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A NOTE ON N-CONTINUUM

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Abstract. In this paper a new concept allied to 'continuum' has been introduced with the name N-continuum. Some very interesting results have been obtained which describe some interesting features of this new concept.

1. Introduction

In [6] S. Ganguly and T. Bandyopadhyay introduced a new type of space called 'H-continuum' by combining together the concepts of H-closedness and θ-connectedness ; the study was further continued in [5].

In the present paper, we utilize the concept of N-closedness [4] and δ-connectedness [8] to give rise to another continuum-like concept, called N-continuum and study some of its properties.

For such study a locally nearly compact [2] space has been utilized ; in this context, concepts of δ-component and δ-quasicomponent have been introduced. Finally, it has been shown that the two coincide in a locally nearly compact space.

2. Prerequisites

Let (X,τ) be a topological space. Let $\overline{A}$ and A^0 denote the closure and interior of A respectively in this space. We shall write simply X to denote the topological space (X,τ), if no confusion regarding the topology arises.

2.1. Preliminary definitions

Definition 2.1.1([4]) A subset $A \subseteq X$ is said to be regular open (regular closed) if $A = (\overline{A})^0$ [respectively $A = \overline{(A^0)}$].

Definition 2.1.2.([12]) A point $x \in X$ is said to be a δ-cluster point of $A(\subseteq X)$ if $U \cap A \neq \Phi$, for every regular open neighbourhood (nbd. in short) U of x in X.

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The set of all δ-cluster points of $A(\subseteq X)$ is called the δ-closure of A and we denote this by $\overline{A}^{\delta}$.

A set $A(\subseteq X)$ is said to be δ-closed if $A = \overline{A}^{\delta}$.

A set $A(\subseteq X)$ is said to be δ-open if $X \setminus A$ is δ-closed.

Definition 2.1.3.([4]) A space X is said to be semi-regular if every point of the space has a fundamental system of regularly open nbds.

Definition 2.1.4.([7]) A space X is said to be almost regular if any regularly closed set A and any $x \notin A$ can be strongly separated.

Definition 2.1.5. ([4]) A space X is called nearly-compact if any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X by open sets in X has a finite subfamily $\{U_{\alpha_i} : i = 1, \ldots, n\}$ such that $X = \bigcup_{i=1}^{n} (\overline{U_{\alpha_i}})^0$.

A set $A(\subseteq X)$ is called N-closed if any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by open sets in X has a finite subfamily $\{U_{\alpha_i} : i = 1, \ldots, n\}$ such that $A \subseteq \bigcup_{i=1}^{n} (\overline{U_{\alpha_i}})^0$.

Definition 2.1.6.([2]) A space X is called locally-nearly compact if each point has a nbd. whose closure is N-closed.

Definition 2.1.7.([8]) A pair (P, Q) of nonempty subsets of X is said to be a δ-separation relative to X if $\overline{P}^{\delta} \cap Q = \Phi = \overline{Q}^{\delta} \cap P$.

A subset A of a space X is said to be δ-connected relative to X if there exists no δ-separation (P, Q) relative to X such that $A = P \cup Q$.

Definition 2.1.8.([9]) A function $f : X \longrightarrow Y$ is said to be δ-continuous if for any $x \in X$ and each open nbd. V of $f(x)$ in Y, $\exists$ an open nbd. U of x in X such that $f((\overline{U})^0) \subseteq (\overline{V})^0$.

2.2. Some useful results

Result 2.2.1.([7]) *A space X is almost regular iff for each $x \in X$ and each regular open set U containing x, $\exists$ a regular open set V such that $x \in V \subseteq \overline{V} \subseteq U$.*

Result 2.2.2. *Let $\{A_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of subsets of X. Then*

(i) $\bigcup_{\alpha\in\Lambda} (\overline{A_\alpha}^{\delta}) \subseteq \overline{(\bigcup_{\alpha\in\Lambda} A_\alpha)}^{\delta}$. *Equality holds if Λ is finite.*

(ii) $\overline{(\bigcap_{\alpha\in\Lambda} A_\alpha)}^{\delta} \subseteq \bigcap_{\alpha\in\Lambda} (\overline{A_\alpha}^{\delta})$

(iii) $A \subseteq B(\subseteq X) \Rightarrow \overline{A}^{\delta} \subseteq \overline{B}^{\delta}$.

Proof. Straightforward.

Note 2.2.3. It is easy to see from the above result 2.2.2 that, the collection of all δ-open sets in X form a topology. We denote this topology by $\tau^\star$. We also note that, the collection of all regular open sets form a basis for the topology $\tau^\star$. Thus, each regular open set is δ-open.

Result 2.2.4.([8]) *If (P, Q) be a δ-separation relative to X, $A \subseteq P$, $B \subseteq Q$ then (A, B) is a δ-separation relative to X.*

Result 2.2.5. *Any pair (U, V) of non-empty disjoint open subsets of X is a δ-separation relative to X.*

Proof. Obvious.

