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ON SEMI-SYMMETRIC METRIC CONNECTION IN SUB-RIEMANNIAN MANIFOLD

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Abstract. The authors firstly in this paper define a semi-symmetric metric non-holonomic connection (in briefly, SS-connection) on sub-Riemannian manifolds. An invariant under a SS-connection transformation is obtained. The authors then further give a result that a sub-Riemannian manifold $(M, V_0, g, \bar{\nabla})$ is locally horizontally flat if and only if M is horizontally conformally flat and horizontally Ricci flat.

1. Introduction

In order to formulate a unified field theory, H. Weyl [8] introduced a generalization of Riemannian geometry. Weyl's theory provides an instructive example of non-Riemannian connections. These non-Riemannian connections are exactly the semi-symmetric metric connection which firstly proposed by K.Yano [10] in 1970. The study of various semi-symmetric connections on Riemannian or non-Riemannian manifolds has been an active field over the past seven decades. In particular, since the formidable papers [1, 3, 4, 5, 6, 7] were published in succession, these works had stimulated such research fields to present a scene of prosperity, and demonstrate the importance of this topic.

In this paper we will do a similar argument on sub-Riemannian manifolds, that is, we will introduce a semi-symmetric metric connection (SS-connection) on sub-Riemannian manifolds, and investigate the geometries of sub-Riemannian manifolds equipped with a class of SS-connection(defined below) by combining the idea of K. Yano with the work of Zhao and Jiao [11].

The paper is organized as follows. In Section 2 we collect some necessary definitions and notations about sub-Riemannian manifolds which will be used later . Then we define a class of semi-symmetric metric connection(i.e. SS-connection defined below) based on the unique

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SR-connection. Moreover we find that the horizontal Weyl conformal curvature tensors are kept unchanged under the horizontal projective transformation. A sufficient and necessary condition that a sub-Riemannian manifold $(M, V_0, g, \bar{\nabla})$ is locally horizontally flat is given at the end of Section 3. In section 4, we explain our results by Heisenberg group.

2. Preliminaries

Let (M, V_0, g) be a n-dimensional sub-Riemannian manifold, where V_0 is a ℓ-dimensional sub-bundle, that is the so-called horizontal bundle, g is called the sub-Riemannian metric. In the paper, we denote by $\Gamma(V_0)$ the $C^\infty(M)$ -module of smooth sections on V_0. Also, if not stated otherwise, we use the following ranges for indices: $i, j, k, h, \cdots \in \{1, \cdots, \ell\}$, $\alpha, \beta, \cdots \in \{\ell+1, \cdots, n\}$. The repeated indices with one upper index and one lower index indicates summation over their range.

In order to study the geometry of $\{M, V_0, g\}$, we suppose that there exists a Riemannian metric $\langle \cdot, \cdot \rangle$ and V_1 is taken as the complementary orthogonal distribution to V_0 in TM, then, there holds $V_0 \oplus V_1 = TM$. Here we call V_1 the vertical distribution. Denote by X_0 the projection of the vector field X from TM onto V_0, and by X_1 the projection of the vector field X from TM onto V_1.

Assume that $\{e_i\}$ is a basis of V_0, then the formulas $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$, define ℓ^3 functions as Γ^k_{ij}, we call Γ^k_{ij} the connection coefficients of the non-holonomic connection ∇. It is well known that the Lie bracket $[\cdot, \cdot]$ on M is a Lie algebra structure of smooth tangent vector fields $\Gamma(TM)$, then it is easy to see that the following formula

$$[e_i, e_j]_0 = \Omega^k_{ij} e_k,$$

determine ℓ^3 functions Ω^k_{ij}.

Theorem 2.1 ([2, 9])**.** *Given a sub-Riemannian manifold* (M, V_0, g)*, then there exists a unique non-holonomic connection satisfying*

$$(\nabla_Z g)(X, Y) = Z(g(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = 0, \tag{2.1}$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]_0 = 0. \tag{2.2}$$

Definition 2.1. A non-holonomic connection is said to be metric if it satisfies (2.1) and symmetric if it satisfies (2.2). A non-holonomic connection satisfying (2.1) and (2.2) is called a sub-Riemannian connection, in short, SR-connection.

Remark 2.1. For given sub-Riemannian metric g, it is extended to Riemannian metric $\bar{g}$ in TM. If we denote D by the Levi-civita connection associated with $\bar{g}$, then the SR-connection is exactly the projection of Levi-civita connection D on the horizontal bundle, namely, for any horizontal vectors X, Y, there holds

$$\nabla_X Y = (D_X Y)_0.$$

Theorem 2.1 is the counterpart of the existence and uniqueness of the Levi-Civita connection in Riemannian geometry. It can be regarded as the projection of Levi-Civita connection on the horizontal bundle. We will use this SR-connection to build the relative transformative theories of the semi-symmetric metric connection.

