



ON SOLVABILITY OF A COUPLED HYBRID SYSTEM OF QUADRATIC FRACTIONAL INTEGRAL EQUATIONS

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Abstract. Of concern is studying solvability of the hybrid systems of quadratic fractional integral equations. To this aim applying hybrid fixed point theory due to *Dhage*, existence of at least one positive solution for mentioned systems via so called D-Lipschitzian mappings will be concluded. We illustrate the obtained results by presenting an example.

1. Introduction

The fractional calculus is the theory of arbitrary order integration and differentiation that generalizes the integer order ones in classic differential calculus. Not only in theoretical manner but also as a result of more accurate description of real world phenomena in comparison with classic differential calculus, we can observe the boom of development of theory of fractional calculus in less than three decades in almost whole sciences related to mathematics such as biosciences, medicine, engineering, economy and so on. More details and applications can be found in the monographs and papers [26], [23], [1]-[12], [14]-[20]. On the other hand the theory of integral equations by itself has been introduced as full applicable theory in mathematics and great theories such as geomagnetic theory, transport theory, mechanics and so forth. So we can conclude that combination of fractional calculus and integral equations may introduce more effective tool for analysis and description of topics mentioned above. In this way more interesting applications can be found in references [25], [27] and references cited therein. Let us now introduce in the sequel the main motivation for preparing this paper.

The authors in [22], by means of Dhage hybrid fixed point theory obtained periodic solutions of integral equation

$$x(t) = \sum_{i=1}^n f_i(t, x(a_i(t))) \cdot \int_{\mathbb{R}} k_i(t, s) g_i(s, x(b_i(s))) ds.$$

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The authors in [13], considered the fractional order integral equation

$$x(t) = f(t, x(t)) + g(t, x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s, x(s)) ds, \quad t \in J = [0, 1], \alpha > 0,$$

and via above mentioned theory concluded the existence of at least one solution.

In this paper we consider the coupled hybrid system of quadratic fractional integral equations

$$\begin{cases} u(t) = \sum_{i=1}^n \left\{ k_{1,i}(t, u(t), v(t)) + g_{1,i}(t, u(t), v(t)) \cdot \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_{1,i}(s, u(s), v(s)) ds \right\}, \\ v(t) = \sum_{j=1}^n \left\{ k_{2,j}(t, u(t), v(t)) + g_{2,j}(t, u(t), v(t)) \cdot \int_0^t \frac{(t-s)^{\beta_j-1}}{\Gamma(\beta_j)} f_{2,j}(s, u(s), v(s)) ds \right\}, \end{cases} \quad (1.1)$$

where $t \in \mathbb{J} = [0, T]$, $T, \alpha_i, \beta_j \in \mathbb{R}^+$, $i, j = 1, 2, \dots, n$.

Assume that the following hypotheses are satisfied throughout this paper.

(A₁) $k_{1,i} \in C(\mathbb{J} \times \mathbb{R}^2, \mathbb{R}^+)$ for $i = 1, 2, \dots, n$ and there exist positive constants $L_{1,k_{1,i}}, L_{2,k_{1,i}}, \rho_{k_{1,i}}$ with $L_{1,k_{1,i}} \leq L_{2,k_{1,i}}$ such that

$$\|k_{1,i}(t, u_1, \cdot) - k_{1,i}(t, u_2, \cdot)\|_E \leq \frac{1}{2n} \frac{L_{1,k_{1,i}} \ln(1 + \|u_1 - u_2\|_E)}{L_{2,k_{1,i}} + \ln(1 + \|u_1 - u_2\|_E)}, \quad t \in \mathbb{J}, u_1, u_2 \in C(\mathbb{R}),$$

where $E = C(\mathbb{J}, \mathbb{R})$ is a Banach space endowed with the max-norm, also assume that

$$\sup k_{1,i}(t, u, v) = \rho_{k_{1,i}}, \quad i = 1, 2, \dots, n, t \in \mathbb{J}, u, v \in C(\mathbb{R}).$$