Result 2.2.6.([8]) *If (P, Q) be a δ-separation relative to X and $A \subseteq X$ be δ-closed with $A = P \cup Q$ then, P, Q are δ-closed in X.*

Result 2.2.7([10]) *If A be an N-closed set in a T_2 space X then A is δ-closed.*

Remark 2.2.8. It is clear from definition that a set $A(\subseteq X)$ is N-closed iff every regular open cover of A has a finite subcover. We now give another characterisation of N-closed sets.

Result 2.2.9. A subset A of X is N-closed iff every δ-open cover of A has a finite subcover.

Proof. If A be N-closed, then, obviously, the condition holds since each δ-open set contains a regular open set [by note 2.2.3].

To prove the converse, let $\{U_\alpha : \alpha \in \Lambda\}$ be an arbitrary open cover of A. Since $U_\alpha \subseteq (\overline{U_\alpha})^0, \forall \alpha \in \Lambda$, $\{(\overline{U_\alpha})^0 : \alpha \in \Lambda\}$ is also a cover of A by regular open sets in X. But each regular open set being δ-open [by note 2.2.3] $\{(\overline{U_\alpha})^0 : \alpha \in \Lambda\}$ has a finite subcover [by given condition]. This proves that A is N-closed in X.

Result 2.2.10. *If B be a δ-closed subset of an N-closed set A in a space X then B is also N-closed.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a δ-open cover of B. Since B is δ-closed $\{U_\alpha : \alpha \in \Lambda\} \bigcup \{X \setminus B\}$ is a δ-open cover of X and hence of A i.e. $(\bigcup_{\alpha \in \Lambda} U_\alpha) \cup (X \setminus B) = X \supseteq A$.

Since A is N-closed, by result 2.2.9, the above cover has a finite subcover for A. If this finite subcover does not contain $X \setminus B$, it will be the required finite subcover of B. If it contains $X \setminus B$ then excluding $X \setminus B$ from this family we get the required finite subcover of B. Then using the result 2.2.9 we get the result.

Result 2.2.11.([2]) (i) *A T_2 space X is locally-nearly compact iff for each N-closed set C of X and each regular open set U such that $C \subseteq U$, $\exists$ an open set V in X such that $\overline{V}$ is N-closed and $C \subseteq V \subseteq \overline{V} \subseteq U$.*
(ii) *A T_2 space X is locally-nearly compact iff for each $x \in X$ and each regular open set U such that $x \in U$, $\exists$ an open set V in X such that $\overline{V}$ is N-closed and $x \in V \subseteq \overline{V} \subseteq U$.*

Result 2.2.12. *For a subset A of a space X, the following are equivalent.*

(i) *A is δ-connected relative to X.*
(ii) *If (P, Q) is a δ-separation relative to X and $A \subseteq P \cup Q$ then either $A \subseteq P$ or $A \subseteq Q$.*

(iii) *For any* $x, y \in A$, $\exists$ *a* δ*-connected set* $B \subseteq A$ *relative to* X *such that* $x, y \in B$.

Proof. (i) $\Longleftrightarrow$ (ii) follows from lemma 2.3 [8].
(i) $\Longrightarrow$ (iii): Taking $B = A$ the result follows.
(iii) $\Longrightarrow$ (i): If possible let, A be not δ-connected. Then $\exists$ a δ-separation (P, Q) relative to X such that $A = P \cup Q$. Let $x \in P, y \in Q$. Then by (iii), $\exists$ a δ-connected set $B \subseteq A$ such that $x, y \in B$. Now $(B \cap P, B \cap Q)$ forms a δ-separation of B relative to X [by result 2.2.4] ——— a contradiction.

Result 2.2.13.([8]) *If* $A \subseteq X$ *be* δ*-connected relative to* X *and* $A \subseteq B \subseteq \overline{A}^{\delta}$ *then,* B *is* δ*-connected relative to* X.

Result 2.2.14.([8]) *If* $f : X \longrightarrow Y$ *be* δ*-continuous and* $K(\subseteq X)$ *is* δ*-connected relative to* X *then* $f(K)$ *is* δ*-connected relative to* Y.

Result 2.2.15.([9]) *If* $f : X \longrightarrow Y$ *be* δ*-continuous and* $K(\subseteq X)$ *is N-closed in* X *then* $f(K)$ *is N-closed in* Y.

Result 2.2.16. ([11]) *If* A, B *be two disjoint N-closed sets in a Hausdorff space* X *then* $\exists$ *two disjoint regular open sets* U, V *of* X *such that* $A \subseteq U$ *and* $B \subseteq V$.

