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θ_σ-SUMMABLE SEQUENCES AND SOME MATRIX TRANSFORMATIONS

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Abstract. In this paper we introduce θ_σ-conservative and θ_σ-regular matrices and also give matrices transformation from almost convergent sequence spaces into lacunary invariant convergent sequence spaces.

1. Introduction

Let ℓ_∞ and c denote the Banach spaces of real bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$, respectively.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or σ-mean if and only if (i) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n, (ii) $\phi(e) \geq 0$, where $e = (1, 1, 1, \ldots)$ and, (iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$. For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space c, in sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-means are equal.

When $\sigma(n) = n + 1$, the σ-means are the classical Banach limits on ℓ_∞ and V_σ reduces to $\hat{c}$, the space all almost convergent sequences (see, Lorentz [4]). If $A = (a_{nk})$ is an infinite matrix of complex numbers such that $A_n(x) = \sum_k a_{nk}x_k$ is an almost convergent sequence for every convergent sequence $x = (x_k)$, A is said to be an almost conservative matrix (see, King [3]). When the common value of all Banach limits of $A_n(x)$ is $\lim x$ for all $x \in c$, then the almost conservative matrix A is said to be almost regular.

After, Schaefer [8] defined σ-conservative and σ-regular as following:
An infinite matrix A is said to be σ-conservative if and only if $Ax = \{\sum_k a_{nk}x_k\}_{n\in N} \in V_\sigma$ for all $x \in c$. An infinite matrix A is said to be σ-regular if and only if it is σ-conservative and $\sigma - \lim Ax = \lim x$ for all $x \in c$. The necessary and sufficient conditions for a matrix which is σ-conservative or σ-regular were given by Schacfer [8].

After, Mursaleen [5] gave absolute σ-conservative and absolute σ-regular matrices.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals

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determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al [2] as:

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \quad \text{for some } l \right\}$$

Recently, the concept of lacunary σ-convergence was introduced by Savas [7], which is generalization of the idea of lacunary strong almost convergence due to Das and Mishra [1].

The sequence $x = (x_k)$ is said to be lacunary σ-convergent if $\lim_r t_{rn}(x)$ exists uniformly in n, where

$$t_{rn}(x) = \frac{1}{h_r} \sum_{k \in I_r} x_{\sigma^k(n)}.$$

After that, the lacunary σ-convergence is going to be called as θ_σ-convergent. The spaces of all θ_σ-convergent sequence will be denoted by N_θ^σ.

Quite recently, Nuray [6] introduced the concept θ-almost convergent and defined θ-almost conservative and θ-almost regular as following:

The A is said to be θ-almost conservative if $x \in c$ implies that the A-transform of x is θ-almost convergent. A is said to be θ-almost regular if the A-transform of x is θ-almost convergent to the limit of x for each $x \in c$. Also, the necessary and sufficient for these the matrix gave by Nuray [6].

In the sequel the following notation is used: C denotes the complex numbers and N denotes positive integers. The linear spaces of all continuous linear functional on c is denoted by c^*. We use the special sequences, $e = (1, 1, 1, \ldots)$, $e_k = (0, 0, 0, \ldots, 1, \ldots, 0, 0, 0, \ldots)$ (with 1 in rank k) and $\Delta = \{e, e_0, e_1, \ldots\}$.

Now we give the definitions of θ_σ-conservative and θ_σ-regular matrices and characterize the class $A \in (\hat{c}, N_\theta^\sigma)$.

2. Main Results

The following notations are used throughout this paper. Let

$$t_{rn}(x) = T_{rn}(Ax) = \sum_{k=0}^{\infty} a(r, n, k) x_k,$$

where

$$a(r, n, k) = \frac{1}{h_r} \sum_{j \in I_r} a_{\sigma^j(n), k}$$

Definition 2.1. The matrix A is said to be θ_σ-conservative if $x \in c$ implies that the A-transform of x is θ_σ-convergent. A is said to be θ_σ-regualr if the A-transform of x is θ_σ-convergent to the limit of x for each $x \in c$.

Theorem 2.1. *Let $A = (a_{nk})$ be an infinite matrix and let $\theta = (k_r)$ be a lacunary sequence. Then the matrix A is θ_σ-conservative if and only if*

(i) $\sup_{r,n}\{\sum_{k=0}^{\infty}|a(r,n,k)|\}<\infty$

(ii) *there exists an* $\alpha\in C$ *such that* $\lim_r\sum_{k=0}^{\infty}a(r,n,k)=\alpha$ *uniformly in* n, *and*

(iii) *there exists an* $\alpha_k\in C$, $k=0,1,2,\ldots$ *such that* $\lim_r a(r,n,k)=\alpha_k$ *uniformly in* n.