For sub-Riemannian manifolds, J. A. Schouten first considered the curvature problem of non-holonomic connections(see [2]), he defined a curvature tensor as follows:

Definition 2.2. A horizontal curvature tensor is a mapping $R^H : \Gamma(V_0)\times\Gamma(V_0) \to \Gamma(V_0)$ defined by

$$R^H(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]_0} Z - [[X,Y]_1, Z]_0, \tag{2.3}$$

where $X,\ Y,\ Z \in \Gamma(V_0)$.

Proposition 2.2. *For any horizontal vector fields $X, Y, Z, V, W \in \Gamma(V_0)$,*

(1) $R^H(X,Y)Z + R^H(Y,X)Z = 0$;

(2) $R^H(X,Y)Z + R^H(Y,Z)X + R^H(Z,X)Y = 0$;

(3) $R^H(X,Y,Z,W) + R^H(Y,X,Z,W) = [Z,W]_1 g(Y,X) - g([[Z,W]_1,X]_0,Y) - g([[Z,W]_1,Y]_0,X)$.

where $R^H(X,Y,Z,W) = g(R^H(X,Y)Z,W)$.

Proof. (1), (2) follow from Definition 2.2 and the Jacobi identity. One need to show formula (3).

$$\begin{aligned}
R^H(X,Y,Z,W) + R^H(Y,X,Z,W) &= g(R^H(Z,W)Y,X) + g(R^H(Z,W)X,Y)\\
&= g(\nabla_Z\nabla_W Y,X) - g(\nabla_W\nabla_Z Y,X) - g(\nabla_{[Z,W]_0}Y,X) - g([[Z,W]_1,Y]_0,X)\\
&\quad + g(\nabla_Z\nabla_W X,Y) - g(\nabla_W\nabla_Z X,Y) - g(\nabla_{[Z,W]_0}X,Y) - g([[Z,W]_1,X]_0,Y)\\
&= Zg(\nabla_W Y,X) - g(\nabla_W Y,\nabla_Z X) - Wg(\nabla_Z Y,X) + g(\nabla_Z Y,\nabla_W X) - g([[Z,W]_{1,Y}]_0,X)\\
&\quad + g(\nabla_Z\nabla_W X,Y) - g(\nabla_W\nabla_Z X,Y) - g(\nabla_{[Z,W]_0}X,Y) - g([[Z,W]_1,X]_0,Y)\\
&= Z\{Wg(Y,X) - g(Y,\nabla_W X)\} - Wg(Y,\nabla_Z X) + g(Y,\nabla_W\nabla_Z X) - W\{Zg(Y,X) - g(Y,\nabla_Z X)\}\\
&\quad + Zg(Y,\nabla_Z X) - g(Y,\nabla_Z\nabla_W X) - [Z,W]_0 g(Y,X) - g([[Z,W]_1,Y]_0,X)\\
&\quad + g(\nabla_Z\nabla_W X,Y) - g(\nabla_W\nabla_Z X,Y) - g(\nabla_{[Z,W]_0}X,Y) - g([[Z,W]_1,X]_0,Y)\\
&= -g([[Z,W]_1,Y]_0,X) - g([[Z,W]_1,X]_0,Y) + \{ZW - WZ - [Z,W]_0\}g(Y,X)
\end{aligned}$$

$$= [Z,W]_1 g(Y,X) - g([[Z,W]_1,X]_0,Y) - g([[Z,W]_1,Y]_0,X).$$

This finishes the proof. □

Let $\{e_i\}$ be a basis of V_0, we denote by

$$R^H(e_i,e_j)e_k = (R^H)^h_{ijk}e_h, \nabla_{e_i}e_j = \Gamma^k_{ij}e_k, [e_i,e_j]_0 = \Omega^k_{ij}e_k,$$
$$[e_i,e_j]_1 = M^\alpha_{ij}e_\alpha, [[e_i,e_j]_1,e_k]_0 = M^\alpha_{ij}\Lambda^h_{\alpha k}e_h.$$

Then we know that

$$(R^H)^h_{ijk} = e_i(\Gamma^h_{jk}) - e_j(\Gamma^h_{ik}) + \Gamma^e_{jk}\Gamma^h_{ie} - \Gamma^e_{ik}\Gamma^h_{je} - \Omega^e_{ij}\Gamma^h_{ke} - M^\alpha_{ij}\Lambda^h_{\alpha k}. \tag{2.4}$$

Since ∇ is torsion free, then we get

$$\nabla_{e_i}e_j - \nabla_{e_j}e_i - [e_i,e_j]_0 = 0,$$

so we arrive at

$$\Gamma^k_{ij} - \Gamma^k_{ji} = \Omega^k_{ij}, \tag{2.5}$$

we further have

$$[e_i,e_j] - \Omega^k_{ij}e_k = M^\alpha_{ij}e_\alpha. \tag{2.6}$$

In this basis, the identity (1) and (2) in Proposition 2.2 can be rewritten, respectively, as

$$(R^H)^h_{ijk} = -(R^H)^h_{jik}, \tag{2.7}$$
$$(R^H)^h_{ijk} + (R^H)^h_{jki} + (R^H)^h_{kij} = 0. \tag{2.8}$$

We call (2.8) the first Bianchi identity of the SR-connection ∇.

