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INEQUALITIES FOR POWER SERIES WITH NONNEGATIVE COEFFICIENTS VIA A REVERSE OF JENSEN INEQUALITY

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Abstract. Some inequalities for power series with nonnegative coefficients via a reverse of Jensen inequality obtained by Dragomir & Ionescu in 1994 are given. Applications for some fundamental functions defined by power series are also provided.

1. Introduction

In 1994, Dragomir & Ionescu obtained the following reverse of Jensen's discrete inequality:

Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interior $\mathring{I}$ of the interval I. If $x_i \in \mathring{I}$ and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the inequality:

$$0 \leq \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n w_i \Phi'(x_i) x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i. \tag{1.1}$$

In order to improve Grüss' discrete inequality, Cerone & Dragomir established in 2002 the following result [1]:

$$\left|\sum_{i=1}^n w_i a_i b_i - \sum_{i=1}^n w_i a_i \sum_{i=1}^n w_i b_i\right| \leq \frac{1}{2}(A-a)\sum_{i=1}^n w_i \left|b_i - \sum_{j=1}^n w_j b_j\right| \leq \frac{1}{2}(A-a)\left[\sum_{i=1}^n w_i b_i^2 - \left(\sum_{i=1}^n w_i b_i\right)^2\right]^{1/2}, \tag{1.2}$$

provided $-\infty < a \leq a_i \leq A < \infty$, and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$.

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In addition, if $-\infty < b \le b_i \le B < \infty$, $(i = 1, \ldots, n)$ then we have the string of inequalities

$$\begin{aligned}\left|\sum_{i=1}^{n} w_i a_i b_i - \sum_{i=1}^{n} w_i a_i \sum_{i=1}^{n} w_i b_i\right| &\le \frac{1}{2}(A-a)\sum_{i=1}^{n} w_i \left|b_i - \sum_{j=1}^{n} w_j b_j\right| \\ &\le \frac{1}{2}(A-a)\left[\sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i b_i\right)^2\right]^{1/2} \\ &\le \frac{1}{4}(A-a)(B-b). \end{aligned} \tag{1.3}$$

Utilising these results, we observe that if Φ is differentiable convex on a finite interval, say $[m, M]$, then we have the inequalities:

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\le \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \\ &\le \frac{1}{2}(M-m)\sum_{i=1}^{n} w_i \left|\Phi'(x_i) - \sum_{j=1}^{n} w_j \Phi'(x_j)\right| \\ &\le \frac{1}{2}(M-m)\left[\sum_{i=1}^{n} w_i [\Phi'(x_i)]^2 - \left(\sum_{i=1}^{n} w_i \Phi'(x_i)\right)^2\right]^{1/2} \end{aligned} \tag{1.4}$$

for $x_i \in (m, M)$ $(i = 1, \ldots, n)$.

If the lateral derivatives $\Phi'_+(m)$ and $\Phi'_-(M)$ are finite, then we also have

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\le \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \\ &\le \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right]\sum_{i=1}^{n} w_i \left|x_i - \sum_{j=1}^{n} w_j x_j\right| \\ &\le \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right]\left[\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{i=1}^{n} w_i x_i\right)^2\right]^{1/2} \\ &\le \frac{1}{4}(M-m)\left[\Phi'_-(M) - \Phi'_+(m)\right] \end{aligned} \tag{1.5}$$

for $x_i \in [m, M]$ $(i = 1, \ldots, n)$.

In the recent paper [9], by the use of a refinement of Young's inequality, the authors proved the following result:

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $y,\ z,\ z^{\nu}y^{1-\nu},\ z^{1-\nu}y^{\nu}\in(0,R)$ and $\nu\in[0,1]$ then we have the inequalities:

$$
\begin{aligned}
2\min\{\nu,1-\nu\}\left[f(y)f(z)-f^2\left(\sqrt{yz}\right)\right] &\le f(y)f(z)-f\left(z^{\nu}y^{1-\nu}\right)f\left(z^{1-\nu}y^{\nu}\right)\\
&\le 2\max\{\nu,1-\nu\}\left[f(y)f(z)-f^2\left(\sqrt{yz}\right)\right] \qquad (1.6)
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
2\min\{\nu,1-\nu\}\left[f\left(u^2\right)f\left(t^2\right)-f^2(ut)\right] &\le f\left(u^2\right)f\left(t^2\right)-f\left(u^{2\nu}t^{2(1-\nu)}\right)f\left(t^{2(1-\nu)}u^{2\nu}\right)\\
&\le 2\max\{\nu,1-\nu\}\left[f\left(u^2\right)f\left(t^2\right)-f^2(ut)\right], \qquad (1.7)
\end{aligned}
$$

provided $u^2,\ t^2,\ u^{2\nu}t^{2(1-\nu)},\ t^{2(1-\nu)}u^{2\nu}\in(0,R)$ and $\nu\in[0,1]$.

For other recent results for power series with nonnegative coefficients, see [9] and [10]. For more results on power series inequalities, see [2] and [5]-[8].

Motivated by the above results and utilizing a reverse of Jensen inequality obtained by Dragomir & Ionescu in 1994 we provide in this paper other inequalities for power series with nonnegative coefficients. Applications for some fundamental and special functions are given as well.

