EXACT AND UNIQUE SOLUTION OF A TRANSPORT EQUATION IN A SEMI-INFINITE MEDIUM BY LAPLACE TRANSFORM AND WIENER-HOPF TECHNIQUE

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Abstract. The equation of radiative transfer in non-conservative case for diffuse reflection in a plane-parallel semi-infinite atmosphere with axial symmetry has been solved by Laplace transform and Wiener-Hopf technique. We have determined the emergent intensity in terms of Chandrasekhar’s H-function and the intensity at any optical depth by inversion.

1. Basic Equation and Solution for Emergent Intensity

A parallel beam of radiation of net flux HF per unit area normal to itself is incident on a plane parallel atmosphere of semi-infinite thickness in some specified direction (−µ, φ0). The equation appropriate to the problem (Chandrasekhar [2]; DasGupta [3]) is

\[ \mu \frac{dI(t, \mu)}{dt} = I(t, \mu) - \frac{1}{2} \int_{-1}^{1} p(\mu, \mu') I(t, \mu') d\mu' - \frac{F}{4} e^{-\frac{\mu}{\mu_0}} p(\mu, -\mu_0) \] (1)

where \( I(t, \mu) \) is the intensity characterizing the diffuse radiation field in the direction \( \cos^{-1} \mu \) at the optical depth \( t \). We take (Busbridge [1], art.45; Chandrasekhar [2])

\[ p(\mu, \mu') = \omega_0 + \omega_1 \mu \mu'. \] (2)

The average intensity \( J_0(t) \) and the normal flux \( J_1(t) \) are defined by

\[ J_r(t) = \frac{1}{2} \int_{-1}^{1} \mu'^r I(t, \mu) d\mu; \quad r = 0, 1. \] (3)

Boundary conditions are

\[ I(0, -\mu) = 0, \quad 0 < \mu \leq 1 \] (4)

and

\[ I(t, \mu) e^{-\frac{\mu}{\mu_0}} \to \infty \text{ as } t \to \infty, \quad |\mu| < 1. \] (5)
Equation (1) takes the form

\[ \frac{dI(t, \mu)}{dt} = I(t, \mu) - \omega_0 J_0(t) - \omega_1 \mu J_1(t) - \frac{F}{4} (\omega_0 - \omega_1 \mu \mu) e^{-\frac{t}{\mu}}. \] (6)

Laplace transform of equation (6) gives

\[ (\mu s - 1) I(s, \mu) = \mu s I(0, \mu) - \omega_0 J_0^*(s) - \omega_1 \mu J_1^*(s) - \frac{F}{4} (\omega_0 - \omega_1 \mu \mu) \frac{\mu_0 s}{1 + \mu_0 s}. \] (7)

The formal solution of equation (6) gives, setting \( \mu = \frac{1}{s} \), is

\[ I(0, 1/s) = \omega_0 J_0(s) + \frac{\omega_1}{s} J_1^*(s) + \frac{F}{4} \left( \omega_0 \frac{\mu_0}{s} - \omega_1 \mu \right) \frac{\mu_0 s}{1 + \mu_0 s}. \] (8)

Multiplying equation (8) by \( \frac{1}{2} d\mu \) and \( \frac{1}{2} d\mu \) successively and integrating between \(-1\) and \(1\) and eliminating \( J_0^*(s), J_1^*(s) \)

\[ T(z) I(0, z) = G^+(z) + \frac{F \mu_0}{4(\mu_0 + z)} [\omega_0 - (\omega_1 \mu_0 - \omega_0 \omega_1 \mu_0) z] \] (9)

where

\[ T(z) = 1 - 2z^2 \int_0^1 \frac{U(x)}{x^2 - x^2} dx \] (10)

\[ U(x) = \frac{1}{2} [\omega_0 + \omega_1 (1 - \omega_0)x^2] \] (11)

\[ \int_0^1 U(x) dx < \frac{1}{2} \] (12)

and

\[ G^+(z) = \frac{1}{2} \int_0^1 \frac{x}{x - z} [\omega_0 + \omega_1 (1 - \omega_0)x] I(0, x) dx. \] (13)

We now proceed to solve the integral equation (9).

Following Busbride [1] we have

\[ \frac{I(0, z)}{H(z)} (z + \mu_0) \frac{k - z}{k} = c_0 + c_1 z + c_2 z^2 \] (14)

where

\[ H(z) = 1 + z H(z) \int_0^1 \frac{U(x) H(x)}{x + z} dx. \] (15)

Equation (14) gives the emergent intensity as

\[ I(0, z) = \frac{k(c_0 + c_1 z + c_2 z^2)}{(k - z)(z + \mu_0)} H(z). \] (16)
2. Intensity at any Optical Depth

The radiation intensity at any optical depth \( t \) is given by

\[
I(t, \mu) = \frac{1}{2\pi i} \lim_{\delta \to 0} \int_{c-i\delta}^{c+i\delta} \frac{I^*(s, \mu)}{s} e^{st} ds, \quad c > 0. \tag{17}
\]

The integrand of equation (17) has simple poles at \( s = -\frac{1}{\mu_0} \) and \( s = \pm k; s = 0 \) is not a pole. Again

\[
\lim_{s \to -\frac{1}{\mu}} \left( s - \frac{1}{\mu} \right) \frac{I^*(s, \mu)}{s} e^{st} = 0 \tag{18}
\]