3. Example

We know that every compact space is nearly-compact and every nearly-compact space is H-closed. But the converse is not true in general. However, if the space be semi-regular and almost-regular then, the above three concepts become identical. So first of all we need a suitable example of a space which is neither semi-regular nor almost-regular.

Example.([1]) Let $X = \{(x, y) \in \pi : x, y \in \mathcal{Q} \text{ and } y \leq 0\}$, where π is the Euclidean plane equipped with a cartesian co-ordinate system and $\mathcal{Q}$ denotes the set of all rational numbers.

Let, $X' = \{(x, 0) \in \pi : x \in \mathcal{Q}\}$. Then $X' \subset X$.

Let $\tau_{X'}$ be the subspace topology on X' relative to the usual topology inherited from the plane.

Let, $\mathcal{E}$ be the collection of all open intervals lying on the x-axis. We fix an irrational number α >0. For each $U \in \mathcal{E}$ we define, $U^+ = \{(x', y') \in X \setminus X' : \text{the line } y - y' = \alpha(x - x') \text{ intersects } U\}$ and $U^- = \{(x', y') \in X \setminus X' : \text{the line } y - y' = -\alpha(x - x') \text{ intersects } U\}$. Also we define, $B(z; U, V) = \{z\} \bigcup (U \times \{0\} \bigcap X') \bigcup (V \times \{0\} \bigcap X')$, where $U, V \in \mathcal{E}$ and $z \in U^+ \bigcap V^-$. We now define, $\mathcal{B} = \{B(z; U, V) : U, V \in \mathcal{E},\ z \in U^+ \bigcap V^-\} \bigcup \tau_{X'}$. It is easy to verify that $\mathcal{B}$ is a basis for some topology τ' (say) on X and (X, τ') is Hausdorff. We note that, $(x', y') \in U^+ \Longleftrightarrow x' - \frac{y'}{\alpha} \in U$ and $(x', y') \in U^- \Longleftrightarrow x' + \frac{y'}{\alpha} \in U$.

Note 3.1. In the sequel we have identified $A \subseteq \mathcal{R}$ with $A \times \{0\}$, where A is any subset of the real line $\mathcal{R}$; the context shall speak for itself.

Proposition 3.2. $\overline{B(z;U,V)}^X = (\overline{U}\cap X')\cup(\overline{V}\cap X')\cup(\overline{U}^+\cup\overline{U}^-)\cup(\overline{V}^+\cup\overline{V}^-)$, *[where $\overline{U}$ denotes the closure of U in the usual topology on the real line $\mathcal{R}$ and $\overline{B}^X$ denotes the closure of B in X].*

Proof. Let us denote the R.H.S. by A. Then, $(x,y)\in A$ with $y=0 \Longrightarrow (x,0)\in(\overline{U}\cap X')\cup(\overline{V}\cap X')$.
Let, W be any open nbd. of $(x,0)$ in X.

If $(x,0)\in\overline{U}\cap X'$ then $W\cap U\neq\Phi \Longrightarrow W\cap B(z;U,V)\neq\Phi$.

If $(x,0)\in\overline{V}\cap X'$ then $W\cap V\neq\Phi \Longrightarrow W\cap B(z;U,V)\neq\Phi$.

Thus, $(x,0)\in\overline{B(z;U,V)}^X$. Now, let $(x,y)\in A$ with $y\neq 0$. Then $(x,y)\in(\overline{U}^+\cup\overline{U}^-\cup\overline{V}^+\cup\overline{V}^-)$. Let, $B((x,y);M,N)$ be any open nbd. of (x,y) in X. So, $(x,y)\in\overline{U}^+ \Longrightarrow x-\frac{y}{\alpha}\in\overline{U}$. Again, $x-\frac{y}{\alpha}\in M$. Since M is an open interval it follows that, $M\cap U\neq\Phi$. Consequently, $B((x,y);M,N)\cap B(z;U,V)\neq\Phi$ and hence $(x,y)\in\overline{B(z;U,V)}^X$. Similarly, if $(x,y)\in\overline{U}^-$ or $\overline{V}^+$ or $\overline{V}^-$ arguing same as above we have, $(x,y)\in\overline{B(z;U,V)}^X$. Thus $A\subseteq\overline{B(z;U,V)}^X$———(i)