2. Power inequalities

The most important power series with nonnegative coefficients are:

$$
\begin{aligned}
&\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C},\quad \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\ z\in D(0,1),\\
&\ln\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{1}{n}z^n,\ z\in D(0,1),\quad \cosh z=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n},\ z\in\mathbb{C}, \qquad (2.1)\\
&\sinh z=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1},\ z\in\mathbb{C}.
\end{aligned}
$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$
\begin{aligned}
&\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),\\
&\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1},\ z\in D(0,1), \qquad (2.2)\\
&\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),
\end{aligned}
$$

$$ {}_2F_1(\alpha,\beta,\gamma,z) := \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0 \qquad z \in D(0,1), $$

where Γ is *Gamma function.*

The following result for powers holds:

Theorem 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p \ge 1$, $0 < \alpha < R$ and $x > 0$ with $\alpha x^p, \alpha x^{p-1} < R$, then*

$$ 0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \le p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f(\alpha x^{p-1})}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]. \tag{2.3} $$

Moreover, if $0 < x \le 1$, then

$$ \begin{aligned} 0 &\le \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \le p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f(\alpha x^{p-1})}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right] \\ &\le \frac{1}{2}p\left(\frac{f(\alpha x^{2(p-1)})}{f(\alpha)} - \left[\frac{f(\alpha x^{p-1})}{f(\alpha)}\right]^2\right)^{1/2} \le \frac{1}{4}p \end{aligned} \tag{2.4} $$

and

$$ \begin{aligned} 0 &\le \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \le p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f(\alpha x^{p-1})}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right] \\ &\le \frac{1}{2}p\left(\frac{f(\alpha x^2)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^2\right)^{1/2} \le \frac{1}{4}p. \end{aligned} \tag{2.5} $$

Proof. If we write the inequality (1.1) for the convex function $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(x) = x^p$, $p \ge 1$, then we have

$$ 0 \le \sum_{i=1}^{n} w_i x_i^p - \left(\sum_{i=1}^{n} w_i x_i\right)^p \le p\left(\sum_{i=1}^{n} w_i x_i^p - \sum_{i=1}^{n} w_i x_i^{p-1} \sum_{i=1}^{n} w_i x_i\right) \tag{2.6} $$

for any $w_i, x_i \ge 0$ $(i = 1,\dots,n)$ with $\sum_{i=1}^{n} w_i = 1$.

If $0 < \alpha < R$ and $k \ge 1$, then by (2.6) we have

$$ \begin{aligned} 0 &\le \frac{1}{\sum_{j=0}^{k} a_j\alpha^j} \sum_{j=0}^{k} a_j\alpha^j (x^p)^j - \left(\frac{1}{\sum_{j=0}^{k} a_j\alpha^j} \sum_{j=0}^{k} a_j\alpha^j x^j\right)^p \\ &\le p\left[\frac{1}{\sum_{j=0}^{k} a_j\alpha^j} \sum_{j=0}^{k} a_j\alpha^j (x^p)^j - \frac{1}{\sum_{j=0}^{k} a_j\alpha^j} \sum_{j=0}^{k} a_j\alpha^j (x^{p-1})^j \frac{1}{\sum_{j=0}^{k} a_j\alpha^j} \sum_{j=0}^{k} a_j\alpha^j x^j\right]. \end{aligned} \tag{2.7} $$

Since all series whose partial sums involved in the inequality (2.7) are convergent, then by letting $k \to \infty$ in (2.7) we deduce (2.3).

Now, if $x_i \in [m, M] \subset [0, \infty)$, $(i = 1, \ldots, n)$, then by (1.4) for the convex function $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(x) = x^p$, $p \geq 1$ we have

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i x_i^p - \left(\sum_{i=1}^{n} w_i x_i\right)^p \\
&\leq p\left(\sum_{i=1}^{n} w_i x_i^p - \sum_{i=1}^{n} w_i x_i^{p-1} \sum_{i=1}^{n} w_i x_i\right) \\
&\leq \frac{1}{2} p (M - m) \sum_{i=1}^{n} w_i \left| x_i^{p-1} - \sum_{j=1}^{n} w_j x_j^{p-1} \right| \\
&\leq \frac{1}{2} p (M - m) \left[\sum_{i=1}^{n} w_i x_i^{2(p-1)} - \left(\sum_{i=1}^{n} w_i x_i^{p-1}\right)^2\right]^{1/2} \\
&\leq \frac{1}{2} p (M - m) \left(M^{p-1} - m^{p-1}\right). \qquad (2.8)
\end{aligned}
$$

If $0 < x \leq 1$, then $0 < x^j \leq 1$ for $j = 0, \ldots, k$ and by (2.8) we have

$$
\begin{aligned}
0 &\leq \frac{1}{\sum_{j=0}^{k} a_j \alpha^j} \sum_{j=0}^{k} a_j \alpha^j \left(x^p\right)^j - \left(\frac{1}{\sum_{j=0}^{k} a_j \alpha^j} \sum_{j=0}^{k} a_j \alpha^j x^j\right)^p \\
&\leq p\left[\frac{1}{\sum_{j=0}^{k} a_j \alpha^j} \sum_{j=0}^{k} a_j \alpha^j \left(x^p\right)^j - \frac{1}{\sum_{j=0}^{k} a_j \alpha^j} \sum_{j=0}^{k} a_j \alpha^j \left(x^{p-1}\right)^j \frac{1}{\sum_{j=0}^{k} a_j \alpha^j} \sum_{j=0}^{k} a_j \alpha^j x^j\right] \\
&\leq \frac{1}{2} p\left[\frac{1}{\sum_{j=0}^{k} a_j \alpha^j} \sum_{j=0}^{k} a_j \alpha^j \left[x^{2(p-1)}\right]^j - \left(\frac{1}{\sum_{j=0}^{k} a_j \alpha^j} \sum_{j=0}^{k} a_j \alpha^j \left(x^{p-1}\right)^j\right)^2\right]^{1/2} \\
&\leq \frac{1}{4} p. \qquad (2.9)
\end{aligned}
$$

Since all series whose partial sums involved in the inequality (2.9) are convergent, then by letting $k \to \infty$ in (2.9) we deduce (2.4).

