ (see [2], p. 46 and 49), and consequently $q \notin V_1$.

Since $p \in \operatorname{fr} G_0$ and G_0 does not consist of just one orbit, there exists a point $r_1 \in V_1 \cap H_0$ such that $r_1 \in G_0$. Let K_1 be a section of V_1 passing through p_0 and r_1 (as K_1 we can take the preimage of the straight line containing $\psi(p_0)$ and $\psi(r_1)$). Then by Proposition 1.4, $s \in G_0$ for every $s \in K_1 \cap D_{r_1 p}$. Hence by Proposition 1.1 we have $V_1 \cap D_{r_1 p} \subset G_0$. Let B_1 be the preimage of a closed ball with center at $\psi(p_0)$ contained in $(-\infty,+\infty) \times (-1,1)$ which has no common point with $(-\infty,+\infty) \times \{\psi(r_1)\}$. Then $B_1 \cap H_0 \subset V_1 \cap D_{r_1 p}$ and $\operatorname{fr} B_1$ is a Jordan curve having exactly two common points with C_p. Denote these points by z_1 and z_2.

Since $q \notin V_1$ and $B_1 \subset V_1$, we have $q \notin B_1$. Hence the endpoint q of L_0 lies outside of the Jordan curve $\operatorname{fr} B_1$, whereas the second endpoint p_0 of L_0 lies inside of $\operatorname{fr} B_1$. Thus by the Jordan Theorem there exists a point $x_1 \in L_0 \cap \operatorname{fr} B_1$. By the fact that p_0 is the only point of L_0 belonging to C_p, we have $x_1 \neq z_1$ and $x_1 \neq z_2$. Since $L_0 \subset \operatorname{cl} H_0$, the point x_1 belongs to the arc of $\operatorname{fr} B_1$ having z_1 and z_2 as its endpoints which is contained in $\operatorname{cl} H_0$. Since every point of this arc different from z_1 and z_2 belongs to $V_1 \cap D_{r_1 p}$ and $V_1 \cap D_{r_1 p} \subset G_0$, we have $x_1 \in G_0$. Hence q_0 belongs to the subarc of L_0 having q and x_1 as its endpoints, since q_0 is the first point of L_0 belonging to $\operatorname{cl} G_0$. Thus $q_0 \notin C_p$, and consequently $q_0 \in \operatorname{cl} G_0 \setminus C_p$.

Now we show that $q = q_0$. Suppose, on the contrary, that $q \neq q_0$. Since $q_0 \in \operatorname{fr} G_0$, we have by Proposition 1.5 that the set $\operatorname{cl} G_0 \setminus C_{q_0}$ is contained in exactly one of the components of $\mathbb{R}^2 \setminus C_{q_0}$. Denote this component by F_0. In particular, $p \in F_0$, since $q_0 \notin C_p$. In the same way as before we can find a neighbourhood V_2 of q_0 which is a union of orbits such that $\{f^t|_{V_2} : t \in \mathbb{R}\}$ is conjugate with the flow of translations and a homeomorphic preimage B_2 of a closed ball such that $B_2 \cap F_0 \subset V_2 \cap D_{r_2 q_0} \subset G_0$ for some $r_2 \in V_2 \cap F_0 \cap G_0$, $q_0 \in \operatorname{int} B_2$, $q \notin B_2$ and $\operatorname{fr} B_2$ is a Jordan curve having exactly two common points with C_{q_0}. Denote these points by y_1 and y_2.

Denote by L_1 the subarc of L_0 having q and q_0 as its endpoints. By the Jordan Theorem, there exists a point $x_2 \in L_1 \cap \operatorname{fr} B_2$. It follows from Propositions 1.1 and 1.2 that $\operatorname{cl} G_0$ is a union of orbits. Hence by the fact that $y_1, y_2 \in C_{q_0}$, we have $y_1, y_2 \in \operatorname{cl} G_0$. Since q_0 is the only point of L_1 belonging to $\operatorname{cl} G_0$ and $q_0 \in \operatorname{int} B_2$, we have $x_2 \neq y_1$ and $x_2 \neq y_2$. Moreover, the point q_0 is the first point of L_1 belonging to C_{q_0}, since it is the first point of L_1 belonging to $\operatorname{cl} G_0$. Hence $L_1 \setminus \{q_0\}$ is contained in one of the components of $\mathbb{R}^2 \setminus C_{q_0}$. By the definition of $J(p)$ we have $J(p) \subset F_0 \cup C_{q_0}$. Since $p \in F_0$ and F_0 is invariant, we get from the assumption $q \in J(p)$ that $q \in F_0 \cup C_{q_0}$. Consequently $L_1 \setminus \{q_0\} \subset F_0$. Hence $x_2 \in K_2 \setminus \{y_1, y_2\}$, where K_2 is the arc of $\operatorname{fr} B_2$ having y_1 and y_2 as its endpoints which is contained in F_0. Since this arc is a subset of G_0, we have $x_2 \in G_0$. But this contradicts the fact that q_0 is the first point of L_1 belonging to $\operatorname{cl} G_0$. Thus $q = q_0$, and consequently $q \in \operatorname{cl} G_0$.