Proof. Suppose that A is θ_σ-conservative for all n. Let

$$t_{rn}(x)=\sum_{k=0}^{\infty}a(r,n,k)x_k$$

We can write

$$|t_{rn}(x)|\leq\sum_{k=0}^{\infty}|a(r,n,k)|\,\|x\|$$

Since $t_{rn}(x)$ is the linear functional on c, hence $t_{rn}\in c^*$. Since A is θ_σ-conservative $\lim_{r\to\infty}t_{rn}(x)=t(x)$ uniformly in n. It follows that $\{t_{rn}(x)\}_{r\in N}$ is bounded for $x\in c$ and all n. Hence $\{\|t_{rn}\|\}$ is bounded by uniform boundedness principle. For each $p\in N$, define the sequence $u=(u_k)$ by

$$u_k=\begin{cases}sign\,a(r,n,k); & 0\leq k\leq p\\ 0; & p>k\end{cases}$$

Then $u\in c$, $\|u\|=1$ for all n, and

$$t_{rn}(u)=\sum_{k=0}^{p}|a(r,n,k)|.$$

Hence $|t_{rn}(u)|\leq\|t_{rn}\|\,\|u\|=\|t_{rn}\|$. Therefore $\sum_{k=0}^{\infty}|a(r,n,k)|\leq\|t_{rn}\|$, so that (i) follows.

Since e and e_k are convergent sequences, $k=0,1,2,\ldots$, $\lim_{r\to\infty}t_{rn}(e)$ and $\lim_{r\to\infty}t_{rn}(e_k)$ must exist uniformly in n. Hence (ii) and (iii) must hold.

Now suppose that (i)-(iii) hold. Put

$$t_{rn}(x)=\sum_{k=0}^{\infty}a(r,n,k)x_k.$$

Then we can write, for all n,

$$|t_{rn}(x)|\leq\sum_{k=0}^{\infty}|a(r,n,k)|\,\|x\|.$$

Therefore $|t_{rn}(x)|\leq R_n\|x\|$ by (i), where R_n is a constant independent of r. Hence $t_{rn}\in c^*$ and the sequence $\{\|t_{rn}\|\}$ is bounded for each n. So, (ii) and (iii) imply that $\lim_{r\to\infty}t_{rn}(e)$ and $\lim_{r\to\infty}t_{rn}(e_k)$ exist for n, $k=0,1,2,\ldots$. Since $\{e,e_0,e_1,\ldots\}$ is a fundamental set in c, it follows that $\lim_{r\to\infty} t_{rn}(x) = t_n(x)$ exists and $t_n \in c^*$. Therefore t_n has the form

$$t_n(x) = \lambda \left[t_n(e) - \sum_{k=0}^{\infty} t_n(e_k) \right] + \sum_{k=0}^{\infty} x_k t_n(e_k)$$

where $\lambda = \lim_k x_k$. But $t_n(e) = \alpha$ and $t_n(e_k) = \alpha_k$, $k = 0, 1, 2, \ldots$, by (ii) and (iii), respectively. Hence $\lim_{r\to\infty} t_{rn}(x) = t_n(x)$ exists for each $x \in c$, $n = 0, 1, 2, \ldots$, with

$$t(x) = \lambda \left[\alpha - \sum_{k=0}^{\infty} \alpha_k \right] + \sum_{k=0}^{\infty} x_k \alpha_k \tag{2.1}$$

Since $t_{rn} \in c^*$ for each r and n, it has the form

$$t_{rn}(x) = \lambda \left[t_{rn}(e) - \sum_{k=0}^{\infty} t_{rn}(e_k) \right] + \sum_{k=0}^{\infty} x_k t_{rn}(e_k) \tag{2.2}$$

It is easy to see from (2.1) and (2.2) the convergence of $\{t_{rn}(x)\}$ to $t(x)$ is uniform in n, since $\lim_{r\to\infty} t_{rn}(e) = \alpha$ and $\lim_{r\to\infty} t_{rn}(e_k) = \alpha_k$, uniformly in n. Therefore A is θ_σ-conservative. This completes the proof.