Now, if $x_i \in [m, M] \subset [0, \infty)$, $(i = 1, \ldots, n)$, then by (1.5) for the convex function $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(x) = x^p$, $p \geq 1$ we have

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i x_i^p - \left(\sum_{i=1}^{n} w_i x_i\right)^p \\
&\leq p\left(\sum_{i=1}^{n} w_i x_i^p - \sum_{i=1}^{n} w_i x_i^{p-1} \sum_{i=1}^{n} w_i x_i\right) \\
&\leq \frac{1}{2} p \left(M^{p-1} - m^{p-1}\right) \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right| \\
&\leq \frac{1}{2} p \left(M^{p-1} - m^{p-1}\right) \left[\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{i=1}^{n} w_i x_i\right)^2\right]^{1/2}
\end{aligned}
$$

$$\leq \frac{1}{4}(M-m)\left(M^{p-1}-m^{p-1}\right). \tag{2.10}$$

Finally, by utilizing a similar argument as above, we obtain the inequality (2.5). The details are omitted. □

Remark 1. We observe that, the second inequality in (2.3) is equivalent to

$$\frac{f(\alpha x)}{f(\alpha)}\left(p\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^{p-1}\right)\leq (p-1)\frac{f(\alpha x^{p})}{f(\alpha)} \tag{2.11}$$

or to

$$f(\alpha x)\left(pf\left(\alpha x^{p-1}\right)[f(\alpha)]^{p-2}-[f(\alpha x)]^{p-1}\right)\leq (p-1)f\left(\alpha x^{p}\right)[f(\alpha)]^{p-1}, \tag{2.12}$$

provided that $p\geq 1$, $0<\alpha<R$ and $x>0$ with $\alpha x^{p},\alpha x^{p-1}<R$.

Moreover, if $0<x\leq 1$, then from (2.4) we have

$$\left[\frac{f(\alpha x)}{f(\alpha)}\right]^{p}\leq \frac{f(\alpha x^{p})}{f(\alpha)}\leq \frac{1}{4}p+\left[\frac{f(\alpha x)}{f(\alpha)}\right]^{p}. \tag{2.13}$$

Taking the power $1/p$ and using the inequality $(a+b)^{1/p}\leq a^{1/p}+b^{1/p}$, $p\geq 1$ we get

$$0\leq \left[f\left(\alpha x^{p}\right)\right]^{1/p}[f(\alpha)]^{1-\frac{1}{p}}-f(\alpha x)\leq \frac{1}{4^{1/p}}p^{1/p}f(\alpha). \tag{2.14}$$

Corollary 1. *Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $u,v>0$ with $v^{p}\leq u^{q}<R$, then*

$$\left[\frac{f(uv)}{f(u^{q})}\right]^{p}\leq \frac{f(v^{p})}{f(u^{q})}\leq \frac{1}{4}p+\left[\frac{f(uv)}{f(u^{q})}\right]^{p} \tag{2.15}$$

and

$$0\leq \left[f\left(v^{p}\right)\right]^{1/p}\left[f\left(u^{q}\right)\right]^{1/q}-f(uv)\leq \frac{1}{4^{1/p}}p^{1/p}f\left(u^{q}\right). \tag{2.16}$$

Proof. Follows by taking into (2.13) and (2.14) $\alpha=u^{q}$ and $x=\frac{v}{u^{q/p}}$. The details are omitted. □

Example 1. a) If we write the inequalities (2.4) and (2.5) for the function $\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$, $z\in D(0,1)$, then we have

$$\begin{aligned}0&\leq \frac{1-\alpha}{1-\alpha x^{p}}-\left(\frac{1-\alpha}{1-\alpha x}\right)^{p}\leq p\left[\frac{1-\alpha}{1-\alpha x^{p}}-\frac{(1-\alpha)^2}{\left(1-\alpha x^{p-1}\right)(1-\alpha x)}\right]\\&\leq \frac{1}{2}p\left[\frac{1-\alpha}{1-\alpha x^{2(p-1)}}-\left(\frac{1-\alpha}{1-\alpha x^{p-1}}\right)^{2}\right]^{1/2}\leq \frac{1}{4}p\end{aligned} \tag{2.17}$$

and

$$0\leq \frac{1-\alpha}{1-\alpha x^{p}}-\left(\frac{1-\alpha}{1-\alpha x}\right)^{p}\leq p\left[\frac{1-\alpha}{1-\alpha x^{p}}-\frac{(1-\alpha)^2}{\left(1-\alpha x^{p-1}\right)(1-\alpha x)}\right]$$

$$\leq \frac{1}{2}p\left[\frac{1-\alpha}{1-\alpha x^2}-\left(\frac{1-\alpha}{1-\alpha x}\right)^2\right]^{1/2}\leq \frac{1}{4}p \tag{2.18}$$

for any $\alpha, x \in (0,1)$ and $p \geq 1$.