In (2.8), by taking $j = h = e$ and using (2.7), we get

$$(R^H)^e_{kie} = (R^H)^e_{kei} - (R^H)^e_{iek}. \tag{2.9}$$

It is clear that $(R^H)^e_{kie}$ is an anti-symmetric $(0,2)$ tensor , which is different from Riemannian case. So

$$0 = (R^H)^e_{kie}g^{ki} + (R^H)^e_{ike}g^{ki} = (R^H)^e_{kie}g^{ki} + (R^H)^e_{ike}g^{ik} = 2(R^H)^e_{kie}g^{ki}.$$

Now multiplying g^{ki} at both side of (2.9), then $g^{ki}(R^H)^e_{kei} - (R^H)^e_{iek}g^{ik} = 0$. Similar to the case of Riemannian manifolds, we call $R^H = g^{ik}(R^H)^e_{iek}$ the horizontal scalar curvature, and $(R^H)^e_{iek}$ the horizontal Ricci curvature tensor of horizontal curvature tensors.

3. Main theorems and proofs

In view of the unique SR-connection in sub-Riemannian manifolds, we firstly introduce a very important non-holonomic connection-semi-sub-Riemannian connection. Roughly speaking, a semi-sub-Riemannian connection is a non-holonomic connection with non-vanishing torsion tensor which is compatible with sub-Riemannian metric. Now we give a new definition below

Definition 3.1. A non-holonomic connection is called a semi-sub-Riemannian connection, in short, a SS-connection, if it satisfies

$$\begin{cases} (\bar{\nabla}_Z g)(Y,Z) = Zg(X,Y) - g(\bar{\nabla}_Z X, Y) - g(X, \bar{\nabla}_Z Y) = 0, \forall X,Y,Z \in V_0, \\ \bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_X Y - [X,Y]_0 = \pi(Y)X - \pi(Y)X, \forall X,Y,Z \in V_0. \end{cases} \tag{3.1}$$

where π is a smooth 1-form defined on the horizontal bundle.

Remark 3.1. It's obvious that the SS-connection is a metric connection. It is also called a SS-connection transformation from the transformation's theory. We denote a sub-Riemannian manifold (M, V_0, g) admitting a SS-connection $\bar{\nabla}$ by $(M, V_0, g, \bar{\nabla})$.

By a straight forward calculation, one can derive that the SS-connection $\bar{\nabla}$ is necessarily of the form,

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P, \tag{3.2}$$

where P is a horizontal vector field defined by $g(P,X) = \pi(X)$ for any $X \in V_0$. In local frame $\{e_i\}$, denote by $\pi(e_i) = \pi_i$, $\pi^i = g^{ij}\pi_j$, then we know

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \pi_j - g_{ij}\pi^k, \tag{3.3}$$

and the horizontal curvature tensor of the SS-connection $\bar{\nabla}$ is

$$(\bar{R}^H)^h_{ijk} = e_i(\bar{\Gamma}^h_{jk}) - e_j(\bar{\Gamma}^h_{ik}) + \bar{\Gamma}^e_{jk}\bar{\Gamma}^h_{ie} - \bar{\Gamma}^e_{ik}\bar{\Gamma}^h_{je} - \bar{\Omega}^e_{ij}\bar{\Gamma}^h_{ke} - \bar{M}^\alpha_{ij}\bar{\Lambda}^h_{\alpha k}, \tag{3.4}$$

where

$$[e_i, e_j]_0 = \bar{\Omega}^k_{ij} e_k, [e_i, e_j]_1 = \bar{M}^\alpha_{ij} e_\alpha, [[e_i, e_j]_1, e_k]_0 = \bar{M}^\alpha_{ij}\bar{\Lambda}^h_{\alpha k} e_h,$$

then by using (2.5), (2.6) and (3.3), we have

$$\bar{\Omega}^k_{ij} = \Omega^k_{ij}, \bar{M}^\alpha_{ij} = M^\alpha_{ij}, \bar{\Lambda}^h_{\alpha k} = \Lambda^h_{\alpha k}. \tag{3.5}$$