(A₂) $g_{1,i} \in C(\mathbb{J} \times \mathbb{R}^2, \mathbb{R}^+)$ for $i = 1, 2, \dots, n$ and there exist positive constants $L_{1,g_{1,i}}, L_{2,g_{1,i}}, \rho_{g_{1,i}}$ with $L_{1,g_{1,i}} \leq L_{2,g_{1,i}}$ such that

$$\|g_{1,i}(t, u_1, \cdot) - g_{1,i}(t, u_2, \cdot)\|_E \leq \frac{1}{2nM_i} \frac{L_{1,g_{1,i}} \ln(1 + \|u_1 - u_2\|_E)}{L_{2,g_{1,i}} + \ln(1 + \|u_1 - u_2\|_E)}, \quad t \in \mathbb{J}, u_1, u_2 \in C(\mathbb{R}),$$

where $M_i = \|I_{0^+}^{\alpha_i} f_{1,i}(\cdot, u, v)\|$, also suppose that

$$\sup g_{1,i}(t, u, v) = \rho_{g_{1,i}}, \quad i = 1, 2, \dots, n, t \in \mathbb{J}, u, v \in C(\mathbb{R}).$$

(A₃) $f_{1,i}, f_{2,j} \in C(\mathbb{J} \times \mathbb{R}^2, \mathbb{R}^+)$ for $i, j = 1, 2, \dots, n$ and

$$\sup f_{1,i}(t, u, v) = \theta_{1,i}, \quad \sup f_{2,j}(t, u, v) = \theta_{2,j}, \quad i, j = 1, 2, \dots, n, t \in \mathbb{J}, u, v \in C(\mathbb{R}).$$

(A₄) Replacing i with j and u with v also α with β in conditions (A₁), (A₂) the corresponding conditions will be satisfied for $k_{2,j}, g_{2,j}$ for $j = 1, 2, \dots, n$ in hybrid system (1.1).

2. Preliminaries

This section contains two steps. First we represent some concepts from fractional calculus that will be needed in the sequel and in second one in preparatory manner we will briefly overview the hybrid fixed point theory due to B. C. Dhage.

Definition 2.1 ([23]). The fractional *Riemann–Liouville* integral of order $\alpha > 0$ for function $f \in L^1(0, \infty)$ is defined as

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s} ds$ denotes the Euler gamma function.

Definition 2.2 ([23]). The fractional *Riemann–Liouville* derivative of order $\alpha > 0$ for function $f \in L^1(0, \infty)$ is given by

$$D_{0^+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d^n}{dt^n} \right) \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \tag{2.2}$$

where $n = [\alpha] + 1$.

Lemma 2.3 ([26]). *Let $\alpha > 0$ then*

- (i) *If $\mu > -1$, $\mu \neq \alpha - i$ with $i = 1, 2, \dots, [\alpha] + 1$ and $t > 0$, then $D_{0^+}^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}$.*
- (ii) *$D_{0^+}^\alpha t^{\alpha-i} = 0$, for $i = 1, 2, \dots, [\alpha] + 1$.*
- (iii) *If $u \in L^1(0, \infty)$, then $D_{0^+}^\alpha I_{0^+}^\alpha u(t) = u(t)$, $I_{0^+}^\alpha D_{0^+}^\alpha u(t) = u(t) + \sum_{i=1}^n c_i t^{\alpha-i}$.*
- (iv) *$D_{0^+}^\alpha u(t) = 0 \Leftrightarrow u(t) = \sum_{i=1}^n c_i t^{\alpha-i}$,*

where $c_i \in \mathbb{R}$, $n = [\alpha] + 1$. We notice that replacing α with $-\alpha$ in (i), gives us fractional integration of power functions.