b) If we write the inequalities (2.4) and (2.5) for the function $\exp z = \sum_{n=0}^{\infty}\frac{1}{n!}z^n$, $z \in \mathbb{C}$, then we have

$$\begin{aligned} 0 &\leq \exp\left[\alpha\left(x^p-1\right)\right]-\exp\left[p\alpha\left(x-1\right)\right] \\ &\leq p\left[\exp\left[\alpha\left(x^p-1\right)\right]-\exp\left[\alpha\left(x^{p-1}+x-2\right)\right]\right] \\ &\leq \frac{1}{2}p\left(\exp\left[\alpha\left(x^{2(p-1)}-1\right)\right]-\exp\left[2\alpha\left(x^{p-1}-1\right)\right]\right)^{1/2}\leq \frac{1}{4}p \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} 0 &\leq \exp\left[\alpha\left(x^p-1\right)\right]-\exp\left[p\alpha\left(x-1\right)\right] \\ &\leq p\left[\exp\left[\alpha\left(x^p-1\right)\right]-\exp\left[\alpha\left(x^{p-1}+x-2\right)\right]\right] \\ &\leq \frac{1}{2}p\left(\exp\left[\alpha\left(x^2-1\right)\right]-\exp\left[2\alpha\left(x-1\right)\right]\right)^{1/2}\leq \frac{1}{4}p. \end{aligned} \tag{2.20}$$

for any $\alpha, p > 0$ and $x \in (0,1)$.

3. Exponential inequalities

The following exponential inequality holds:

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $0 < \alpha < R$ and $x, \beta \in \mathbb{R}$ with $\alpha\exp(\beta x) < R$ then*

$$\begin{aligned} 0 &\leq \frac{f\left(\alpha\exp\left(\beta x\right)\right)}{f(\alpha)}-\exp\left[\frac{\alpha\beta x f'(\alpha)}{f(\alpha)}\right] \\ &\leq \alpha\beta x\left[\frac{\exp\left(\beta x\right)f'\left(\alpha\exp\left(\beta x\right)\right)}{f(\alpha)}-\frac{f\left(\alpha\exp\left(\beta x\right)\right)}{f(\alpha)}\frac{f'(\alpha)}{f(\alpha)}\right]. \end{aligned} \tag{3.1}$$

Moreover, if $x \leq 0$, $\beta > 0$ with $\exp(\beta x) < R$ and $0 < \alpha < R$, then

$$\begin{aligned} 0 &\leq \frac{f\left(\alpha\exp\left(\beta x\right)\right)}{f(\alpha)}-\exp\left[\frac{\alpha\beta x f'(\alpha)}{f(\alpha)}\right] \\ &\leq \alpha\beta x\left[\frac{\exp\left(\beta x\right)f'\left(\alpha\exp\left(\beta x\right)\right)}{f(\alpha)}-\frac{f\left(\alpha\exp\left(\beta x\right)\right)}{f(\alpha)}\frac{f'(\alpha)}{f(\alpha)}\right] \\ &\leq \frac{1}{2}\beta|x|\left[\frac{\alpha\left[f'(\alpha)+\alpha f''(\alpha)\right]}{f(\alpha)}-\left(\frac{\alpha f'(\alpha)}{f(\alpha)}\right)^2\right]^{1/2}. \end{aligned} \tag{3.2}$$

Proof. If we write the inequality (1.1) for the convex function $\Phi:\mathbb{R}\to[0,\infty)$, $\Phi(x)=\exp(\beta x)$, then we have

$$\begin{aligned}0 &\le \sum_{i=1}^{n} w_i \exp(\beta x_i) - \exp\left(\beta \sum_{i=1}^{n} w_i x_i\right)\\ &\le \beta\left[\sum_{i=1}^{n} w_i x_i \exp(\beta x_i) - \sum_{i=1}^{n} w_i \exp(\beta x_i)\sum_{i=1}^{n} w_i x_i\right] \qquad (3.3)\end{aligned}$$

for any $w_i \ge 0$ $(i=1,\dots,n)$ with $\sum_{i=1}^{n} w_i = 1$ and $x_i \in \mathbb{R}$ $(i=1,\dots,n)$.

If $0<\alpha<R$ and $k\ge 1$, then by (3.3) for $x_j = jx$, we have

$$\begin{aligned}0 &\le \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left[\exp(\beta x)\right]^j - \exp\left(\frac{\beta x}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j \alpha^j\right)\\ &\le \beta x\left[\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j\left[\exp(\beta x)\right]^j\right.\\ &\quad\left.-\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left[\exp(\beta x)\right]^j \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j\right]. \qquad (3.4)\end{aligned}$$

Observe that the series $\sum_{j=0}^{\infty} j a_j \alpha^j$ is convergent for $0<\alpha<R$ and

$$\sum_{j=0}^{\infty} j a_j\alpha^j = \sum_{j=1}^{\infty} j a_j \alpha^j = \alpha f'(\alpha),\ 0<\alpha<R.$$

Since all series whose partial sums involved in the inequality (3.4) are convergent, then by letting $k\to\infty$ in (3.4) we deduce (3.1).

If we write the inequality (1.4) for the convex function $\Phi:\mathbb{R}\to[0,\infty)$, $\Phi(x)=\exp(\beta x)$, then we have

$$\begin{aligned}0 &\le \sum_{i=1}^{n} w_i \exp(\beta x_i) - \exp\left(\beta \sum_{i=1}^{n} w_i x_i\right)\\ &\le \beta\left[\sum_{i=1}^{n} w_i x_i \exp(\beta x_i) - \sum_{i=1}^{n} w_i \exp(\beta x_i)\sum_{i=1}^{n} w_i x_i\right]\\ &\le \frac{1}{2}\beta\left[\exp(\beta M)-\exp(\beta m)\right]\sum_{i=1}^{n} w_i\left|x_i - \sum_{j=1}^{n} w_j x_j\right|\\ &\le \frac{1}{2}\beta\left[\exp(\beta M)-\exp(\beta m)\right]\left[\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{i=1}^{n} w_i x_i\right)^2\right]^{1/2}, \qquad (3.5)\end{aligned}$$

provided $m\le x_i\le M$, $i\in\{1,\dots,n\}$.