Substituting (3.3) and (3.5) into (3.4) and by straightway computation, we can get the relation between the horizontal curvature tensor of $\bar{\nabla}$ and ∇ as follows

$$(\bar{R}^H)^h_{ijk} = (R^H)^h_{ijk} + \delta^h_j \pi_{ik} - \delta^h_i \pi_{jk} + \pi^h_j g_{ik} - \pi^h_i g_{jk}, \tag{3.6}$$

where

$$\pi_{ik} = \nabla_i \pi_k - \pi_i \pi_k + \frac{1}{2} g_{ik} \pi_h \pi^h, \pi_i^j = \pi_{ik} g^{jk}, \nabla_i \pi_j = e_i(\pi_j) - \Gamma_{ij}^k \pi_k. \tag{3.7}$$

It is not hard to derive that $(\bar{R}^H)_{ijk}^h$ satisfy the following properties,

$$\begin{cases} (\bar{R}^H)_{ijk}^h + (\bar{R}^H)_{jik}^h = 0; \\ (\bar{R}^H)_{ijk}^h + (\bar{R}^H)_{jki}^h + (\bar{R}^H)_{kij}^h = \delta_j^h(\pi_{ik} - \pi_{ki}) + \delta_i^h(\pi_{jk} - \pi_{kj}) + \delta_k^h(\pi_{ij} - \pi_{ji}). \end{cases}$$

The second formula is called the first Bianchi identity of the SS-connection. Contracting j and h in (3.6), we have

$$(\bar{R}^H)_{iek}^e = (R^H)_{iek}^e + (\ell - 2)\pi_{ik} + \alpha g_{ik}, \tag{3.8}$$

where $\alpha = \pi_{ij} g^{ij} = \pi_i^i$. It is no longer symmetric about the two indexes unless $\pi_{ik} = \pi_{ik}$, namely π is closed on the horizontal bundle. Now when multiplying (3.8) by g^{ik} we get

$$(\bar{R})^H = R^H + 2(\ell - 1)\alpha. \tag{3.9}$$

We call $\bar{R}^H$ the horizontal curvature, and hence $(\bar{R}^H)_{iek}^e$ the horizontal Ricci curvature tensors w.r.t. the SS-connection.

For the SS-connection $\bar{\nabla}$, we define the horizontal Weyl conformal curvature tensors by

$$\begin{aligned} \bar{C}_{ijk}^h = (\bar{R}^H)_{ijk}^h &- \frac{1}{\ell - 2}\{\delta_j^h((\bar{R}^H)_{iek}^e - \frac{1}{\ell}(\bar{R}^H)_{ike}^e - \frac{\bar{R}^H}{2(\ell-1)} g_{ik}) - \delta_i^h((\bar{R}^H)_{jek}^e \\ &- \frac{1}{\ell}(\bar{R}^H)_{jke}^e - \frac{\bar{R}^H}{2(\ell-1)} g_{jk}) + g_{ik}((\bar{R}^H)_{jef}^e g^{fh} - \frac{1}{\ell}(\bar{R}^H)_{jfe}^e g^{fh} - \frac{\bar{R}^H}{2(\ell-1)}\delta_j^h) \\ &- g_{jk}((\bar{R}^H)_{ief}^e g^{fh} - \frac{1}{\ell}(\bar{R}^H)_{ife}^e g^{fh} - \frac{\bar{R}^H}{2(\ell-1)}\delta_i^h)\} + \frac{1}{\ell}\delta_k^h(\bar{R}^H)_{ije}^e. \end{aligned} \tag{3.10}$$

Remark 3.2. The horizontal Weyl conformal curvature tensors $\bar{C}_{ijk}^h$ will degenerate into the sub-conformal Weyl curvature tensors defined by [11], if the 1-form π vanishes. It is natural to assume $\ell > 2$ from now.

Theorem 3.1. *The horizontal Weyl conformal curvature tensors are invariants under the SS-connection transformation.*

Proof. In virtue of Equation (3.6), one has

$$(\bar{R}^H)_{ijk}^h + (\bar{R}^H)_{jik}^h = 0,$$

and

$$(\bar{R}^H)_{ijk}^h + (\bar{R}^H)_{jki}^h + (\bar{R}^H)_{kij}^h = \delta_j^h(\pi_{ik} - \pi_{ki}) + \delta_i^h(\pi_{kj} - \pi_{jk}) + \delta_k^h(\pi_{ji} - \pi_{ij}).$$

Let $k = h = e$, one gets

$$(\bar{R}^H)^e_{ije} = (\bar{R}^H)^e_{iej} - (\bar{R}^H)^e_{jei} + (\ell-2)(\pi_{ji} - \pi_{ij}).$$

Considering (3.8), one further obatins

$$(\bar{R}^H)^e_{ije} = (R^H)^e_{iej} - (R^H)^e_{jei} = (R^H)^e_{ije}. \tag{3.11}$$

