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EXACT AND UNIQUE SOLUTION OF A TRANSPORT EQUATION IN A SEMI-INFINITE MEDIUM BY LAPLACE TRANSFORM AND WIENER-HOPF TECHNIQUE

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Abstract. The equation of radiative transfer in non-conservative case for diffuse reflection in a plane-parallel semi-infinite atmosphere with axial symmetry has been solved by Laplace transform and Wiener-Hopf technique. We have determined the emergent intensity in terms of Chandrasekhar's H-function and the intensity at any optical depth by inversion.

1. Basic Equation and Solution for Emergent Intensity

A parallel beam of radiation of net flux HF per unit area nomral to itself is incident on a plane parallel atmosphere of semi-infinite thickness in some specified direction $(-\mu, \phi_0)$. The equation appropriate to the problem (Chandrasekhar [2]; DasGupta [3]) is

$$\mu\frac{dI(t,\mu)}{dt} = I(t,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(t,\mu')d\mu' - \frac{F}{4}e^{-\frac{t}{\mu_0}}p(\mu,-\mu_0) \tag{1}$$

where $I(t,\mu)$ is the intensity characterizing the diffuse radiation field in the direction $\cos^{-1}\mu$ at the optical depth t. We take (Busbridge [1], art.45; Chandrasekhar [2])

$$p(\mu,\mu') = \omega_0 + \omega_1\mu\mu'. \tag{2}$$

The average intensity $J_0(t)$ and the normal flux $J_1(t)$ are defined by

$$J_r(t) = \frac{1}{2}\int_{-1}^{1}\mu^r I(t,\mu)d\mu; \quad r = 0, 1. \tag{3}$$

Boundary conditions are

$$I(0,-\mu) = 0, \quad 0 < \mu \le 1 \tag{4}$$

and

$$I(t,\mu)e^{-\frac{t}{\mu}} \to \infty \text{ as } t \to \infty, \quad |\mu| < 1. \tag{5}$$

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Equation (1) takes the form

$$\mu\frac{dI(t,\mu)}{dt} = I(t,\mu) - \omega_0 J_0(t) - \omega_1 \mu J_1(t) - \frac{F}{4}(\omega_0 - \omega_1\mu\mu_0)e^{-\frac{t}{\mu_0}}. \tag{6}$$

Laplace transform of equation (6) gives

$$(\mu s - 1)I^*(s,\mu) = \mu s I(0,\mu) - \omega_0 J_0^*(s) - \omega_1\mu J_1^*(s) - \frac{F}{4}(\omega_0 - \omega_1\mu\mu_0)\frac{\mu_0 s}{1+\mu_0 s}. \tag{7}$$

The formal solution of equation (6) gives, setting $\mu = \frac{1}{s}$, is

$$I\left(0,\frac{1}{s}\right) = \omega_0 J_0^*(s) + \frac{\omega_1}{s}J_1^*(s) + \frac{F}{4}\left(\omega_0 - \frac{\omega_1\mu_0}{s}\right).\frac{\mu_0 s}{1+\mu_0 s} \tag{8}$$

Multiplying equation (8) by $\frac{1}{2}d\mu$ and $\frac{1}{2}\frac{d\mu}{\mu s-1}$ successively and integrating between -1 and 1 and eliminating $J_0^*(s)$, $J_1^*(s)$

$$T(z)I(0,z) = G^+(z) + \frac{F\mu_0}{4(\mu_0+z)}[\omega_0 - (\omega_1\mu_0 - \omega_0\omega_1\mu_0)z] \tag{9}$$

where

$$T(z) = 1 - 2z^2\int_0^1 \frac{U(x)}{z^2-x^2}dx \tag{10}$$

$$U(x) = \frac{1}{2}[\omega_0 + \omega_1(1-\omega_0)x^2] \tag{11}$$

$$\int_0^1 U(x)dx < \frac{1}{2} \tag{12}$$

and

$$G^+(z) = \frac{1}{2}\int_0^1 \frac{x}{x-z}[\omega_0 + \omega_1(1-\omega_0)xz]I(0,x)dx. \tag{13}$$

We now proceed to solve the integral equation (9).

Following Busbride [1] we have

$$\frac{I(0,z)}{H(z)}(z+\mu_0)\frac{k-z}{k} = c_0 + c_1 z + c_2 z^2 \tag{14}$$

where

$$H(z) = 1 + zH(z)\int_0^1 \frac{U(x)H(x)}{x+z}dx. \tag{15}$$

Equation (14) gives the emergent intensity as

$$I(0,z) = \frac{k(c_0 + c_1 z + c_2 z^2)}{(k-z)(z+\mu_0)}H(z). \tag{16}$$

2. Intensity at any Optical Depth

The radiation intensity at any optical depth t is given by

$$I(t,\mu) = \frac{1}{2\pi i} \lim_{\delta\to\infty} \int_{c-i\delta}^{c+i\delta} \frac{I^*(s,\mu)}{s} e^{st} ds, \quad c > 0. \tag{17}$$

The integrand of equation (17) has simple poles at $s = -\frac{1}{\mu_0}$ and $s = \pm k$; $s = 0$ is not a pole. Again

$$\lim_{s\to\frac{1}{\mu}} \left(s - \frac{1}{\mu}\right) \frac{I^*(s,\mu)}{s} e^{st} = 0 \tag{18}$$