Now, if $\beta>0$, by letting $M=0$ and $m\to-\infty$ then by (3.5) we have

$$0 \le \sum_{i=1}^{n} w_i \exp(\beta x_i) - \exp\left(\beta \sum_{i=1}^{n} w_i x_i\right)$$

$$\leq \beta\left[\sum_{i=1}^{n} w_i x_i \exp(\beta x_i) - \sum_{i=1}^{n} w_i \exp(\beta x_i)\sum_{i=1}^{n} w_i x_i\right]$$
$$\leq \frac{1}{2}\beta\sum_{i=1}^{n} w_i\left|x_i - \sum_{j=1}^{n} w_j x_j\right| \leq \frac{1}{2}\beta\left[\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{i=1}^{n} w_i x_i\right)^2\right]^{1/2}, \tag{3.6}$$

provided $-\infty < x_i \leq 0$.

If $0 < \alpha < R$, $x \leq 0$, $\beta > 0$ and $k \geq 1$, then by (3.3) for $x_j = jx \in (-\infty, 0]$, we have

$$0 \leq \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left[\exp(\beta x)\right]^j - \exp\left(\frac{\beta x}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j\right)$$
$$\leq \beta x\left[\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j\left[\exp(\beta x)\right]^j\right.$$
$$\left. - \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left[\exp(\beta x)\right]^j \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j\right]$$
$$\leq \frac{1}{2}\beta|x|\left[\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j^2 a_j\alpha^j - \left(\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j\right)^2\right]^{1/2}. \tag{3.7}$$

If we denote $g(u) := \sum_{n=0}^{\infty} \alpha_n u^n$, then for $|u| < R$, its radius of convergence, we have

$$\sum_{n=0}^{\infty} n\alpha_n u^n = ug'(u)$$

and

$$\sum_{n=0}^{\infty} n^2\alpha_n u^n = u\left(ug'(u)\right)'.$$

However

$$u\left(ug'(u)\right)' = ug'(u) + u^2 g''(u)$$

and then

$$\sum_{n=0}^{\infty} n^2\alpha_n u^n = ug'(u) + u^2 g''(u).$$

Since all series whose partial sums involved in the inequality (3.7) are convergent, then by letting $k \to \infty$ in (3.7) we deduce (3.2). □

Example 2.

(a) If we write the inequality (3.2) for the function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we have for $x \leq 0$, $\beta > 0$ and $0 < \alpha < 1$, that

$$0 \leq \frac{1-\alpha}{1-\alpha\exp(\beta x)} - \exp\left(\frac{\alpha\beta x}{1-\alpha}\right)$$
$$\leq \alpha\beta x\left[\frac{(1-\alpha)\exp(\beta x)}{\left(1-\alpha\exp(\beta x)\right)^2} - \frac{1}{1-\alpha\exp(\beta x)}\right] \leq \frac{1}{2}\frac{\beta|x|\alpha^{1/2}}{1-\alpha}. \tag{3.8}$$

(b) If we write the inequality (3.1) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, then we have

$$0 \le \exp\left(\alpha\left[\exp\left(\beta x\right) - 1\right]\right) - \exp\left(\alpha\beta x\right) \\ \le \alpha\beta x\left[\exp\left(\alpha\left[\exp\left(\beta x\right) - 1\right] + \beta x\right) - \exp\left(\alpha\left[\exp\left(\beta x\right) - 1\right]\right)\right] \tag{3.9}$$

for any $\alpha > 0$ and $x \le 0$, $\beta > 0$.

3.1. Logarithmic Inequalities

The following logarithmic inequality holds:

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $0 < \alpha < R$, $p > 0$ and $x > 0$ with $\alpha x^p, \alpha x^{-p} < R$, then*

$$0 \le \ln\left(\frac{f(\alpha x^p)}{f(\alpha)}\right) - p\frac{\alpha f'(\alpha)}{f(\alpha)}\ln x \le \frac{f(\alpha x^p)}{f(\alpha)}\frac{f(\alpha x^{-p})}{f(\alpha)} - 1. \tag{3.10}$$

Moreover, if $0 < x \le 1$ with $\alpha x^p, \alpha x^{-p} < R$, then

$$0 \le \ln\left(\frac{f(\alpha x^p)}{f(\alpha)}\right) - p\frac{\alpha f'(\alpha)}{f(\alpha)}\ln x \le \frac{f(\alpha x^p)}{f(\alpha)}\frac{f(\alpha x^{-p})}{f(\alpha)} - 1 \\ \le \frac{1}{2}\left[\frac{f\left(\alpha x^{-2p}\right)}{f(\alpha)} - \left(\frac{f(\alpha x^{-p})}{f(\alpha)}\right)^2\right]^{1/2}. \tag{3.11}$$

Proof. If we write the inequality (1.1) for the convex function $\Phi : (0,\infty) \to \mathbb{R}$, $\Phi(x) = -\ln x$, then we have

$$0 \le \ln\left(\sum_{i=1}^{n} w_i x_i\right) - \sum_{i=1}^{n} w_i \ln(x_i) \le \sum_{i=1}^{n} \frac{w_i}{x_i} \sum_{i=1}^{n} w_i x_i - 1, \tag{3.12}$$

for any $w_i \ge 0$ $(i = 1,\ldots,n)$ with $\sum_{i=1}^{n} w_i = 1$ and $x_i \in (0,\infty)$ $(i = 1,\ldots,n)$.