The substitution of Equations (3.6), (3.8),(3.9) and (3.11)into (3.10) implies

$$\begin{aligned}
\bar{C}^h_{ijk} &= (\bar{R}^H)^h_{ijk} - \frac{1}{\ell-2}\{\delta^h_j((\bar{R}^H)^e_{iek} - \frac{1}{\ell}(\bar{R}^H)^e_{ike} - \frac{\bar{R}^H}{2(\ell-1)}g_{ik}) - \delta^h_i((\bar{R}^H)^e_{jek} \\
&\quad -\frac{1}{\ell}(\bar{R}^H)^e_{jke} - \frac{\bar{R}^H}{2(\ell-1)}g_{jk}) + g_{ik}((\bar{R}^H)^e_{jef}g^{fh} - \frac{1}{\ell}(\bar{R}^H)^e_{jfe}g^{fh} - \frac{\bar{R}^H}{2(\ell-1)}\delta^h_j) \\
&\quad -g_{jk}((\bar{R}^H)^e_{ief}g^{fh} - \frac{1}{\ell}(\bar{R}^H)^e_{ife}g^{fh} - \frac{\bar{R}^H}{2(\ell-1)}\delta^h_i)\} + \frac{1}{\ell}\delta^h_k(\bar{R}^H)^e_{ije} \\
&= (R^H)^h_{ijk} + \delta^h_j\pi_{ik} - \delta^h_i\pi_{jk} + \pi^h_j g_{ik} - \pi^h_i g_{jk} \\
&\quad -\frac{1}{\ell-2}\delta^h_j[(R^H)^e_{iek} + (\ell-2)\pi_{ik} + \alpha g_{ik} - \frac{1}{\ell}(R^H)^e_{ike} - \frac{R^H + 2(\ell-1)\alpha}{2(\ell-1)}g_{ik}] \\
&\quad +\frac{1}{\ell-2}\delta^h_i[(R^H)^e_{jek} + (\ell-2)\pi_{jk} + \alpha g_{jk} - \frac{1}{\ell}(R^H)^e_{jke} - \frac{R^H + 2(\ell-1)\alpha}{2(\ell-1)}g_{jk}] \\
&\quad -\frac{1}{\ell-2}g_{ik}[g^{fh}((R^H)^e_{jef} + (\ell-2)\pi_{jf} + \alpha g_{jf}) - \frac{1}{\ell}g^{fh}(R^H)^e_{jfe} - \frac{R^H + 2(\ell-1)\alpha}{2(\ell-1)}\delta^h_j] \\
&\quad +\frac{1}{\ell-2}g_{jk}[g^{fh}(R^e_{ief} + (\ell-2)\pi_{if} + \alpha g_{if}) - \frac{1}{\ell}g^{fh}(R^H)^e_{ife} - \frac{R^H + 2(\ell-1)\alpha}{2(\ell-1)}\delta^h_i] \\
&\quad +\frac{1}{\ell}\delta^h_k(R^H)^e_{ije} \\
&= (R^H)^h_{ijk} - \frac{1}{\ell-2}\{\delta^h_j((R^H)^e_{iek} - \frac{1}{\ell}(R^H)^e_{ike} - \frac{R^H}{2(\ell-1)}g_{ik}) - \delta^h_i((R^H)^e_{jek} \\
&\quad -\frac{1}{\ell}(R^H)^e_{jke} - \frac{R^H}{2(\ell-1)}g_{jk}) + g_{ik}((R^H)^e_{jef}g^{fh} - \frac{1}{\ell}(R^H)^e_{jfe}g^{fh} - \frac{R^H}{2(\ell-1)}\delta^h_j) \\
&\quad -g_{jk}((R^H)^e_{ief}g^{fh} - \frac{1}{\ell}(R^H)^e_{ife}g^{fh} - \frac{R^H}{2(\ell-1)}\delta^h_i)\} + \frac{1}{\ell}\delta^h_k(R^H)^e_{ije} \\
&= C^h_{ijk}.
\end{aligned}$$

This finishes the proof. □

Definition 3.2. A sub-Riemannian manifold $(M, V_0, g, \bar{\nabla})$ is locally horizontally flat if and only if the horizontal curvature tensors associated with the SS-connection $\bar{\nabla}$ equal zero, i.e. $(\bar{R}^H)^h_{ijk} = 0$.

Theorem 3.2. *A sub-Riemannian manifold $(M, V_0, g, \bar{\nabla})$ is locally horizontally flat if and only if M is horizontally conformally flat and horizontally Ricci flat.*

Proof. If $(M, V_0, g, \bar{\nabla})$ is locally horizontally flat, then $(\bar{R}^H)^h_{ijk} = 0$, w.r.t. the SS-connection, that is, there holds

$$(R^H)^h_{ijk} = \delta^h_i \pi_{jk} - \delta^h_j \pi_{ik} + \pi^h_i g_{jk} - \pi^h_j g_{ik}, \tag{3.12}$$

let $j = h = e$, we obtain

$$(R^H)^e_{iek} = (2-\ell)\pi_{ik} - \alpha g_{ik}. \tag{3.13}$$

Multiplying the Equation (3.13) by g^{ik} we get $R^H = (R^H)^e_{iek} g^{ik} = 2(1-\ell)\alpha$, so we have

$$\alpha = \frac{R^H}{2(1-\ell)}. \tag{3.14}$$

Substituting (3.14) into (3.13), we get

$$\pi_{ik} = \frac{1}{2-\ell}((R^H)^e_{iek} - \frac{R^H}{2(\ell-1)} g_{ik}). \tag{3.15}$$