In what follows we will apply the Banach space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$:

$$\begin{aligned} \mathfrak{B} &= E \times E, \quad E = \{u \mid u \in C(\mathbb{J}, \mathbb{R})\}, \\ \|(u, v)\|_{\mathfrak{B}} &= \|u\|_E + \|v\|_E, \quad \|u\|_E = \max_{t \in \mathbb{J}} \{u(t) \mid u \in E\}. \end{aligned}$$

Define set $S \subset \mathfrak{B}$ as

$$\begin{aligned} S &= \{(u, v) \in \mathfrak{B} \mid u(t), v(t) \geq 0, t \in \mathbb{J}, \|(u, v)\|_{\mathfrak{B}} \leq r\}, \\ &= \{u, v \in E \mid u(t), v(t) \geq 0, t \in \mathbb{J}, \|u\|_E + \|v\|_E \leq r\}. \end{aligned} \tag{2.3}$$

Definition 2.4. We define the integral operators $T_1, T_2 : E \rightarrow E$ by

$$\begin{aligned} T_{1,u}(t) &= \sum_{i=1}^n \left\{ k_{1,i}(t, u(t), v(t)) + g_{1,i}(t, u(t), v(t)) \cdot \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_{1,i}(s, u(s), v(s)) ds \right\}, \\ T_{2,v}(t) &= \sum_{j=1}^n \left\{ k_{2,j}(t, u(t), v(t)) + g_{2,j}(t, u(t), v(t)) \cdot \int_0^t \frac{(t-s)^{\beta_j-1}}{\Gamma(\beta_j)} f_{2,j}(s, u(s), v(s)) ds \right\}. \end{aligned} \tag{2.4}$$

Now we can define the operator $\mathfrak{T} : \mathfrak{B} \rightarrow \mathfrak{B}$ as follows

$$\mathfrak{T}(u, v) = (T_{1,u}, T_{2,v}) \quad (2.5)$$

Definition 2.5 ([1]). Let X be a normed vector space. A mapping $T : X \rightarrow X$ is said to be D-Lipschitzian, provided there exists a continuous and nondecreasing function $\psi_T : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for $x, y \in X$

$$\|Tx - Ty\| \leq \psi_T(\|x - y\|), \quad \psi_T(0) = 0.$$

The function ψ_T is called a D-function of T on X .

Remark 2.6 ([24]). If $\phi, \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are two D-functions, then i) $\phi + \psi$, ii) $\lambda\phi$, for $\lambda > 0$, and iii) $\phi\psi$ are also D-functions on \mathbb{R}^+ . Frequently used D-functions are $\phi(r) = kr$, with $k > 0$, $\phi(r) = \frac{Lr}{K+r}$, with $0 \leq L \leq K$, $\phi(r) = r - \ln(1+r)$, $\phi(r) = \ln(1+r)$, etc. So we conclude that the inequalities in hypotheses $(A_1) - (A_4)$ are reasonable.

Definition 2.7 ([21]). Let X be a normed space and suppose $S \subset X$. A finite set of N balls $B(x_n, \epsilon)$ with $x_n \in X$ and $\epsilon > 0$ is said to be a finite ϵ -covering of S , provided that every element of S lies inside one of the balls $B(x_n, \epsilon)$, i.e.

$$S \subset \bigcup_{n=1}^N B(x_n, \epsilon).$$

The set of centers $\{x_n\}$ of a finite ϵ -covering is called a finite ϵ -net for S .

Definition 2.8 ([21]). Let X be a normed space. A set $S \subset X$ is said to be a *Totally Bounded* if and only if it has a finite ϵ -covering for every $\epsilon > 0$.

