A NOTE ON THE LEAST (NORMALIZED) LAPLACIAN EIGENVALUE OF SIGNED GRAPHS

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Abstract. Let $\Gamma = (G, \sigma)$ be a connected signed graph, and $L(\Gamma)$ be its Laplacian and $\mathscr{L}(\Gamma)$ its normalized Laplacian with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, respectively. It is known that a signed graph Γ is balanced if and only if $\lambda_n = 0$ (or $\mu_n = 0$). We show that λ_n and μ_n measure how much Γ is far from being balanced by proving that

$$\mu_n(\Gamma) \leq \min\{\frac{2\epsilon(\Gamma)}{m}, \frac{\nu(\Gamma)}{\nu(\Gamma) + \nu_1(\Gamma)}\},$$
$$\lambda_n(\Gamma) \leq \min\{\lambda_1(\Gamma') : \Gamma - \Gamma' \text{ is balanced}\},$$

where $\nu(\Gamma)$ (resp. $\epsilon(\Gamma)$) denotes the frustration number (resp. the frustration index) of Γ, that is the minimum number of vertices (resp. edges) to be deleted such that the signed graph is balanced.

1. Introduction

A signed graph $\Gamma = (G, \sigma)$ consists of an underlying graph $G = (V, E)$ and a mapping $\sigma : E \rightarrow \{+1, -1\}$, where $V := \{v_1, v_2, \cdots, v_n\}$ and edge $v_i v_j$ is denoted by e_{ij}. Let $d_G(v)$ denote the vertex degree of v in G and for simplicity $d_G(v)$ is written as d_v when the graph G can be understood from the context.

Signed graphs were introduced by Heider [9] in 1946 to describe sentiment relations between people pertaining to a same social group and to provide a systematic statement of social balance theory. Cartwright and Harary [4] formalized Heider's theory stating that a balanced social group, i.e., a balanced signed graph, could be partitioned into two mutually hostile subgroups each having internal solidarity. Due to its applications in social network,

portfolio analysis in risk management, biological systems and so on, in which the common element among these applications is the fact that all of them are defined in a collaborative vs. conflicting environment that can be modeled over a signed graph, signed graphs have continued to be a very attractive discrete structure studied extensively.

Let C be a cycle of a signed graph $\Gamma = (G, \sigma)$. The sign of C is denoted by $sgn(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is $+$ (or $-$) is called *positive* (or *negative*). A signed graph is called *balanced* if all its cycles are positive. Suppose $\theta : V \to \{+1, -1\}$ is any sign function. Switching Γ by θ means forming a new signed graph $\Gamma^\theta = (G, \sigma^\theta)$ whose underlying graph is the same as G, but whose sign function is defined by $\sigma^\theta(e_{ij}) = \theta(v_i)\sigma(e_{ij})\theta(v_j)$. Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graph with the same underlying graph G. Γ_1 and Γ_2 are called *switching equivalent*, written $\Gamma_1 \simeq \Gamma_2$, if there exists a switching function θ such that $\Gamma_2 = \Gamma_1^\theta$. Switching leaves many signed-graphic characteristics invariant, such as the set of positive cycles.

The *Laplacian matrix* $L := L(\Gamma) = (L_{uv})$ of a signed graph Γ of order n is the $n \times n$ matrix L defined as follows:

$$L_{uv} = \begin{cases} d_u & \text{if } u = v, \\ -\sigma(uv) & \text{if } uv \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Hou et al.[11] described $L(\Gamma)$ by means of its quadratic form:

$$x^T L(\Gamma) x = \sum_{v_i v_j \in E(\Gamma)} (x_i - \sigma(v_i v_j) x_j)^2, \tag{1.1}$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$.

Lemma 1.1 ([17]). *Let Γ be a connected signed graph and $L(\Gamma)$ be its Laplacian matrix. Then Γ is balanced if and only if $\det(L(\Gamma)) = 0$.*

Two matrices M_1 and M_2 of order n are called *signature similar* if there exists a signature matrix, that is, a diagonal matrix $S = \text{diag}(s_1, s_2, \ldots, s_n)$ with diagonal entries $s_i = \pm 1$ such that $M_2 = SM_1S$.