Theorem 2.2. *Let $A = (a_{nk})$ be an infinite matrix and let $\theta = (k_r)$ be a lacunary sequence. Then the matrix A is θ_σ-regular if and only if*

a) $\sup_{r,n} \{\sum_{k=0}^{\infty} |a(r,n,k)|\} < \infty$
b) $\lim_r \sum_{k=0}^{\infty} a(r,n,k) = 1$ *uniformly in n, and*
c) $\lim_r a(r,n,k) = 0$ *uniformly in n, $k = 0, 1, 2, \ldots$.*

Proof. Suppose that A is θ_σ-regular. Then A is θ_σ-conservative so that (a) must hold by Theorem 2.1. (b) and (c) must hold since the A-transform of the sequences e_k and e must be θ_σ-convergent to 0 and 1, respectively.

Now suppose that (a), (b) and (c) hold. Then A is θ_σ-conservative by Theorem 2.1. Therefore $\lim_{r\to\infty} t_{rn}(x) = t_n(x)$ uniformly in n for each $x \in c$. The representation (2.1) gives $t(x) = \lim_k x_k$. Hence A is θ_σ-regular.

We are going to give that the results concerning the theorems 2.1 and 2.2.

If $\sigma(n) = n + 1$, the theorems 2.1 and 2.2 reduce to the results of Nuray [6]. When $\theta = 2^r$, the theorems above reduce to the results of Schaefer [8], and if $\sigma(n) = n + 1$ and $\theta = 2^r$, then the theorems (2.1) and (2.2) reduce to the results of King [3]. We characterize the matrix transformation $A \in (\hat{c}, N_\theta^\sigma)$. We have

Theorem 2.3. *Let the matrix A be θ_σ-regular. $A \in (\hat{c}, N_\theta^\sigma)$ if and only if*

$$\lim_{r\to\infty} \sum_{k=0}^{\infty} |a(r,n,k) - a(r,n,k+1)| = 0 \tag{2.3}$$

uniformly in n.

Proof. Suppose that (2.3) holds. Let (x_k) be almost convergent and $\lim x_k = L$. For any arbitrary $\varepsilon > 0$ we can find a natural number p such that

$$\frac{1}{p}(x_k + x_{k+1} + \cdots + x_{k+p-1}) = L + \alpha_k, \quad |\alpha_k| < \varepsilon,\ k = 0, 1, 2, \ldots$$

the term above, multiplying by $a(r,n,k)$ and adding we have

$$\frac{1}{p}\sum_{k=1}^{\infty} a(r,n,k)(x_k + x_{k+1} + \cdots + x_{k+p-1}) = LA_{rn} + \sum_{k=1}^{\infty} a(r,n,k)\alpha_k \tag{2.4}$$

Since $A_{rn} = \sum_{k=1}^{\infty} a(r,n,k) \to 1$ and $a(r,n,k) \to 0$ as $r \to \infty$, uniformly in n, respectively, we have

$$\begin{aligned}
&\frac{1}{p}\sum_{k=1}^{\infty} a(r,n,k)(x_k + x_{k+1} + \cdots + x_{k+p-1}) \\
&= \frac{1}{p}\left(\sum_{k=1}^{\infty} a(r,n,k)x_k + \sum_{k=2}^{\infty} a(r,n,k-1)x_k + \cdots + \sum_{k=p-1}^{\infty} a(r,n,k-p+1)x_k\right) \\
&= \frac{1}{p}\sum_{k=1}^{p-2} a(r,n,k)(x_k + x_{k+1} + \cdots + x_{k+p-2}) \\
&\quad + \frac{1}{p}\sum_{k=p-1}^{\infty} x_k(a(r,n,k) + \cdots + a(r,n,k-p-1)) \\
&= o(1) + \frac{1}{p}\sum_{k=p-1}^{\infty} x_k(a(r,n,k-p-1) + \cdots + a(r,n,k))
\end{aligned}$$

In this case, we have

$$\begin{aligned}
&\frac{1}{p}\sum_{k=1}^{\infty} a(r,n,k)(x_k + x_{k+1} + \cdots + x_{k+p-1}) \\
&= o(1) + y_{rn} + \frac{1}{p}\sum_{k=p-1}^{\infty} x_k[(a(r,n,k-p+1) + \cdots + a(r,n,k)) - a(r,n,k)]
\end{aligned} \tag{2.5}$$

where $y_{rn} = \sum_{k=1}^{\infty} a(r,n,k)x_k$.