Theorem 2.9 (Hausdorff compactness criterion [21]). *Assume that X be a normed space. A set $S \subset X$ is compact if and only if it is closed and totally bounded.*

Theorem 2.10 (Dhage fixed point theorem [1]). *Assume that S be a nonempty closed convex and bounded subset of Banach algebra X . Let $A, C : X \rightarrow X$ and $B : S \rightarrow X$ be three operators with the following properties:*

- (i) A, C are D-Lipschitzian with D-functions ϕ_A and ϕ_C respectively.
- (ii) B is completely continuous.
- (iii) $x = Ax + CxB y \Rightarrow x \in S$, for all $y \in S$.
- (iv) $\phi_A(r) + M\phi_C(r) < r$, for $r > 0$ where $M = \|B(S)\|$.

Then the equation $Ax + CxBx = x$ has a solution in S .

3. Main results

Theorem 3.1. *Suppose that the conditions $(A_1) - (A_4)$ hold. Then the fractional coupled system (1.1) has at least one positive solution in S .*

Proof. We are going to the represent of the proof in three steps as follows:

(S₁) Suppose that

$$\begin{aligned} A_{1,i}u(t) &= \sum_{i=1}^n k_{1,i}(t, u(t), v(t)), \\ C_{1,i}u(t) &= g_{1,i}(t, u(t), v(t)), \quad i = 1, 2, \dots, n, \\ B_{1,i}u(t) &= \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_{1,i}(s, u(s), v(s)) ds, \quad i = 1, 2, \dots, n, \end{aligned} \tag{3.1}$$

$$\begin{aligned} A_{2,j}v(t) &= \sum_{j=1}^n k_{2,j}(t, u(t), v(t)), \\ C_{2,j}v(t) &= g_{2,j}(t, u(t), v(t)), \quad j = 1, 2, \dots, n, \\ B_{2,j}v(t) &= \int_0^t \frac{(t-s)^{\beta_j-1}}{\Gamma(\beta_j)} f_{2,j}(s, u(s), v(s)) ds, \quad j = 1, 2, \dots, n. \end{aligned} \tag{3.2}$$

Define

$$A_{1,i,2,j}(u, v)(t) = \begin{pmatrix} \sum_{i=1}^n k_{1,i}(t, u(t), v(t)) \\ \sum_{j=1}^n k_{2,j}(t, u(t), v(t)) \end{pmatrix}, \tag{3.3}$$

$$C_{1,i,2,j}(u, v)(t) = \begin{pmatrix} g_{1,i}(t, u(t), v(t)) & 0 \\ 0 & g_{2,j}(t, u(t), v(t)) \end{pmatrix}, \quad i, j = 1, 2, \dots, n, \tag{3.4}$$

$$B_{1,i,2,j}(u, v)(t) = \begin{pmatrix} \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_{1,i}(s, u(s), v(s)) ds \\ \int_0^t \frac{(t-s)^{\beta_j-1}}{\Gamma(\beta_j)} f_{2,j}(s, u(s), v(s)) ds \end{pmatrix}, \quad i, j = 1, 2, \dots, n. \tag{3.5}$$

Now we can transform operator $\mathfrak{T}(u, v)$ defined by (2.5) to the operator

$$\mathfrak{T}(u, v)(t) = A_{1,i,2,j}(u, v)(t) + \sum_i \sum_j C_{1,i,2,j}(u, v)(t) B_{1,i,2,j}(u, v)(t), \tag{3.6}$$

such that above double summation acts on $(i, j) = (1, 1), (2, 2), \dots, (n, n)$.

Indeed we shall show in this step that $\mathfrak{T}(u, v)$ is a nonlinear D-Lipschitzian with D-function

$$\psi_{u,v} = \sum_{i=1}^n \psi_{k_{1,i}} + \sum_{j=1}^n \psi_{k_{2,j}} + \sum_i \sum_j \{\psi_{g_{1,i}} M_i + \psi_{g_{2,j}} M_j\}, \tag{3.7}$$

where $(i, j) = (1, 1), (2, 2), \dots, (n, n)$ and $\psi_{k_{1,i}}, \psi_{k_{2,j}}, \psi_{g_{1,i}}, \psi_{g_{2,j}}$ are D-functions corresponding to the nonlinear D-Lipschitzian $k_{1,i}, k_{2,j}, g_{1,i}, g_{2,j}$. To this aim by means of