Lemma 1.2 ([11]). *Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be signed graphs on the same underlying graph G. Then $\Gamma_1 \simeq \Gamma_2$ if and only if $L(\Gamma_1)$ and $L(\Gamma_2)$ are signature similar.*

The *normalized Laplacian* of Γ is the $n \times n$ matrix $\mathscr{L} := \mathscr{L}(\Gamma) = (\mathscr{L}_{uv})$ given by

$$\mathscr{L}_{uv} = \begin{cases} 1 & \text{if } u = v \text{ and } d_u \neq 0, \\ -\sigma(uv) \dfrac{1}{\sqrt{d_u d_v}} & \text{if } uv \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Let D denote the diagonal matrix with the (u, u)th entry having value d_u. For a general graph, we have $\mathscr{L} = D^{-1/2} L D^{-1/2}$ with the convention that $D^{-1}(u, u) = 0$ if $d_u = 0$, where $A = (\sigma(uv) a_{uv})$ is the adjacency matrix of Γ (i.e., $a_{uv} = 1$ if u is adjacent to v, and 0 otherwise). By the quadratic form of Laplacian $L(\Gamma)$ and normalized Laplacian $\mathscr{L}(\Gamma)$, they are positive semi-definite and so their eigenvalues are nonnegative and denoted by $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma) \geq 0$ and $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) \geq 0$, respectively.

The following characterization of balanced signed graphs is well known [17, 12, 14].

Lemma 1.3. *Let $\Gamma = (G, \sigma)$ be a signed graph. Then the following conditions are equivalent:*

(1) Γ *is balanced.*

(2) $\Gamma = (G, \sigma) \sim (G, +)$, *where $(G, +)$ is the signed graph with all positive edges.*

(3) $V(\Gamma)$ *can be partitioned into two color classes $V_1 \cup V_2$ such that every edge between V_1 and V_2 is negative and every edge within V_1 or V_2 is positive.*

(4) $\lambda_n(\Gamma) = 0$.

(5) $\mu_n(\Gamma) = 0$.

The Maximum Balanced Subgraph Problem (MBSP) is the problem of finding a subgraph of a signed graph that is balanced and maximizes the cardinality of its vertex set. The MBSP is known to be an NP-hard problem although the problem of detecting balance in signed graphs can be solved in polynomial time. Figueiredo and Frota [6] first discussed applications of the MBSP and the efficient solution of the MBSP. In [2], some parameters called frustration number and frustration index measuring how far a signed graph is to be balanced have been investigated in terms of the least eigenvalue of Laplacian of signed graph. More results on the spectra of signed graphs can be found in [1, 2, 3, 6, 11, 12, 16, 13, 14, 5, 7, 8].

In this paper, we consider the corresponding problem to bound frustration number and frustration index of signed graphs in terms of its least eigenvalue of normalized Laplacian of signed graphs. Further, we show that the least eigenvalue of Laplacian can be bounded up using the spectral radius of edge-induced subgraph whose deletion makes it being balanced.

2. Main results

As in [2], the *frustration number* $\nu(\Gamma)$ and the *frustration index* $\epsilon(\Gamma)$ of signed graph Γ are defined as the smallest number of vertices and edges whose deletion leads to a balanced graph, respectively. It is known that $\nu(\Gamma) \leq \epsilon(\Gamma)$ always holds. In [2], it is shown that $\lambda_n(\Gamma) \leq \epsilon(\Gamma)$ (proved first in [16]) and $\lambda_n(\Gamma) \leq \nu(\Gamma)$. We would mention here that $\lambda_n(\Gamma)$ has been called the "algebraic frustration". It is in analogy with the algebraic connectivity and algebraic bipartiteness.

Some notations are needed here. For a signed graph Γ with two proper vertex subset $S, T \subseteq V(\Gamma)$, let $E_\Gamma(S, T)$ ($E_\Gamma^\pm(S, T)$) be the set of (positive or negative) edges with one end in S and another end in T, and the subscript Γ will usually be dropped off when it can be understood from the context. When $S = T$, $E_\Gamma(S, T)$ ($E_\Gamma^\pm(S, T)$) will be denoted by $E_\Gamma(S)$ ($E_\Gamma^\pm(S)$). Denote by $G[S]$ the induced subgraph of G by S. Let $Vol(G)$ denote the sum of the degrees of all vertices in G.