Corollary 2.2. *Let G_0 be an equivalence class which does not consist of just one orbit. Let H_0 be the component of $\mathbb{R}^2 \setminus C_p$ which contains $\operatorname{cl} G_0 \setminus C_p$. Assume that $p \in \operatorname{fr} G_0$, $q \in H_0$ and $q \in J(p)$. Then $q \in \operatorname{fr} G_0$.*

Proof. By Theorem 2.1, we have that $q \in \operatorname{cl} G_0$. On account of Proposition 1.3 the point q cannot belong to $\operatorname{int} G_0$. Thus $q \in \operatorname{fr} G_0$.

3. Properties of Boundary Orbits

In this section we prove that if two points lying on two different boundary orbits of a class are contained in the same component of the complement of an orbit of this class, then each of these two points is contained in the first prolongational limit set of the other one.

Proposition 3.1. *Let G_0 be an equivalence class which does not consist of just one orbit. Let $p \in \operatorname{fr} G_0$, $q \in \operatorname{fr} G_0$ and $C_p \neq C_q$. Assume that p and q belong to the same component of $\mathbb{R}^2 \setminus C_{r_0}$ for an $r_0 \in G_0$. Then p and q belong to the same component of $\mathbb{R}^2 \setminus C_r$ for every $r \in \operatorname{int} G_0$.*

Proof. Fix an $r \in \operatorname{int} G_0 \setminus C_{r_0}$. Denote by H_0 the component of $\mathbb{R}^2 \setminus C_{r_0}$ which contains p and q. If $r \notin H_0$, then p and q belong to the component of $\mathbb{R}^2 \setminus C_r$ which contains r_0. Now assume that $r \in H_0$. Then by Proposition 1.6 neither p nor q belongs to $D_{r r_0}$, since $p \notin \operatorname{int} G_0$ and $q \notin \operatorname{int} G_0$. Thus p and q belong to the same component of $\mathbb{R}^2 \setminus C_r$, since $p, q \in H_0 \setminus (D_{r r_0} \cup C_r)$.

Theorem 3.2. *Let G_0 be an equivalence class which does not consist of just one orbit. Let $p \in \operatorname{fr} G_0$, $q \in \operatorname{fr} G_0$ and $C_p \neq C_q$. Assume that p and q belong to the same component of $\mathbb{R}^2 \setminus C_r$ for an $r \in G_0$. Then $p \in J(q)$.*

Proof. First we prove that $p \notin G_0$ and $q \notin G_0$. Suppose, on the contrary, that one of the points p, q belongs to G_0. Let $p \in G_0$. By Proposition 1.5, q and r are contained in the same component of $\mathbb{R}^2 \setminus C_p$. On the other hand, by our assumption, q and p are contained in the same component of $\mathbb{R}^2 \setminus C_r$. Thus $q \in D_{pr}$. Hence by Proposition 1.6 $q \in \operatorname{int} G_0$, since $p \in G_0$. But this contradicts the assumption that $q \in \operatorname{fr} G_0$.

On account of the Whitney-Bebutov theorem (see [2], p. 52) there exists a neighbourhood V_1 of p which is a union of orbits such that $\{f^t|_{V_1} : t \in \mathbb{R}\}$ is conjugate with the flow of translations, i.e. there exists a homeomorphism ψ_1 of V_1 onto $(-\infty,+\infty)\times(-1,1)$ such that $\psi_1(p)=(0,0)$ and for every orbit C contained in V_1 the image of C under ψ_1 is a horizontal straight line in the open set $(-\infty,+\infty)\times(-1,1)$. Without loosing of generality we can assume that $r\notin V_1$. Since $p\in \operatorname{fr} G_0$, there exists a point $s_1\in V_1$ such that $s_1\in G_0$. Then by Proposition 1.5 s_1 and r are contained in the same component of $\mathbb{R}^2\setminus C_p$.