conditions $(A_1), (A_4)$ we have

$$\begin{aligned}
 & \|A_{1,i,2,j}(u_1, v_1) - A_{1,i,2,j}(u_2, v_2)\|_{\mathfrak{B}} \\
 & \leq \sum_{i=1}^n \|k_{1,i}(\cdot, u_1, v_1) - k_{1,i}(\cdot, u_2, v_2)\|_E + \sum_{j=1}^n \|k_{2,j}(\cdot, u_1, v_1) - k_{2,j}(\cdot, u_2, v_2)\|_E \\
 & \leq \sum_{i=1}^n \frac{L_{1,k_{1,i}} \ln(1 + \|u_1 - u_2\|_E)}{2n(L_{2,k_{1,i}} + \ln(1 + \|u_1 - u_2\|_E))} + \sum_{j=1}^n \frac{L_{1,k_{2,j}} \ln(1 + \|v_1 - v_2\|_E)}{2n(L_{2,k_{2,j}} + \ln(1 + \|v_1 - v_2\|_E))} \\
 & \leq \frac{\ln(1 + \|u_1 - u_2\|_E)}{2} + \frac{\ln(1 + \|v_1 - v_2\|_E)}{2} \\
 & \leq \frac{\ln(1 + \|(u_1, u_2) - (v_1, v_2)\|_{\mathfrak{B}})}{2} + \frac{\ln(1 + \|(u_1, u_2) - (v_1, v_2)\|_{\mathfrak{B}})}{2} \\
 & = \ln(1 + \|(u_1, u_2) - (v_1, v_2)\|_{\mathfrak{B}}). \tag{3.8}
 \end{aligned}$$

Thus $k_{1,i}, k_{2,j}$ are two D-Lipschitzian with corresponding D-functions

$$\psi_{k_{1,i}}(r) = \frac{L_{1,k_{1,i}} \ln(1+r)}{2n(L_{2,k_{1,i}} + \ln(1+r))}, \quad \psi_{k_{2,j}}(r) = \frac{L_{1,k_{2,j}} \ln(1+r)}{2n(L_{2,k_{2,j}} + \ln(1+r))}, \quad i, j = 1, 2, \dots, n. \tag{3.9}$$

Equivalently we have been proved that $A_{1,i,2,j}(u, v)$ is a D-Lipschitzian with corresponding D-function

$$\psi_{A_{1,i,2,j}}(r) = \left(\sum_{i=1}^n \psi_{k_{1,i}} + \sum_{j=1}^n \psi_{k_{2,j}} \right) r. \tag{3.10}$$

Similarly by means of conditions $(A_2), (A_4)$ we conclude that

$$\begin{aligned}
 & \sum_i \sum_j \|C_{1,i,2,j}B_{1,i,2,j}(u_1, v_1) - C_{1,i,2,j}B_{1,i,2,j}(u_2, v_2)\|_{\mathfrak{B}} \\
 & \leq \sum_i \sum_j \left\{ \|g_{1,i}(\cdot, u_1, v_1) - g_{1,i}(\cdot, u_2, v_2)\|_E \|I_{0^+}^{\alpha_i} f_{1,i}\|_E + \|g_{2,j}(\cdot, u_1, v_1) \right. \\
 & \quad \left. - g_{2,j}(\cdot, u_2, v_2)\|_E \|I_{0^+}^{\beta_j} f_{2,j}\|_E \right\} \\
 & \leq \sum_i \sum_j \left\{ \frac{L_{1,g_{1,i}} \ln(1 + \|u_1 - u_2\|_E) M_i}{2nM_i(L_{2,g_{1,i}} + \ln(1 + \|u_1 - u_2\|_E))} + \frac{L_{1,g_{2,j}} \ln(1 + \|v_1 - v_2\|_E) M_j}{2nM_j(L_{2,g_{2,j}} + \ln(1 + \|v_1 - v_2\|_E))} \right\} \\
 & \leq \frac{\ln(1 + \|u_1 - u_2\|_E)}{2} + \frac{\ln(1 + \|v_1 - v_2\|_E)}{2} \\
 & \leq \frac{\ln(1 + \|(u_1, u_2) - (v_1, v_2)\|_{\mathfrak{B}})}{2} + \frac{\ln(1 + \|(u_1, u_2) - (v_1, v_2)\|_{\mathfrak{B}})}{2} \\
 & = \ln(1 + \|(u_1, u_2) - (v_1, v_2)\|_{\mathfrak{B}}). \tag{3.11}
 \end{aligned}$$