If $0 < \alpha < R$ and $k \ge 1$, then by (3.12) for $x_j = (x^p)^j$, we have

$$0 \le \ln\left(\frac{1}{\sum_{j=0}^{k} a_j \alpha^j}\sum_{j=0}^{k} a_j \alpha^j \left(x^p\right)^j\right) - \frac{p\ln x}{\sum_{j=0}^{k} a_j \alpha^j}\sum_{j=0}^{k} j a_j \alpha^j \\ \le \frac{1}{\sum_{j=0}^{k} a_j \alpha^j}\sum_{j=0}^{k} a_j \alpha^j \left(x^p\right)^j \frac{1}{\sum_{j=0}^{k} a_j \alpha^j}\sum_{j=0}^{k} a_j \frac{\alpha^j}{(x^p)^j} - 1. \tag{3.13}$$

Since all series whose partial sums involved in the inequality (3.13) are convergent, then by letting $k \to \infty$ in (3.13) we deduce (3.10).

From the inequality (1.4) we have

$$0 \le \ln\left(\sum_{i=1}^{n} w_i x_i\right) - \sum_{i=1}^{n} w_i \ln(x_i) \le \sum_{i=1}^{n} \frac{w_i}{x_i} \sum_{i=1}^{n} w_i x_i - 1$$

$$\leq \frac{1}{2}(M-m)\sum_{i=1}^{n} w_i \left| \frac{1}{x_i} - \sum_{j=1}^{n} \frac{w_j}{x_j} \right|$$
$$\leq \frac{1}{2}(M-m)\left[\sum_{i=1}^{n} \frac{w_i}{x_i^2} - \left(\sum_{i=1}^{n} \frac{w_i}{x_i}\right)^2\right]^{1/2} \tag{3.14}$$

for any $w_i \geq 0$ $(i = 1,\ldots,n)$ with $\sum_{i=1}^{n} w_i = 1$ and $x_i \in [m,M] \subset (0,\infty)$ $(i = 1,\ldots,n)$.

If $0 < x \leq 1$, then $0 < x^p \leq 1$ and if we apply the inequality (3.14) for $x_j = (x^p)^j \in (0,1]$ we have

$$0 \leq \ln\left(\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j (x^p)^j\right) - \frac{p\ln x}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j \alpha^j$$
$$\leq \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j (x^p)^j \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j \frac{\alpha^j}{(x^p)^j} - 1$$
$$\leq \frac{1}{2}\left[\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j \frac{\alpha^j}{(x^{2p})^j} - \left(\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j \frac{\alpha^j}{(x^p)^j}\right)^2\right]^{1/2}. \tag{3.15}$$

Since all series whose partial sums involved in the inequality (3.15) are convergent, then by letting $k \to \infty$ in (3.15) we deduce (3.11). □

Corollary 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v > 0$ with $v^p, u^q, \frac{u^{2q}}{v^p} < R$, then*

$$0 \leq \ln\left(\frac{f(v^p)}{f(u^q)}\right) - \frac{u^q f'(u^q)}{f(u^q)}\ln\frac{v^p}{u^q} \leq \frac{f(v^p)}{f(u^q)}\frac{f\left(\frac{u^{2q}}{v^p}\right)}{f(u^q)} - 1. \tag{3.16}$$

If $v^p \leq u^q < R$ and $\frac{u^{2q}}{v^p}, \frac{u^{3q}}{v^{2p}}$, then

$$0 \leq \ln\left(\frac{f(v^p)}{f(u^q)}\right) - \frac{u^q f'(u^q)}{f(u^q)}\ln\frac{v^p}{u^q} \leq \frac{f(v^p)}{f(u^q)}\frac{f\left(\frac{u^{2q}}{v^p}\right)}{f(u^q)} - 1$$
$$\leq \frac{1}{2}\left[\frac{f\left(\frac{u^{3q}}{v^{2p}}\right)}{f(u^q)} - \left(\frac{f\left(\frac{u^{2q}}{v^p}\right)}{f(u^q)}\right)^2\right]^{1/2}. \tag{3.17}$$

Proof. Follows by taking into (3.10) and (3.11) $\alpha = u^q$ and $x = \frac{v}{u^{q/p}}$. The details are omitted. □

Example 3.

(a) If we write the inequality (3.11) for the function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we have for $0 < \alpha, x < 1$ and $p > 0$ that

$$0 \leq \ln\left(\frac{1-\alpha}{1-\alpha x^p}\right) - \frac{p\alpha}{1-\alpha}\ln x \leq \frac{(1-\alpha)^2}{(1-\alpha x^p)(1-\alpha x^{-p})} - 1 \tag{3.18}$$

$$\leq \frac{1}{2}\left[\frac{1-\alpha}{1-\alpha x^{-2p}}-\left(\frac{1-\alpha}{1-\alpha x^{-p}}\right)^{2}\right]^{1/2}.$$

(b) If we write the inequality (3.10) for the function $\exp z=\sum_{n=0}^{\infty}\frac{1}{n!}z^{n}$, $z\in\mathbb{C}$, then we have

$$0\leq\alpha\left(x^{p}-1\right)-p\alpha\ln x\leq\exp\left[\alpha\left(x^{p}+x^{-p}-2\right)\right]-1 \tag{3.19}$$

for $\alpha,p,x>0$.