Take as K_0 the image under ψ_1^{-1} of the segment in $(-\infty,+\infty)\times(-1,1)$ from $\psi_1(s_1)$ to $\psi_1(p)$. Then K_0 has just one common point with every orbit contained in $D_{ps_1}\cap V_1$, $K_0\setminus\{p,s_1\}\subset D_{ps_1}$ and $C_p\mid C_x\mid C_{s_1}$ for all $x\in K_0\setminus\{p,s_1\}$. Then by Proposition 1.4 we have $K_0\setminus\{p\}\subset G_0$. In the same way we can find a neighbourhood V_2 of q which is a union of orbits such that $\{f^t|_{V_2}\colon t\in\mathbb{R}\}$ is conjugate with the flow of translations and $r\notin V_2$, a point $s_2\in V_2\cap G_0$ and an arc L_0 from s_2 to q such that L_0 has just one common point with every orbit contained in $D_{qs_2}\cap V_2$, $L_0\setminus\{q,s_2\}\subset D_{qs_2}$ and $L_0\setminus\{q\}\subset G_0$.

Fix $\varepsilon_1,\varepsilon_2>0$ such that $\operatorname{cl}B(p,\varepsilon_1)\cap \operatorname{cl}B(q,\varepsilon_2)=\emptyset$, $s_1,s_2\notin \operatorname{cl}B(p,\varepsilon_1)$, $s_1,s_2\notin \operatorname{cl}B(q,\varepsilon_2)$, $\operatorname{cl}B(p,\varepsilon_1)\cap C_r=\emptyset$, $\operatorname{cl}B(q,\varepsilon_2)\cap C_r=\emptyset$, $\operatorname{cl}B(p,\varepsilon_1)\cap C_q=\emptyset$ and $\operatorname{cl}B(q,\varepsilon_2)\cap C_p=\emptyset$. Denote by p_0 the last point of K_0 which belongs to $\operatorname{fr}B(p,\varepsilon_1)$ and let K_1 be the subarc of K_0 having p_0 and p as its endpoints. Then $K_1\setminus\{p\}\subset G_0$ and $K_1\setminus\{p_0\}\subset B(p,\varepsilon_1)$. Take a positive $\varepsilon_3\le\varepsilon_2$ such that $\operatorname{cl}B(q,\varepsilon_3)\cap C_{p_0}=\emptyset$. Denote by q_0 the last point of L_0 which belongs to $\operatorname{fr}B(q,\varepsilon_3)$. Let L_1 be the subarc of L_0 having q_0 and q as its endpoints. Then $L_1\setminus\{q\}\subset G_0$ and $L_1\setminus\{q_0\}\subset B(q,\varepsilon_3)$.

On account of Proposition 1.7 the relation $\cdot\mid\cdot\mid\cdot$ holds for the orbits C_{p_0},C_{q_0},C_r, since $p_0,q_0,r\in G_0$. Suppose that $C_{p_0}\mid C_{q_0}\mid C_r$. Then $\operatorname{cl}B(q,\varepsilon_3)\subset D_{rp_0}$, since $\operatorname{cl}B(q,\varepsilon_3)\cap C_{p_0}=\emptyset$, $\operatorname{cl}B(q,\varepsilon_3)\cap C_r=\emptyset$ and $q_0\in \operatorname{cl}B(q,\varepsilon_3)$. Hence by Proposition 1.6 we have $q\in \operatorname{int}G_0$, which is impossible. Consequently, $C_{q_0}\mid C_{p_0}\mid C_r$, since q_0 and p_0 belong to the same component of $\mathbb{R}^2\setminus C_r$.

From the definition of p_0 and q_0 it follows that p, q, p_0 and q_0 belong to the same component of of $\mathbb{R}^2\setminus C_r$. Suppose that $p\in D_{rq_0}$. Then by Proposition 1.6 we have $p\in \operatorname{int}G_0$, which is impossible. Thus $p\notin D_{rq_0}$. Moreover, $p\notin C_{q_0}$, since $p\notin G_0$ and $q_0\in G_0$. Consequently p belongs to the component of $\mathbb{R}^2\setminus C_{q_0}$ which does not contain r. Thus p_0 and p lies in different components of $\mathbb{R}^2\setminus C_{q_0}$, which means that $C_p\mid C_{q_0}\mid C_{p_0}$. In the same way we can show that $C_q\mid C_{q_0}\mid C_{p_0}$. Hence L_1 and p are contained in the same component of $\mathbb{R}^2\setminus C_{p_0}$. On the other hand, by Proposition 1.5, L_1 and p_0 are contained in the same component of $\mathbb{R}^2\setminus C_p$, since $L_1\subset \operatorname{cl}G_0$, $p_0\in G_0$ and $q\notin C_p$. Thus $L_1\subset D_{pp_0}$.