for $(i, j) = (1, 1), (2, 2), \dots, (n, n)$. Hence we deduce that both of operators $g_{1,i}, g_{2,j}$ are D-Lipschitzian with corresponding D-functions

$$\psi_{g_{1,i}}(r) = \frac{L_{1,g_{1,i}} \ln(1+r)}{2nM_i(L_{2,g_{1,i}} + \ln(1+r))}, \quad \psi_{g_{2,j}}(r) = \frac{L_{1,g_{2,j}} \ln(1+r)}{2nM_j(L_{2,g_{2,j}} + \ln(1+r))}, \quad i, j = 1, 2, \dots, n. \tag{3.12}$$

It can be that $\sum_i \sum_j C_{1,i,2,j} B_{1,i,2,j}(u, v)$ is a D-Lipschitzian with corresponding D-function

$$\psi_{C_{1,i,2,j}}(r) = \sum_i \sum_j [\psi_{g_{1,i}} M_i + \psi_{g_{2,j}} M_j] r, \quad (i, j) = (1, 1), (2, 2), \dots, (n, n). \tag{3.13}$$

At last, by means of (3.8)-(3.13) one can observe that $\mathfrak{T}(u, v)$ defined by (3.6) is a D-Lipschitzian with corresponding D-function $\psi_{u,v}$ defined by (3.7). So (S_1) is complete now.

(S₂) In this step we must prove that the operator $B_{1,i,2,j}(u, v)$ defined by (3.5) is completely continuous on S defined by (2.3). In this way firstly applying the Hausdorff compactness criterion given by Theorem 2.9, we will prove that S is a compact subset of Banach space \mathfrak{B} .

It is clear that $S \subset \mathfrak{B}$ is a cone in \mathfrak{B} . Let us define

$$\begin{aligned} S_u &= \{u \in E \mid \|u\|_E \leq \frac{r}{2}, t \in \mathbb{J}\}, \\ S_v &= \{v \in E \mid \|v\|_E \leq \frac{r}{2}, t \in \mathbb{J}\}. \end{aligned} \tag{3.14}$$

Clearly S_u, S_v is closed. Therefore S_u, S_v are complete. Thus as a result of equicontinuity of $u(t), v(t)$, the *Arzela – Ascoli* theorem implies that S_u, S_v are relatively compact. Hence Theorem 2.9 ensures that S_u, S_v are totally bounded. Thus via Definition 2.8 we conclude that there exist two finite ϵ -coverings as

$$\mathfrak{U}_\epsilon(u_i), \mathfrak{U}_\epsilon(v_j) \quad i = 1, 2, 3, \dots, l_1, \quad j = 1, 2, 3, \dots, l_2,$$

such that

$$\begin{aligned} S_u &\subset \bigcup_{i=1}^{l_1} \mathfrak{U}_\epsilon(u_i), \\ S_v &\subset \bigcup_{j=1}^{l_2} \mathfrak{U}_\epsilon(v_j), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \mathfrak{U}_\epsilon(u_i) &= \{u \in S_u \mid \|u - u_i\|_E < \epsilon\}, \\ \mathfrak{U}_\epsilon(v_j) &= \{v \in S_v \mid \|v - v_j\|_E < \epsilon\}. \end{aligned} \tag{3.16}$$

Define

$$S_{ij} = \{(u, v) \in S_u \times S_v \mid u \in \mathfrak{U}_\epsilon(u_i), v \in \mathfrak{U}_\epsilon(v_j)\}.$$

It is easy to see that $S \subset S_u \times S_v \subset \bigcup_{i,j} S_{ij}, 1 \leq i \leq l_1, 1 \leq j \leq l_2$.