The following logarithmic inequality also holds:

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty}a_{n}z^{n}$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $0<\alpha<R$, $p>0$ and $x>0$ with $\alpha x^{p}<R$, then*

$$\begin{aligned}0&\leq\frac{p\alpha x^{p}f'(\alpha x^{p})}{f(\alpha)}\ln x-\frac{f(\alpha x^{p})}{f(\alpha)}\ln\left(\frac{f(\alpha x^{p})}{f(\alpha)}\right)\\&\leq p\alpha\left[\frac{x^{p}f'(\alpha x^{p})}{f(\alpha)}-\frac{f(\alpha x^{p})}{f(\alpha)}\frac{f'(\alpha)}{f(\alpha)}\right]\ln x,\end{aligned} \tag{3.20}$$

or, equivalently

$$\frac{p\alpha f'(\alpha)}{f(\alpha)}\ln x\leq\ln\left(\frac{f(\alpha x^{p})}{f(\alpha)}\right)$$

i.e., the first inequality in (3.10).

Moreover, if $0<x\leq1$ we also have

$$\begin{aligned}0&\leq\frac{p\alpha x^{p}f'(\alpha x^{p})}{f(\alpha)}\ln x-\frac{f(\alpha x^{p})}{f(\alpha)}\ln\left(\frac{f(\alpha x^{p})}{f(\alpha)}\right)\\&\leq p\alpha\left[\frac{x^{p}f'(\alpha x^{p})}{f(\alpha)}-\frac{f(\alpha x^{p})}{f(\alpha)}\frac{f'(\alpha)}{f(\alpha)}\right]\ln x\\&\leq\frac{1}{2}p\left|\ln x\right|\left[\frac{\alpha\left[f'(\alpha)+\alpha f''(\alpha)\right]}{f(\alpha)}-\left(\frac{\alpha f'(\alpha)}{f(\alpha)}\right)^{2}\right]^{1/2}.\end{aligned} \tag{3.21}$$

Proof. If we write the inequality (1.1) for the convex function $\Phi:(0,\infty)\to\mathbb{R}$, $\Phi(x)=x\ln x$, then we have

$$\begin{aligned}0&\leq\sum_{i=1}^{n}w_{i}x_{i}\ln(x_{i})-\sum_{i=1}^{n}w_{i}x_{i}\ln\left(\sum_{i=1}^{n}w_{i}x_{i}\right)\\&\leq\sum_{i=1}^{n}w_{i}\left[\ln(x_{i})+1\right]x_{i}-\sum_{i=1}^{n}w_{i}\left[\ln(x_{i})+1\right]\sum_{i=1}^{n}w_{i}x_{i}\end{aligned} \tag{3.22}$$

for any $w_{i}\geq0$ $(i=1,\ldots,n)$ with $\sum_{i=1}^{n}w_{i}=1$ and $x_{i}\in(0,\infty)$ $(i=1,\ldots,n)$.

If $0<\alpha<R$ and $k\geq 1$, then by (3.22) for $x_j=(x^p)^j$, we have

$$
\begin{aligned}
0 &\leq \frac{p\ln x}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j\left(x^p\right)^j \\
&\quad -\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left(x^p\right)^j\ln\left(\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left(x^p\right)^j\right) \\
&\leq \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left(jp\ln x+1\right)\left(x^p\right)^j \\
&\quad -\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left(jp\ln x+1\right)\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left(x^p\right)^j \\
&= \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\left[p\ln x\sum_{j=0}^{k} j a_j\alpha^j\left(x^p\right)^j+\sum_{j=0}^{k} a_j\alpha^j\left(x^p\right)^j\right] \\
&\quad -\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\left[p\ln x\sum_{j=0}^{k} j a_j\alpha^j+\sum_{j=0}^{k} a_j\alpha^j\right]\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j\left(x^p\right)^j. \qquad (3.23)
\end{aligned}
$$

Since all series whose partial sums involved in the inequality (3.23) are convergent, then by letting $k\to\infty$ in (3.23) we deduce

$$
\begin{aligned}
0 &\leq \frac{p\alpha x^p f'\left(\alpha x^p\right)}{f(\alpha)}\ln x-\frac{f\left(\alpha x^p\right)}{f(\alpha)}\ln\left(\frac{f\left(\alpha x^p\right)}{f(\alpha)}\right) \\
&\leq \frac{1}{f(\alpha)}\left[p\alpha x^p f'\left(\alpha x^p\right)\ln x+f\left(\alpha x^p\right)\right]-\frac{1}{f(\alpha)}\left[p\alpha f'(\alpha)\ln x+f(\alpha)\right]\frac{f\left(\alpha x^p\right)}{f(\alpha)},
\end{aligned}
$$

which is equivalent to (3.20).

If we write the inequality (1.4) for the convex function $\Phi:(0,\infty)\to\mathbb{R}$, $\Phi(x)=x\ln x$, then we have

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i x_i\ln(x_i)-\sum_{i=1}^{n} w_i x_i\ln\left(\sum_{i=1}^{n} w_i x_i\right) \\
&\leq \sum_{i=1}^{n} w_i\left[\ln(x_i)+1\right]x_i-\sum_{i=1}^{n} w_i\left[\ln(x_i)+1\right]\sum_{i=1}^{n} w_i x_i \\
&\leq \frac{1}{2}(M-m)\sum_{i=1}^{n} w_i\left|\ln x_i-\sum_{j=1}^{n} w_j\ln x_i\right| \\
&\leq \frac{1}{2}(M-m)\left[\sum_{i=1}^{n} w_i\left[\ln(x_i)+1\right]^2-\left(\sum_{i=1}^{n} w_i\left[\ln(x_i)+1\right]\right)^2\right]^{1/2} \qquad (3.24)
\end{aligned}
$$

for any $w_i\geq 0$ $(i=1,\dots,n)$ with $\sum_{i=1}^{n} w_i=1$ and $x_i\in[m,M]\subset(0,\infty)$ $(i=1,\dots,n)$.