Now the absolute value of the sum on the right hand side of (2.5) is not larger than

$$\begin{aligned}
&\left|\frac{1}{p}\sum_{k=p-1}^{\infty} x_k[(a(r,n,k-p+1) + \cdots + a(r,n,k)) - pa(r,n,k)]\right| \\
&\leq \frac{1}{p}\sum_{k=p-1}^{\infty} |(a(r,n,k-p+1) + \cdots + a(r,n,k)) - pa(r,n,k)|\,\|x\|
\end{aligned}$$

$$\leq \frac{\|x\|}{p} \sum_{p=0}^{p-1} \sum_{k=1}^{\infty} |(a(r,n,k-p) - a(r,n,k)|$$

$$\leq \frac{p-1}{2} \|x\| \sum_{k=1}^{\infty} |(a(r,n,k) - a(r,n,k+1)|$$

for all n. From (2.4) and (2.5), we have

$$y_{rn} = LA_{rn} + \sum_{k=p}^{\infty} a(r,n,k)\alpha_k + o(1).$$

Since $\sup_{r,n} \sum_k |a(r,n,k)| = M$, we can write $|\sum_k a(r,n,k)\alpha_k| \leq M\varepsilon$. Taking $LA_{rn} = L + o(1)$. From here, we have, for all n,

$$|y_{rn} - L| = \left| LA_{rn} - L + \sum_{k=p-1}^{\infty} a(r,n,k)\alpha_k + o(1) \right|$$

$$\leq \left(\left| \sum_{k=p-1}^{\infty} a(r,n,k)\alpha_k \right| + o(1) \right) \leq (M+1)\varepsilon$$

Therefore $\lim_{r\to\infty} y_{rn} = L$ uniformly in n. This means that the condition (2.3) is sufficient.

We now assume that (2.3) does not hold. We shall construct a sequence (x_k) for which $\lim x_k = 0$ but which is not summable by the matrix A_{rn}. According to our assumption, there is an $\varepsilon > 0$, such that for an infinitely many r

$$\sum_{k=0}^{\infty} |a(r,n,k) - a(r,n,k+1)| > 8\varepsilon$$

For every such r we either have

$$\sum_{l=0}^{\infty} |a(r,n,2l) - a(r,n,2l+1)| > 4\varepsilon$$

or

$$\sum_{l=0}^{\infty} |a(r,n,2l+1) - a(r,n,2l+2)| > 4\varepsilon$$

for all n. We now construct three increasing sequences of natural numbers (r_j), (p_j) and (q_j) where $q_{-1} = 0 < p_1 < q_1 < p_2 < \cdots$. We first choose r_1, p_1 and q_1 such that, for all n,

$$|a(r_1,n,0)| < \frac{\varepsilon}{2}$$

$$\sum_{l=0}^{\frac{q_1-p_1-1}{2}} |a(r_1,n,p_1+2l)-a(r_1,n,p_1+2l+1)| > 2\varepsilon$$
$$\sum_{k=q_1+1}^{\infty} |a(r_1,n,k)| < \frac{\varepsilon}{2}$$

If the numbers (r_v), (p_v) and (q_v), $v = 1,2,3,\ldots,j-1$, are already known, (r_j), (p_j), (q_j) (where $q_{j-1} < p_j < q_j$ and one of the numbers p_j, q_j even. The other, odd) are chosen such that, for all n,

$$\sum_{k=0}^{q_{j-1}} |a(r_1,n,k)| < \frac{\varepsilon}{2}$$
$$\sum_{l=0}^{\frac{q_j-p_j-1}{2}} |a(r_j,n,p_j+2l)-a(r_j,n,p_j+2l+1)| > 2\varepsilon$$
$$\sum_{k=q_j+1}^{\infty} |a(r_j,n,k)| < \frac{\varepsilon}{2}.$$

We now defined the sequence $x=(x_k)$ as following

$$x_k = \begin{cases} x_{p_j+2l} & = \quad (-1)^j sgn(a(r_j,n,p_j+2l)-a(r_j,n,p_j+2l+1)) \\ x_{p_j+2l+1} & = \quad -x_{p_j+2l} \\ x_k & = 0; \quad q_{j-1} < k < q_j,\ j=1,2,\ldots \quad \text{and} \quad l=0,1,2,\ldots,\frac{q_j-p_j-1}{2} \end{cases}$$

Under these conditions, we havew for our sequence

$$\begin{aligned} |y_{r_j,n}| &= \Big|\sum_k a(r_j,n,k)x_k\Big| \\ &\geq \sum_{l=0}^{\frac{q_j-p_j-1}{2}} |a(r_j,n,p_j+2l)-a(r_j,n,p_j+2l+1)| - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\ &> \varepsilon \end{aligned}$$

and $sign(y_{r_j,n}) = (-1)^j$ for all n. Hence it follows that the sequene y_{rn} for all n, diverges. It is easy to see that $\lim x_k = 0$. This completes the proof.

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