In fact if we take $(u_{ij}, v_{ij}) \in S_{ij}$, then $S_u \times S_v$ can be covered by finite 4ϵ -covering

$$\mathfrak{U}_{4\epsilon}(u_{ij}, v_{ij}) = \{(u, v) \in S_u \times S_v \mid \|(u, v) - (u_{ij}, v_{ij})\|_{\mathfrak{B}} < 4\epsilon\}.$$

In other means for every $(u, v) \in S_u \times S_v$, there exist indices i, j such that

$$u \in \mathfrak{U}_\epsilon(u_i), \quad v \in \mathfrak{U}_\epsilon(v_j).$$

Therefore

$$\begin{aligned} |u - u_{ij}| &\leq |u - u_i| + |u_i - u_{ij}| < \epsilon + \epsilon = 2\epsilon, \\ |v - v_{ij}| &\leq |v - v_i| + |v_i - v_{ij}| < \epsilon + \epsilon = 2\epsilon. \end{aligned} \tag{3.17}$$

(3.17) implies that $\|(u, v) - (u_{ij}, v_{ij})\|_{\mathfrak{B}} < 4\epsilon$. Hence S has a finite 4ϵ -covering. Therefore using Theorem 2.9 we conclude that S is compact.

Turning back to the definition of $B_{1,i,2,j}(u, v)$ given by (3.5) and considering the condition (A_3) , we conclude that $B_{1,i,2,j}(u, v)$ is continuous on S . Thus $B_{1,i,2,j}(S)$ is completely continuous on S . This completes the (S_2)

(S₃) In the last step we are going to show that if

$$\begin{aligned} u_* &= \sum_{i=1}^n \{k_{1,i}(\cdot, u_*, v) + g_{1,i}(\cdot, u_*, v) \cdot I_{0^+}^{\alpha_i} f_{1,i}(\cdot, u_*, v)\}, \\ v_* &= \sum_{j=1}^n \{k_{2,j}(\cdot, u, v_*) + g_{2,j}(\cdot, u, v_*) \cdot I_{0^+}^{\beta_j} f_{2,j}(\cdot, u, v_*)\}, \end{aligned} \tag{3.18}$$

then $(u_*, v_*) \in S$ for all $(u, v) \in S$. By means of conditions $(A_1) - (A_4)$ and Applying Lemma 2.3, it is easy to check that

$$\begin{aligned} T_{1,u}(t) &\leq \sum_{i=1}^n \frac{\Gamma(\alpha_i + 1)\rho_{k_{1,i}} + \rho_{g_{1,i}}\theta_{1,i}T^{\alpha_i}}{\Gamma(\alpha_i + 1)} = r_1, \\ T_{2,v}(t) &\leq \sum_{j=1}^n \frac{\Gamma(\beta_j + 1)\rho_{k_{2,j}} + \rho_{g_{2,j}}\theta_{2,j}T^{\beta_j}}{\Gamma(\beta_j + 1)} = r_2. \end{aligned} \tag{3.19}$$

On the other hand $\|\mathfrak{T}(u, v)\|_{\mathfrak{B}} = \|T_{1,u}\|_E + \|T_{2,v}\|_E$. So we have

$$\|\mathfrak{T}(u, v)\|_{\mathfrak{B}} \leq r = 2 \max\{r_1, r_2\}.$$

Equivalently, we have proved that $\mathfrak{T}(S) \subset S$, that is if (3.18) be satisfied, then $(u_*, v_*) \in S$ for all $(u, v) \in S$. So (S_3) is completed.