If Γ is unbalanced, then there exists a signed subgraph $S \subset \Gamma$, with $|S| = v(\Gamma)$, such that $\Gamma - S$ is balanced. Note that $\Gamma - S$ might be not connected, say $\Gamma - S = S_1 + \cdots + S_k$ is balanced where each S_i is a connected balanced signed subgraph of Γ. It is easy to see that $|S| \leq n - 2$. Further, in this case, define $\psi(S) = \max\limits_{1 \leq i \leq k} \frac{Vol(G[S_i])}{|S_i|}$. Maximizing $\psi(S)$ over all such S such that $|S| = v(\Gamma)$ and $\Gamma - S$ is balanced, we get a parameter and denote it by $v_1(\Gamma)$.

In what follows, we give our main result that measures how much Γ is far from being balanced via frustration number and the frustration index in terms of the least eigenvalue of normalized Laplacian.

Theorem 2.1. *Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices and m edges. Then $\mu_n(\Gamma) \leq \min\{\frac{2\epsilon(\Gamma)}{m}, \frac{v(\Gamma)}{v(\Gamma) + v_1(\Gamma)}\}$.*

Proof. If Γ is balanced, then $\mu_n(\Gamma) = 0$ and $v(\Gamma) = \epsilon(\Gamma) = 0$, and there is nothing to prove. Then suppose that Γ is unbalanced thereafter. Let $F \subseteq E(G)$, with $|F| = \epsilon(\Gamma) \geq 0$, be a minimum set of edges for which $\Gamma - F$ is balanced, and let V_1, V_2 be the two color classes of $\Gamma - F$ guaranteed by (3) of Lemma 1.3. Let us define the following vector X on the vertices of G, $X = (x_1, x_2, \cdots, x_n)^\top$ where $x_i = 1$ if $v_i \in V_1$, $x_i = -1$ if $v_i \in V_2$. Observe that $(x_i - \sigma(e_{ij})x_j)^2 = 0$ whenever $e_{ij} \in E(G) \setminus F$. In addition, $F = E^-(V_1) \cup E^-(V_2) \cup E^+(V_1, V_2)$, so $(x_i - \sigma(e_{ij})x_j)^2 = 4$ whenever $e_{ij} \in F$.

Hence, let $f = D^{\frac{1}{2}}X$, we have

$$
\begin{aligned}
\mu_n(\Gamma) &\leq \frac{f^T \mathscr{L}(\Gamma)f}{f^T f} = \frac{X^T L(\Gamma)X}{X^T DX} = \frac{\sum_{e_{ij} \in E(G)}(x_i - \sigma(e_{ij})x_j)^2}{\sum_{v_i \in V(G)} d_i x_i^2} \\
&= \frac{\sum_{e_{ij} \in E(G) \setminus F}(x_i - \sigma(e_{ij})x_j)^2 + \sum_{e_{ij} \in F}(x_i - \sigma(e_{ij})x_j)^2}{\sum_{v_i \in V(G)} d_i x_i^2} \\
&= \frac{2\epsilon(\Gamma)}{m}.
\end{aligned}
$$

Now, let $S \subseteq V(G)$ with $|S| = v(\Gamma) \leq n - 2$ such that $\Gamma - S = S_1 + \cdots + S_k$ is balanced with each S_i being connected and balanced. For each balanced component S_i, again by Lemma 1.3, let S_i^1 be the first color class and S_i^2 the second color class. Define the vector $X_i = (x_1, x_2, \ldots, x_n)$ on $V(\Gamma)$ such that $x_v = 1$ if $v \in S_i^1$, $x_v = -1$ if $v \in S_i^2$, and $x_v = 0$ if $v \in \Gamma - S_i$, for some fixed $1 \leq i \leq k$. Let $\bar{\Gamma} = (\bar{G}, \bar{\sigma})$ be the signed graph obtained from Γ such that the underlying graph $\bar{G}$ is obtained from G by adding some edges, say a edges, between S and S_i, such that each

vertex in S is adjacent to each vertex in S_i, and $\bar{\sigma}$ is obtained from σ by randomly giving any sign on the added edges.