Therefore, $s = \frac{1}{\mu}$f is not a pole of the integrand of equation (17). The pole $s = -\frac{1}{\mu_0}$ is on the singular line and the residue must be calculated there. Hence the integrand of equation (17) is regular for $(-\infty, -1)^c$. Therefore, by Cauchy's residue theorem, equation (17) gives

$$I(t,\mu) = R_1 + R_2 + R_3 + \frac{1}{2\pi i} \int_{HE} \frac{I^*(s,\mu)}{s} e^{st} ds + \frac{1}{2\pi i} \int_{DA} \frac{I^*(s,\mu)}{s} e^{st} ds \tag{19}$$

where

$$R_1 = -2\frac{ke^{-\frac{t}{\mu_0}}}{\mu+\mu_0} \cdot \frac{\omega_0 - \omega_1(1-\omega_0)\mu\mu_0}{\omega_0 + \omega_1(1-\omega_0)\mu_0^2} \cdot \frac{c_0 - c_1\mu_0 + c_2\mu_0^2}{H(\mu_0)(k+\mu_0)} \cdot \frac{X(-\mu_0)}{Z(-\mu_0)} \tag{20}$$

$$R_2 = \frac{\omega_0 + \omega_1(1-\omega_0)k\mu}{\omega_0 + \omega_1(1-\omega_0)k^2} \cdot \frac{c_0 + c_1 k + c_2 k^2}{(k+\mu_0)(\mu - k)} \cdot kH(k)e^{\frac{t}{k}} \tag{21}$$

$$R_3 = \frac{k^2}{2} e^{-\frac{t}{k}} \frac{\omega_0 - \omega_1(1-\omega_0)k\mu}{\omega_0 + \omega_1(1-\omega_0)k^2} \cdot \frac{c_0 - c_1 k + c_2 k^2}{(k+\mu)(\mu_0 - k)} \cdot \frac{1}{H(k)[\frac{d}{ds}T(\frac{1}{s})]_{s=\frac{1}{k}}} \tag{22}$$

where

$$\frac{d}{ds}\left[T\left(\frac{1}{s}\right)\right]_{s=\frac{1}{k}} = \frac{k^3}{k^2-1}[(3k^2-1)(1-\omega_0)\omega_1 + \omega_0] + \frac{k^2}{2}\left[\omega_0 + 3\omega_1(1-\omega_0)k^2 \ln\frac{k-1}{k+1}\right] \tag{23}$$

3. Determination of the Constants c_0, c_1, c_2

We rewrite the equation (9) in the form

$$\left(\frac{k-z}{k}\right) T(z)I(0,z)(\mu_0+z) = \frac{k-z}{k}(\mu_0+z)G^+(z) + \frac{F\mu_0}{4}[\omega_0 - \omega_1(1-\omega_0)\mu_0 z]\frac{k-z}{k}. \tag{24}$$

Again $\frac{1}{H(-z)} \to (1-h_0) - \frac{h_1}{z} - \frac{h_2}{z^2} - \cdots$ as $z \to \infty$

where

$$h_r = \int_0^1 x^r U(x)H(x)dx; \quad r = 0, 1, 2, \dots. \tag{25}$$

We substitute the expression for $I(0, z)$ from equation (16) in equation (24) and equating the coefficients of z^0, z and z^2 from bothsides we obtain

$$\begin{aligned}&\left[(1-h_0) + \frac{k\mu_0}{2}\omega_1(1-\omega_0)\alpha_1 + (k-\mu_0)d_1 - d_2\right]c_0\\&+\left[-h_1 + \frac{k\mu_0}{2}\omega_1(1-\omega_0)\alpha_2 + (k-\mu_0)d_2 - d_3\right]c_1\\&+\left[-h_2 + \frac{k\mu_0}{2}\omega_1(1-\omega_0)\alpha_3 + (k-\mu_0)d_3 - d_4\right]c_2 = \frac{1}{4}F\omega_0\mu_0\end{aligned} \tag{26}$$

$$\begin{aligned}&\left[\frac{1}{2}(k-\mu_0)\omega_1(1-\omega_0)\alpha_1 - d_1\right]c_0 + \left[-(1-h_0) + \frac{1}{2}(k-\mu_0)\omega_1(1-\omega_0)\alpha_2 + d_2\right]c_1\\&+\left[h_1 + \frac{1}{2}(k-\mu_0)\omega_1(1-\omega_0)\alpha_3 + d_3\right]c_2 = \frac{1}{4}F\omega_0\left[\frac{\omega_0}{k} + \omega_1(1-\omega_0)\mu_0\right]\end{aligned} \tag{27}$$

$$\begin{aligned}&\frac{1}{2}\omega_1(1-\omega_0)\alpha_1 c_0 + \frac{1}{2}\omega_1(1-\omega_0)\alpha_2 c_1 + \left[\frac{1}{2}\omega_1(1-\omega_0)\alpha_3 - (1-h_0)\right]c_2\\&= \frac{1}{4}F\frac{\mu_0^2}{k}\omega_1(1-\omega_0)\end{aligned} \tag{28}$$

where

$$\alpha_r = \int_0^1 \frac{x^r H(x)}{(k-x)(\mu_0+x)}dx; \quad r = 1, 2, 3. \tag{29}$$

and

$$d_r = \int_0^1 \frac{x^r U(x)H(x)}{(k-x)(\mu_0+x)}dx; \quad r = 1, 2, 3, 4. \tag{30}$$

Equations (26), (27) and (28) determine the constants c_0, c_1, c_2 explicitly.

References

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