Since all of the conditions (i)–(iv) in Theorem 2.10 hold, then the coupled hybrid system of fractional quadratic integral equations (1.1) has at least one positive solution in S . □

4. An example

Let us consider the coupled hybrid system of FQIEs

$$\begin{cases} u(t) = \sum_{i=1}^n \left\{ k_{1,i}(t, u(t), v(t)) + g_{1,i}(t, u(t), v(t)) \cdot \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_{1,i}(s, u(s), v(s)) ds \right\}, \\ v(t) = \sum_{j=1}^n \left\{ k_{2,j}(t, u(t), v(t)) + g_{2,j}(t, u(t), v(t)) \cdot \int_0^t \frac{(t-s)^{\beta_j-1}}{\Gamma(\beta_j)} f_{2,j}(s, u(s), v(s)) ds \right\}. \end{cases} \tag{4.1}$$

Take $t \in \mathbb{J} = [0, 4]$ and $\alpha_i = \frac{2i-1}{2}$, $\beta_j = \frac{2j+1}{4}$ for $i, j = 1, 2, \dots, n$. Setting

$$L_{p,k_{1,i}} = L_{p,k_{2,i}} = L_{p,g_{1,j}} = L_{p,g_{2,j}} = 1, \quad p = 1, 2, \quad i, j = 1, 2, \dots, n,$$

and

$$\left\{ \begin{array}{l} k_{1,i}(t, u, v) = \frac{\ln(1 + |u|)}{2n(1 + \exp(i + t^2))}, \quad g_{1,i}(t, u, v) = \frac{\ln|u|}{\left(10n\pi \frac{4^{\frac{2i-1}{2}}}{\Gamma\left(\frac{2i+1}{2}\right)}\right)(1 + t^2)}, \quad i = 1, 2, \dots, n, \\ k_{2,j}(t, u, v) = \frac{\ln(1 + |v|)}{2n(1 + \exp(j + t^2))}, \quad g_{2,j}(t, u, v) = \frac{\ln|v|}{\left(2n\pi^2 \frac{4^{\frac{2j+1}{4}}}{\Gamma\left(\frac{2j+5}{4}\right)}\right)(1 + t^2)}, \quad j = 1, 2, \dots, n, \\ f_{1,i}(t, u, v) = 4 \frac{(\tan^{-1}(i + t + u))^2 + (\cot^{-1}(i + t + v))^2}{8}, \quad i = 1, 2, \dots, n, \\ f_{2,j}(t, u, v) = \frac{(\sin^{-1}(j + t + u))^2 + (\cos^{-1}(j + t + v))^2}{8}, \quad j = 1, 2, \dots, n, \end{array} \right. \quad (4.2)$$

and after a direct calculation, we have

$$\sup \{f_{1,i}(t, u, v) \mid t \in \mathbb{J}, u, v \in C(\mathbb{R}), i = 1, 2, \dots, n\} = 5\pi, \quad M_i = 5\pi \frac{4^{\frac{2i-1}{2}}}{\Gamma\left(\frac{2i+1}{2}\right)}, \quad i = 1, 2, \dots, n,$$

$$\sup \{f_{2,j}(t, u, v) \mid t \in \mathbb{J}, u, v \in C(\mathbb{R}), j = 1, 2, \dots, n\} = \pi^2, \quad M_j = \pi^2 \frac{4^{\frac{2j+1}{4}}}{\Gamma\left(\frac{2j+5}{4}\right)}, \quad j = 1, 2, \dots, n.$$

By means of above setting all of conditions $(A_1) - (A_4)$ hold. So according on the Theorem 3.1 we deduce that coupled hybrid system of FQIEs (4.1) has at least one positive solution in S .

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