Similarly, we substitute (3.15) into (3.12), we have

$$\begin{aligned}(R^H)^h_{ijk} = &-\frac{1}{\ell-2}(\delta^h_i (R^H)^e_{jek} - \delta^h_j (R^H)^e_{iek} + g_{jk}(R^H)^e_{ief} g^{fh} - g_{ik}(R^H)^e_{jef} g^{fh}) \\ &+ \frac{R^H}{(\ell-2)(\ell-1)}(g_{jk}\delta^h_i - g_{ik}\delta^h_j),\end{aligned} \tag{3.16}$$

and $(R^H)^e_{ije} = 0$, which means $C^h_{ijk} = 0$. Hence one has $\bar{C}^h_{ijk} = 0$ because of Theorem 3.1.

Conversely, since M is horizontally conformally flat, $\bar{C}^h_{ijk} = 0$, then $C^h_{ijk} = 0$ in view of Theorem 3.1, and

$$\begin{aligned}(R^H)^h_{ijk} = \frac{1}{\ell-2}\{&\delta^h_j((R^H)^e_{iek} - \frac{1}{\ell}(R^H)^e_{ike} - \frac{R^H}{2(\ell-1)} g_{ik}) - \delta^h_i((R^H)^e_{jek} \\ &- \frac{1}{\ell}(R^H)^e_{jke} - \frac{R^H}{2(\ell-1)} g_{jk}) + g_{ik}((R^H)^e_{jef} g^{fh} - \frac{1}{\ell}(R^H)^e_{jfe} g^{fh} - \frac{R^H}{2(\ell-1)}\delta^h_j) \\ &- g_{jk}((R^H)^e_{ief} g^{fh} - \frac{1}{\ell}(R^H)^e_{ife} g^{fh} - \frac{R^H}{2(\ell-1)}\delta^h_i)\} - \frac{1}{\ell}\delta^h_k (R^H)^e_{ije}.\end{aligned}$$

By contracting with k and h, one obtains $(R^H)^e_{ije} = 0$, and hence $(R^H)^e_{iej} = (R^H)^e_{jei}$ because of the first Bianchi identity of the SR-connection. Therefore $\pi_{ik} = \frac{1}{2-\ell}((R^H)^e_{iek} - \frac{R^H}{2(\ell-1)} g_{ik})$ is symmetric, and hence one has the first Bianchi identity of the SS-connection

$$(\bar{R}^H)^h_{ijk} + (\bar{R}^H)^h_{jki} + (\bar{R}^H)^h_{kij} = 0,$$

one further gets by contracting k and h,

$$(\bar{R}^H)^e_{ije} = (\bar{R}^H)^e_{iej} - (\bar{R}^H)^h_{jei}.$$

On the other hand, $\pi_{ik} = \frac{1}{2-\ell}((R^H)^e_{iek} - \frac{R^H}{2(\ell-1)} g_{ik})$ means $(\bar{R}^H)^e_{iek} = 0$ based on the fact (3.8) and (3.9), and $\bar{C}^h_{ijk} = 0$ can derive $(\bar{R}^H)^e_{ike} = 0$, so one has $\bar{R}^H = 0$, and hence

$$\begin{aligned}(\bar{R}^H)^h_{ijk} &= \bar{C}^h_{ijk} + \frac{1}{\ell-2}\{\delta^h_j((\bar{R}^H)^e_{iek} - \frac{1}{\ell}(\bar{R}^H)^e_{ike} - \frac{\bar{R}^H}{2(\ell-1)} g_{ik}) - \delta^h_i((\bar{R}^H)^e_{jek} \\ &\quad - \frac{1}{\ell}(\bar{R}^H)^e_{jke} - \frac{\bar{R}^H}{2(\ell-1)} g_{jk}) + g_{ik}((\bar{R}^H)^e_{jef} g^{fh} - \frac{1}{\ell}(\bar{R}^H)^e_{jfe} g^{fh} - \frac{\bar{R}^H}{2(\ell-1)}\delta^h_j) \\ &\quad - g_{jk}((\bar{R}^H)^e_{ief} g^{fh} - \frac{1}{\ell}(\bar{R}^H)^e_{ife} g^{fh} - \frac{\bar{R}^H}{2(\ell-1)}\delta^h_i)\} - \frac{1}{\ell}\delta^h_k(\bar{R}^H)^e_{ije} \\ &= 0,\end{aligned}$$

where the second equality follows from Equations (3.11) and Theorem 3.1.