Note that $E(G) \subseteq E(\bar{G})$, and $(x_u - \bar{\sigma}(uv)x_v)^2 = 1$ for any edge of $E_{\bar{G}}(S, S_i)$ and $(x_u - \bar{\sigma}(uv)x_v)^2 = 0$ otherwise. Similarly, $(x_u - \sigma(uv)x_v)^2 = 1$ for any edge of $E_G(S, S_i)$ and $(x_u - \sigma(uv)x_v)^2 = 0$ otherwise. Since adding each edge between S and S_i contributes one to the degree of vertex in S_i, we have

$$
\begin{aligned}
\mu_n(\Gamma) &\leq \frac{X^T L(\Gamma) X}{X^T D X} = \frac{\sum_{e_{ij} \in E(G)} (x_i - \sigma(e_{ij})x_j)^2}{\sum_{v_i \in V(G)} d_i x_i^2} = \frac{|E_G(S, S_i)|}{\sum_{v_i \in S_i} d_G(v_i)} \\
&\leq \frac{|E_G(S, S_i)| + a}{\sum_{v_i \in S_i} d_G(v_i) + a} = \frac{|S| \cdot |S_i|}{\sum_{v_i \in S_i} d_{\bar{G}}(v_i)} = \frac{|S| \cdot |S_i|}{|S| \cdot |S_i| + Vol(G[S_i])} \\
&= \frac{|S|}{|S| + \frac{Vol(G[S_i])}{|S_i|}},
\end{aligned}
$$

where the second inequality holds because $|E_G(S, S_i)| \leq \sum_{v_i \in S_i} d_G(v_i)$ and the inequality $\frac{c}{d} \leq \frac{c+a}{d+a}$ holds when $c \leq d$. Then the inequality $\mu_n(\Gamma) \leq \frac{\nu(\Gamma)}{\nu(\Gamma) + \nu_1(\Gamma)}$ follows. $\square$

Next we consider the Laplacian matrix of a signed graph $\Gamma = (G, \sigma)$ with n vertices and m edges. For each edge of $e = uv$ of G, we choose one of u or v to be the head of e and them other to be the tail. We call this an orientation of Γ. The *incidence matrix* $C = C(\Gamma)$ afforded by a fixed orientation of Γ is the $n \times m$ matrix $C = (c_{ve})$, where

$$
c_{ve} = \begin{cases}
1 & \text{if } v \text{ is the head of } e, \\
-1 & \text{if } v \text{ is the tail of } e \text{ and } \sigma(e) = +, \\
1 & \text{if } v \text{ is the tail of } e \text{ and } \sigma(e) = -, \\
0 & \text{otherwise.}
\end{cases}
$$

Note that the rows and columns of C are indexed by vertices and edges of Γ, respectively. While C depends on the orientation of Γ, CC^T does not, and it is easy to verify that $CC^T = L(\Gamma) = D(\Gamma) - A(\Gamma)$.

If Γ is unbalanced, there exists $F \subseteq E(G)$ such that $|F| = \epsilon(\Gamma) = k$ and $\Gamma - F$ is balanced. Without loss of generality, assume that last columns are indexed by F and so C can be written in the form of $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$, where C_2 is a $n \times \epsilon(\Gamma)$ matrix. Thus we have $L(\Gamma) = CC^T = C_1 C_1^T + C_2 C_2^T$, and let $L_i = C_i C_i^T$ for $i = 1, 2$. Note that L_i can be viewed as the Laplacian of signed graph Γ_i for $i = 1, 2$, and Γ_1 and Γ_2 be signed subgraphs induced by the edge sets $E(\Gamma) - F$ and F, respectively.

Theorem 2.2 ([10])**.** *Let A, B be $n \times n$ Hermitian matrices, and let the eigenvalues of A, B and $A + B$ be arranged in decreasing order $\lambda_1(\cdot) \geq \lambda_2(\cdot) \geq \cdots \geq \lambda_n(\cdot)$. Then for every pair of integers j, k such that $1 \leq j, k \leq n$ and $j + k > 1$ we have*

$$
\lambda_{j+k-1}(A + B) \leq \lambda_j(A) + \lambda_k(B).
$$

Consider the Laplacian $L(\Gamma) = L_1 + L_2$, and we have $\lambda_n(\Gamma) = \lambda_n(L(\Gamma)) \leq \lambda_n(L_1) + \lambda_1(L_2)$. By the choice of F, $\Gamma - F$ is balanced and so $\lambda_n(L_1) = 0$ by Lemma 1.3. Below, we let Γ' denote signed subgraph induced by some edge set F in Γ. Thus we come to the following result.

Theorem 2.3. *Let Γ be a signed graph with n vertices. Then $\lambda_n(\Gamma) \leq \min\{\lambda_1(\Gamma') : \Gamma - \Gamma' \text{ is balanced}\}$.*

Note that when Γ is balanced, Γ' is then an empty graph and so $\lambda_1(\Gamma') = 0 = \lambda_n(\Gamma)$.