Now, if we let $m\to 0+$ and $M=1$ in (3.24) we get

$$
0\leq \sum_{i=1}^{n} w_i x_i\ln(x_i)-\sum_{i=1}^{n} w_i x_i\ln\left(\sum_{i=1}^{n} w_i x_i\right)
$$

$$\le \sum_{i=1}^{n} w_i [\ln(x_i)+1] x_i - \sum_{i=1}^{n} w_i [\ln(x_i)+1] \sum_{i=1}^{n} w_i x_i$$
$$\le \frac{1}{2}\sum_{i=1}^{n} w_i \left|\ln x_i - \sum_{j=1}^{n} w_j \ln x_i\right|$$
$$\le \frac{1}{2}\left[\sum_{i=1}^{n} w_i [\ln(x_i)+1]^2 - \left(\sum_{i=1}^{n} w_i [\ln(x_i)+1]\right)^2\right]^{1/2} \tag{3.25}$$

for any $w_i \ge 0$ $(i=1,\dots,n)$ with $\sum_{i=1}^{n} w_i = 1$ and $x_i \in (0,1]$ $(i=1,\dots,n)$.

If $0 < x \le 1$, then $0 < x^p \le 1$ and if we apply the inequality (3.25) for $x_j = (x^p)^j \in (0,1]$, we have

$$0 \le \frac{p\ln x}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j \alpha^j (x^p)^j - \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j (x^p)^j \ln\left(\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j (x^p)^j\right)$$
$$\le \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\left[p\ln x\sum_{j=0}^{k} j a_j\alpha^j (x^p)^j + \sum_{j=0}^{k} a_j\alpha^j (x^p)^j\right]$$
$$- \frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\left[p\ln x\sum_{j=0}^{k} j a_j\alpha^j + \sum_{j=0}^{k} a_j\alpha^j\right]\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j (x^p)^j$$
$$\le \frac{1}{2}\left[\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j [jp\ln x+1]^2 - \left(\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j [jp\ln x+1]\right)^2\right]^{1/2}$$
$$= \frac{1}{2}\left[\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} a_j\alpha^j \left(j^2p^2(\ln x)^2 + 2jp\ln x\right) + 1 - \left(\frac{p\ln x}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j + 1\right)^2\right]^{1/2}$$
$$= \frac{1}{2}\left[\frac{p^2(\ln x)^2}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j^2 a_j\alpha^j + \frac{2p\ln x}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j + 1 - \left(\frac{p\ln x}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j + 1\right)^2\right]^{1/2}$$
$$= \frac{1}{2}p|\ln x|\left[\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j^2 a_j\alpha^j - \left(\frac{1}{\sum_{j=0}^{k} a_j\alpha^j}\sum_{j=0}^{k} j a_j\alpha^j\right)^2\right]^{1/2}. \tag{3.26}$$

Since all series whose partial sums involved in the inequality (3.26) are convergent, then by letting $k \to \infty$ in (3.26) we deduce (3.21). □

Corollary 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v > 0$ with $v^p \le u^q < R$, then*

$$0 \le \frac{f(v^p)}{f(u^q)}\ln\left(\frac{f(u^q)}{f(v^p)}\right) - \frac{v^p f'(v^p)}{f(u^q)}\ln\left(\frac{u^q}{v^p}\right)$$
$$\le u^q\left[\frac{f(v^p)}{f(u^q)}\frac{f'(u^q)}{f(u^q)} - \frac{\frac{v^p}{u^q}f'(v^p)}{f(u^q)}\right]\ln\left(\frac{u^q}{v^p}\right)$$

$$\leq \frac{1}{2}\left[\frac{u^q\left[f'\left(u^q\right)+u^q f''\left(u^q\right)\right]}{f\left(u^q\right)}-\left(\frac{u^q f'\left(u^q\right)}{f\left(u^q\right)}\right)^2\right]^{1/2}\ln\left(\frac{u^q}{v^p}\right). \tag{3.27}$$

Example 4.

(a) If we write the inequality (3.21) for the function $\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n$, $z\in D(0,1)$, then we have for $\alpha, x\in(0,1)$ and $p>0$ that

$$\begin{aligned}0&\leq \frac{p\alpha x^p(1-\alpha)}{(1-\alpha x^p)^2}\ln x-\frac{1-\alpha}{(1-\alpha x^p)}\ln\left(\frac{1-\alpha}{1-\alpha x^p}\right)\\&\leq p\alpha\left[\frac{x^p(1-\alpha)}{(1-\alpha x^p)^2}-\frac{1}{1-\alpha x^p}\right]\ln x\leq\frac{1}{2}\frac{p\alpha^{1/2}}{1-\alpha}|\ln x|.\end{aligned} \tag{3.28}$$

(b) If we write the inequality (3.21) for the function $\exp z=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$, $z\in\mathbb{C}$, then we have

$$\begin{aligned}0&\leq\left[p\alpha x^p\ln x-\alpha\left(x^p-1\right)\right]\exp\left[\alpha\left(x^p-1\right)\right]\\&\leq p\alpha\left(x^p-1\right)\exp\left[\alpha\left(x^p-1\right)\right]\ln x\leq\frac{1}{2}p|\ln x|\alpha^{1/2}\end{aligned} \tag{3.29}$$

for $x\in(0,1)$ and $\alpha, p>0$.

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