This completes the proof of Theorem 3.2. □

4. Examples

Let $M = H^n$ be a Heisenberg group with the noncommutative law

$$x \circ y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n + \frac{1}{2}\Sigma^n_{i,j=1}(x_i y_{n+j} - x_{n+j} y_i)).$$

for any $x = (x_i, x_{n+i}, x_{2n+1})$, $y = (y_i, y_{n+i}, y_{2n+1})$. The left invariant vectors are given by

$$e_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial x_{2n+1}}, e_{n+i} = \frac{\partial}{\partial x_{n+i}} + \frac{x_i}{2}\frac{\partial}{\partial x_{2n+1}}, e_{2n+1} = \frac{\partial}{\partial x_{2n+1}}.$$

Take the horizontal bundle V_0 spanned by e_i, e_{n+i}. Consider $V_1 = span\{e_{2n+1}\}$, and g as the Riemannian metric which $\{e_i, e_{n+i}, e_{2n+1}\}$ is an orthonomal basis. We note that the only non-trivial commutator is

$$[e_i, e_{n+j}] = -\delta_{ij} e_{2n+1}. \tag{4.1}$$

We construct the Levi-civita connection compatible with the Riemannian metric g via the usual Kozul formula

$$\begin{cases} D_{e_i} e_{n+j} = -\frac{1}{2}\delta_{ij} e_{2n+1}, D_{e_i} e_{2n+1} = \frac{1}{2} e_{n+i}, \\ D_{e_{n+i}} e_j = \frac{1}{2}\delta_{ij} e_{2n+1}, D_{e_{n+i}} e_{2n+1} = -\frac{1}{2} e_i, \\ D_{e_{2n+1}} e_i = -\frac{1}{2} e_{n+i}, D_{e_{2n+1}} e_{n+i} = \frac{1}{2} e_i, \end{cases}$$

the left covariant derivatives vanish. So the unique SR-connection is

$$\nabla_{e_i} e_{n+j} = (D_{e_i} e_{n+j})_0 = 0,$$

and hence for all $X, Y \in V_0$ with $Y = \Sigma^n_{i=1}(Y^i e_i + Y^{n+i} e_{n+i})$,

$$\nabla_X Y = \Sigma^n_{i=1}(X(Y^i) e_i + X(Y^{n+i}) e_{n+i}).$$

If we denote the horizontal vector field Z by $Z = \Sigma_{k=1}^{2n} Z^k e_k$, then the horizontal curvature tensor can be given exactly as

$$\begin{aligned} R^H(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]_0} Z - [[X,Y]_1, Z]_0 \\ &= \Sigma_{k=1}^{2n}(XY(Z^k)e_k - YX(Z^k)e_k - [X,Y]_0(Z^k)e_k \\ &\quad -[X,Y]_1(Z^k)e_k - Z^k[[X,Y]_1, e_k]_0) \\ &= \Sigma_{k=1}^{2n}([X,Y]-[X,Y]_0-[X,Y]_1)(Z^k)e_k - \Sigma_{k=1}^{2n} Z^k[[X,Y]_1,e_k]_0 \\ &= 0, \end{aligned} \tag{4.2}$$

where the last equality follows from Equation (4.1) and

$$\begin{aligned} [X,Y] &= \Sigma_{i,j=1}^n (X^i e_i(Y^j)e_j + X^iY^j e_ie_j + X^{n+i}e_{n+i}(Y^j)e_j + X^{n+i}Y^j e_{n+i}e_j \\ &\quad + X^i e_i(Y^{n+j})e_{n+j} + X^iY^{n+j}e_ie_{n+j} + X^{n+i}e_{n+i}(Y^{n+j})e_{n+j} + X^{n+i}Y^{n+j}e_{n+i}e_{n+j}) \\ &\quad -\Sigma_{i,j=1}^n (Y^je_j(X^i)e_i + X^iY^je_je_i + Y^{n+j}e_{n+j}(X^i)e_j + Y^{n+j}X^ie_{n+j}e_i \\ &\quad + Y^je_j(X^{n+i})e_{n+i} + Y^jX^{n+i}e_je_{n+i} + Y^{n+j}e_{n+j}(X^{n+i})e_{n+i} + Y^{n+j}X^{n+i}e_{n+j}e_{n+i}) \\ &= \Sigma_{i,j=1}^n (X^ie_i(Y^j)e_j + X^{n+i}e_{n+i}(Y^j)e_j + X^ie_i(Y^{n+j})e_{n+j} + X^{n+i}e_{n+i}(Y^{n+j})e_{n+j} \\ &\quad - Y^je_j(X^i)e_i - Y^{n+j}e_{n+j}(X^i)e_j - Y^je_j(X^{n+i})e_{n+i} - Y^{n+j}e_{n+j}(X^{n+i})e_{n+i} \\ &\quad + X^iY^j[e_i,e_j] + X^{n+i}Y^j[e_{n+i},e_j] + X^iY^{n+j}[e_i,e_{n+j}] + X^{n+i}Y^{n+j}[e_{n+i},e_{n+j}]), \end{aligned}$$