Therefore, \( s = -\frac{1}{\mu} \) is not a pole of the integrand of equation (17). The pole \( s = -\frac{1}{\mu_0} \) is on the singular line and the residue must be calculated there. Hence the integrand of equation (17) is regular for \((-\infty, -1)^c\). Therefore, by Cauchy’s residue theorem, equation (17) gives

\[
I(t, \mu) = R_1 + R_2 + R_3 + \frac{1}{2\pi i} \int_{HE} \frac{I^*(s, \mu)}{s} e^{st} ds + \frac{1}{2\pi i} \int_{DA} \frac{I^*(s, \mu)}{s} e^{st} ds \tag{19}
\]

where

\[
R_1 = -2k e^{-\frac{s}{\mu_0}} \frac{\omega_0 - \omega_1(1 - \omega_0)\mu_0}{\omega_0 + \omega_1(1 - \omega_0)\mu_0^2} \cdot \frac{c_0 - c_1k + c_2\mu_0^2}{H(\mu)(k + \mu_0)} \cdot \frac{X(-\mu_0)}{Z(-\mu_0)} \tag{20}
\]

\[
R_2 = \frac{\omega_0 + \omega_1(1 - \omega_0)k\mu}{\omega_0 + \omega_1(1 - \omega_0)k^2} \cdot \frac{c_0 + c_1k + c_2k^2}{(k + \mu)(\mu - k)} \cdot kH(k)e^\frac{2k}{\omega_0 + \omega_1(1 - \omega_0)k^2} \tag{21}
\]

\[
R_3 = \frac{k^2 e^{-\frac{s}{\mu_0}}}{\omega_0 + \omega_1(1 - \omega_0)k^2} \cdot \frac{c_0 - c_1k + c_2k^2}{(k + \mu)(\mu_0 - k)} \cdot \frac{1}{H(k)|\frac{k}{\omega_0 + \omega_1(1 - \omega_0)k^2}|^\frac{3k}{s} - 1} \tag{22}
\]

where

\[
\frac{d}{ds} \left[ T \left( \frac{1}{s} \right) \right]_{s=\frac{1}{k}} = \frac{k^3}{k^2 - 1} \left[ (3k^2 - 1)(1 - \omega_0)\omega_1 + \omega_0 \right] + \frac{k^2}{2} \left[ \omega_0 + 3\omega_1(1 - \omega_0)k^2 \ln \frac{k - 1}{k + 1} \right] \tag{23}
\]

3. Determination of the Constants \( c_0, c_1, c_2 \)

We rewrite the equation (9) in the form

\[
\left( \frac{k - z}{k} \right) T(z) I(0, z)(\mu_0 + z) = \frac{k - z}{k} (\mu_0 + z) G^+(z) + \frac{F_{\mu_0}}{4} [\omega_0 - \omega_1(1 - \omega_0)\mu_0 z] \frac{k - z}{k}. \tag{24}
\]

Again \( \frac{1}{H(-z)} \to (1 - h_0) - \frac{h_1}{z} - \frac{h_2}{z^2} - \cdots \) as \( z \to \infty \)
where

\[ h_r = \int_0^1 x^r U(x) H(x) dx; \quad r = 0, 1, 2, \ldots \]  \hspace{1cm} (25)

We substitute the expression for \( I(0, z) \) from equation (16) in equation (24) and equating the coefficients of \( z^0, z \) and \( z^2 \) from both sides we obtain

\[
\begin{align*}
&\left[ (1 - h_0) + \frac{k \mu_0}{2} \omega_1 (1 - \omega_0) \alpha_1 + (k - \mu_0) d_1 - d_2 \right] c_0 \\
&+ \left[ -h_1 + \frac{k \mu_0}{2} \omega_1 (1 - \omega_0) \alpha_2 + (k - \mu_0) d_2 - d_3 \right] c_1 \\
&+ \left[ -h_2 + \frac{k \mu_0}{2} \omega_1 (1 - \omega_0) \alpha_3 + (k - \mu_0) d_3 - d_4 \right] c_2 = \frac{1}{4} F \omega_0 \mu_0 \\
&\left[ \frac{1}{2} (k - \mu_0) \omega_1 (1 - \omega_0) \alpha_1 - d_1 \right] c_0 + \left[ -(1 - h_0) + \frac{1}{2} (k - \mu_0) \omega_1 (1 - \omega_0) \alpha_2 + d_2 \right] c_1 \\
&+ \left[ h_1 + \frac{1}{2} (k - \mu_0) \omega_1 (1 - \omega_0) \alpha_3 + d_3 \right] c_2 = \frac{1}{4} F \omega_0 \left[ \frac{\omega_0}{k} + \omega_1 (1 - \omega_0) \mu_0 \right] \\
&\frac{1}{2} \omega_1 (1 - \omega_0) \alpha_0 c_0 + \frac{1}{2} \omega_1 (1 - \omega_0) \alpha_2 c_1 + \left[ \frac{1}{2} \omega_1 (1 - \omega_0) \alpha_3 - (1 - h_0) \right] c_2 \\
&= \frac{1}{4} F \frac{\mu_0^2}{k} \omega_1 (1 - \omega_0) \hspace{1cm} (26)
\end{align*}
\]

Equations (26), (27) and (28) determine the constants \( c_0, c_1, c_2 \) explicitly.

**References**


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