Take a sequence $(q_n)_{n\in\mathbb{N}}$ such that $q_n\to q$ as $n\to\infty$, $q_n\neq q$ and $q_n\in L_1$ for all n. Then $q_n\in D_{pp_0}\cap G_0$, since $L_1\subset D_{pp_0}$ and $L_1\setminus\{q\}\subset G_0$. By Proposition 1.4 we have $C_{p_0}\mid C_{q_n}\mid C_p$. Hence $C_{q_n}\cap K_1\neq\emptyset$ for every n. Since K_1 has at most one common point with each orbit, the set $C_{q_n}\cap K_1$ consists of just one point for every n. Denote this point by p_n. Thus there exists a sequence (t_n) such that $p_n=f^{t_n}(q_n)$. Since $\operatorname{cl}B(p,\varepsilon_1)\cap \operatorname{cl}B(q,\varepsilon_3)=\emptyset$, we have $K_1\cap L_1=\emptyset$. Hence either $t_n>0$ for all n or $t_n<0$ for all n. We assume without loosing of generality that $t_n>0$ for all n.

Now we show that $p_n\to p$ as $n\to\infty$. Fix an $\varepsilon>0$. Let $\varepsilon\le\varepsilon_1$. Denote by z the last point of K_1 which belongs to $\operatorname{fr}B(p,\varepsilon)$. Denote by K_2 the subarc of K_1 having z and p as its endpoints. Then $K_2\setminus\{z\}\subset B(p,\varepsilon)$. Let $V_3:=V_2\cap H_z$, where H_z denotes the component of $\mathbb{R}^2\setminus C_z$ which contains q. Since V_3 is a neighbourhood of q and $q_n\to q$ as $n\to\infty$, there exists an n_0 such that $q_n\in V_3$ for all $n>n_0$. In the same way as before (taking z instead of p_0 and q_n instead of

q_0) we get that $C_z \mid C_{q_n} \mid C_q$ and $C_z \mid C_{q_n} \mid C_p$. Hence $p \in H_z$ and consequently $K_2 = K_1 \cap H_z$. Thus for all $n > n_0$ we have $p_n \in K_2$, since $C_{q_n} \in H_z$ for all $n > n_0$ and $p_n \in C_{q_n}$. Hence by the fact that $K_2 \setminus \{z\} \subset B(p,\varepsilon)$ we have $p_n \in B(p,\varepsilon)$ for all $n > n_0$.

To finish the proof we show that there exists a subsequence (t_{n_k}) of the sequence (t_n) which tends to $+\infty$ as $n_k \to \infty$. Suppose, on the contrary, that there exists a positive integer α such that $t_n \le \alpha$ for all n. Denote by J the Jordan curve which is a union of L_1, $f^{\alpha}(L_1)$, the subarc of C_{q_0} with endpoints q_0 and $f^{\alpha}(q_0)$ and the subarc of C_q with endpoints q and $f^{\alpha}(q)$. Let $B := J \cup \operatorname{ins} J$, where by $\operatorname{ins} J$ we mean the bounded component of $\mathbb{R}^2 \setminus J$. Then B is a closed set and $f^{t_n}(q_n) \in B$ for every n. Hence by the fact that $p = \lim_{n\to\infty} f^{t_n}(q_n)$ we have $p \in B$. But this is impossible, since $B \subset G_0 \cup C_q$ and $(G_0 \cup C_q) \cap C_p = \emptyset$. Thus there exists a subsequence (t_{n_k}) of the sequence (t_n) such that $\lim_{k\to\infty} t_{n_k} = +\infty$. Hence $\lim_{k\to\infty} q_{n_k} = q$ and $\lim_{k\to\infty} f^{t_{n_k}}(q_{n_k}) = p$, since $\lim_{n\to\infty} q_n = q$ and $\lim_{n\to\infty} f^{t_n}(q_n) = p$. Consequently $p \in J(q)$.

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