Now we intend to obtain some upper bound for the spectral radius of Laplacian of signed graphs, which together with the above Theorem 2.3 will imply some upper bound for the least eigenvalue of Laplacian of signed graphs.

For a vertex v of Γ, denote by $m_v = (1/d_v)\sum_{uv \in E(\Gamma)} d_u$, which is called the average 2-degree of the vertex v. Zhang [19] obtained that $\lambda_1(G) \leq \max\{d_i + \sqrt{d_i m_i} : v_i \in V(G)\}$ for unsigned graphs. Next we shall generalize it to signed graphs.

Theorem 2.4. *Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices. Then*

$$\lambda_1(\Gamma) \leq \max\left\{d_i + \sqrt{d_i m_i} \mid v_i \in V(\Gamma)\right\}.$$

Proof. Let $X = (x_j, v_j \in V(\Gamma))^T$ be an eigenvector corresponding to $\lambda_1(\Gamma)$. Thus $L(\Gamma)X = \lambda_1(\Gamma)X$. Hence for any $v_i \in V(\Gamma)$,

$$\lambda_1(\Gamma)x_i = d_i x_i - \sum_{v_i v_j \in E(\Gamma)} \sigma(v_i v_j)x_j = \sum_{v_i v_j \in E(\Gamma)} (x_i - \sigma(v_i v_j)x_j).$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\lambda_1^2(\Gamma)x_i^2 &\leq \left(\sum_{v_i v_j \in E(\Gamma)} 1^2\right)\left(\sum_{v_i v_j \in E(\Gamma)} (x_i - \sigma(e_{ij})x_j)^2\right) \\
&= d_i^2 x_i^2 - 2d_i x_i \sum_{v_j : v_j \sim v_i} \sigma(e_{ij})x_j + d_i \sum_{v_i v_j \in E(\Gamma)} x_j^2 \\
&= d_i^2 x_i^2 + 2d_i x_i^2(\lambda_1(\Gamma) - d_i) + d_i \sum_{v_i v_j \in E(\Gamma)} x_j^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{v_i \in V(\Gamma)} \lambda_1^2(\Gamma)x_i^2 &\leq \sum_{v_i \in V(\Gamma)} (2d_i \lambda_1(\Gamma) - d_i^2)x_i^2 + \sum_{v_i \in V(\Gamma)} d_i \sum_{v_i v_j \in E(\Gamma)} x_j^2 \\
&= \sum_{v_i \in V(\Gamma)} (2d_i \lambda_1(\Gamma) - d_i^2)x_i^2 + \sum_{v_i \in V(\Gamma)} d_i m_i x_i^2.
\end{aligned}$$

Therefore, we have

$$\sum_{v_i \in V(\Gamma)} (\lambda_1^2(\Gamma) - 2d_i \lambda_1(\Gamma) + d_i^2 - d_i m_i)x_i^2 \leq 0.$$

Then there must exist a vertex v_i such that

$$\lambda_1^2(\Gamma) - 2d_i\lambda_1(\Gamma) + d_i^2 - d_i m_i \le 0,$$

which implies $\lambda_1(\Gamma) \le d_i + \sqrt{d_i m_i}$ and we are done. $\square$

By Theorems 2.3 and 2.4, in most cases the obtained value is less than the frustration index $\epsilon(\Gamma)$, i.e. the edge number of Γ'. For example, if Γ' is triangle-free and $d_v \le |E(\Gamma')| - \sqrt{|E(\Gamma')|}$ for every $v \in V(\Gamma')$, then by Theorem 2.4 we have $\lambda_1(\Gamma') \le d_i + \sqrt{d_i m_i} \le |E(\Gamma')| - \sqrt{|E(\Gamma')|} + \sqrt{|E(\Gamma')|} = |E(\Gamma')| = \epsilon(\Gamma)$. That is what we get in the following corollary.

Corollary 2.5. *Let Γ be a signed graph with n vertices and Γ' is triangle-free and $d_v \le |E(\Gamma')| - \sqrt{|E(\Gamma')|}$ for every $v \in V(\Gamma')$. Then $\lambda_n(\Gamma) \le \lambda_1(\Gamma') \le \epsilon(\Gamma)$.*

In this sense, we have obtained a better bound for the least eigenvalue of Laplacian of signed graphs than that $\lambda_n(\Gamma) \le \epsilon(\Gamma)$ in [2].

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