so

$$\begin{aligned} [X,Y]_1 &= \Sigma_{i,j=1}^n (X^iY^j[e_i,e_j]_1 + X^{n+i}Y^j[e_{n+i},e_j]_1 + X^iY^{n+j}[e_i,e_{n+j}]_1 \\ &\quad + X^{n+i}Y^{n+j}[e_{n+i},e_{n+j}]_1) \\ &= \Sigma_i^n (X^{n+i}Y^i - X^iY^{n+i})e_{2n+1}. \end{aligned}$$

Hence the corresponding horizontal Weyl conformal curvature tensors $C^h_{ijk} = 0$.

Now we define a SS-connection by

$$\bar\nabla_X Y = \Sigma_{i,j,k=1}^{2n}(X^ie_i(Y^k) + Y^j\pi_jX^k - X^iY^jg_{ij}\pi^k)e_k,$$

the horizontal curvature tensors are given, based on Equation (4.2), by,

$$(\bar R^H)^h_{ijk} = \delta^h_j\pi_{ik} - \delta^h_i\pi_{jk} + \pi^h_j g_{ik} - \pi^h_i g_{jk}.$$

By contracting k and h, one obtains $(\bar R^H)^e_{ije} = 0$, and

$$(\bar R^H)^e_{iek} = 2(n-1)\pi_{ik} + \alpha g_{ik};\ (\bar R^H)^e_{jei} - (\bar R^H)^e_{iej} = 2(2n-1)(\pi_{ij} - \pi_{ji}), \tag{4.3}$$

so if $\pi_{ij} = \pi_{ji}$, one can define the horizontal Ricci tensor and horizontal curvature with respect to the SS-connection.

To show H^n is a horizontal flat manifold, one need to show the horizontal Weyl conformal curvature tensors $\bar{C}^h_{ijk}$ equal zero. In fact,

$$\begin{aligned}
\bar{C}^h_{ijk} &= (\bar{R}^H)^h_{ijk} - \frac{1}{2(n-1)}\{\delta^h_j((\bar{R}^H)^e_{iek} - \frac{1}{2n}(\bar{R}^H)^e_{ike} - \frac{\bar{R}^H}{2(2n-1)}g_{ik}) - \delta^h_i((\bar{R}^H)^e_{jek} \\
&\quad - \frac{1}{2n}(\bar{R}^H)^e_{jke} - \frac{\bar{R}^H}{2(2n-1)}g_{jk}) + g_{ik}((\bar{R}^H)^e_{jef}g^{fh} - \frac{1}{2n}(\bar{R}^H)^e_{jfe}g^{fh} - \frac{\bar{R}^H}{2(2n-1)}\delta^h_j) \\
&\quad - g_{jk}((\bar{R}^H)^e_{ief}g^{fh} - \frac{1}{2n}(\bar{R}^H)^e_{ife}g^{fh} - \frac{\bar{R}^H}{2(2n-1)}\delta^h_i)\} + \frac{1}{2n}\delta^h_k(\bar{R}^H)^e_{ije} \\
&= (\bar{R}^H)^h_{ijk} - \frac{1}{2(n-1)}\{\delta^h_j(2(n-1)\pi_{ik} + \alpha g_{ik}) - \frac{\bar{R}^H}{2(n-1)}g_{ik}\delta^h_j - \delta^h_i(2(n-1)\pi_{jk} + \alpha g_{jk}) \\
&\quad + \frac{\bar{R}^H}{2(2n-1)}g_{jk}\delta^h_i + g_{ik}(2(n-1)\pi_{jf} + \alpha g_{jf})g^{fh} - \frac{\bar{R}^H}{2(2n-1)}g_{ik}\delta^h_j \\
&\quad - g_{jk}(2(n-1)\pi_{if} + \alpha g_{if})g^{fh} + \frac{\bar{R}^H}{2(2n-1)}g_{jk}\delta^h_i\} \\
&= -\frac{\alpha}{n-1}g_{ik}\delta^h_j + \frac{\alpha}{n-1}g_{jk}\delta^h_i + \frac{\bar{R}^H}{2(n-1)(n-1)}g_{ik}\delta^h_j - \frac{\bar{R}^H}{2(2n-1)(n-1)}g_{jk}\delta^h_i \\
&= 0,
\end{aligned}$$

where the last equality follows from Equation (4.3).

Therefore Heisenberg group H^n is a horizontally flat manifold and the horizontal Weyl conformal curvature tensor is a variant under the horizontal projective transformation.

Remark 4.1. It is not hard to show our results are also true for Carnot group.

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