Conversely, let $(x,y)\notin A$. If $y=0$ then $(x,0)\notin(\overline{U}\cap X')\cup(\overline{V}\cap X')$.
$\Longrightarrow \exists$ open intervals W_1,W_2 containing $(x,0)$ in X such that $W_1\cap U=\Phi$ and $W_2\cap V=\Phi \Longrightarrow (W_1\cap W_2)\cap B(z;U,V)=\Phi$. Therefore $(x,0)\notin\overline{B(z;U,V)}^X$. If $y\neq 0$ then $(x,y)\notin(\overline{U}^+\cup\overline{U}^-\cup\overline{V}^+\cup\overline{V}^-) \Longrightarrow x-\frac{y}{\alpha}\notin\overline{U}\cup\overline{V}$ and $x+\frac{y}{\alpha}\notin\overline{U}\cup\overline{V}=\overline{U\cup V}$. So, $\exists\ W_1\in\mathcal{E}$ containing $x-\frac{y}{\alpha}$ and $W_2\in\mathcal{E}$ containing $x+\frac{y}{\alpha}$ such that $W_1\cap(U\cup V)=\Phi$ and $W_2\cap(U\cup V)=\Phi \Longrightarrow B((x,y);W_1,W_2)\cap B(z;U,V)=\Phi$. $\Longrightarrow (x,y)\notin\overline{B(z;U,V)}^X$.
Therefore, $\overline{B(z;U,V)}^X\subseteq A$———(ii)

From (i) and (ii) the result follows.

Proposition 3.3. *For each $U\in\mathcal{E}$, $\overline{U\cap X'}^X=(\overline{U}\cap X')\cup(\overline{U}^+\cup\overline{U}^-)$.*

Proof. Let us denote the R.H.S. by A. Then, $(x,y)\in A$ with $y=0\Longrightarrow(x,0)\in\overline{U}\cap X'$. Let W be any open nbd. of $(x,0)$ in X. Then $W\cap U\neq\Phi\Longrightarrow W\cap(U\cap X')\neq\Phi\Longrightarrow (x,0)\in\overline{U\cap X'}^X$. Now, $(x,y)\in A$ with $y\neq 0\Longrightarrow(x,y)\in(\overline{U}^+\cup\overline{U}^-)\Longrightarrow x+\frac{y}{\alpha}\in\overline{U}$ or $x-\frac{y}{\alpha}\in\overline{U}$. Let $B((x,y);W_1,W_2)$ be any open nbd. of (x,y) in X. So, $x-\frac{y}{\alpha}\in W_1$ and $x+\frac{y}{\alpha}\in W_2$.
Therefore, $W_1\cap U\neq\Phi$ or $W_2\cap U\neq\Phi\Longrightarrow W_1\cap(U\cap X')\neq\Phi$ or $W_2\cap(U\cap X')\neq\Phi$.
$\Longrightarrow B((x,y);W_1,W_2)\cap(U\cap X')\neq\Phi$. Consequently, $(x,y)\in\overline{U\cap X'}^X$.
Thus, $A\subseteq\overline{U\cap X'}^X$.——— (i)

Conversely, let $(x,y)\notin A$. If $y=0$, then $(x,0)\notin\overline{U}\cap X'$.
$\Longrightarrow\exists$ an open nbd. W of $(x,0)$ in X such that $W\cap U=\Phi$.
$\Longrightarrow W\cap(U\cap X')=\Phi\Longrightarrow(x,0)\notin\overline{U\cap X'}^X$.

If $y\neq 0$, then $(x,y)\notin(\overline{U}^+\cup\overline{U}^-)\Longrightarrow x-\frac{y}{\alpha}\notin\overline{U}$ and $x+\frac{y}{\alpha}\notin\overline{U}$.
$\Longrightarrow\exists$ open intervals W_1,W_2 containing $x-\frac{y}{\alpha}$, $x+\frac{y}{\alpha}$ respectively such that $W_1\cap U=\Phi$ and $W_2\cap U=\Phi$.
$\Longrightarrow W_1\cap(U\cap X')=\Phi$ and $W_2\cap(U\cap X')=\Phi$.

$\implies B((x,y);W_1,W_2) \cap (U \cap X') = \Phi$.
$\implies (x,y) \notin \overline{U \cap X'}^X$.
Therefore, $\overline{U \cap X'}^X \subseteq A$ ———(ii)

From (i) and (ii) the result follows.

Proposition 3.4. (i) $(\overline{U \cap X'}^X)^0 = (U \cap X') \cup (U^+ \cap U^-)$, for any $U \in \mathcal{E}$.
(ii) $(\overline{B(z;U,V)}^X)^0 = (U \cap X') \cup (V \cap X') \cup (U^+ \cap U^-) \cup (V^+ \cap V^-) \cup (U^+ \cap V^-) \cup (U^- \cap V^+)$, *for any* $U, V \in \mathcal{E}$ *with* $U^+ \cap V^- \neq \Phi$.

Proof. (i) Since $U \cap X'$ is open in X so $U \cap X' \subseteq (\overline{U \cap X'}^X)^0$. Let, $(x,y) \in U^+ \cap U^-$. Then $B((x,y);U,U) \subseteq \overline{U \cap X'}^X = (\overline{U} \cap X') \cup (\overline{U}^+ \cup \overline{U}^-)$. Therefore $(x,y) \in (\overline{U \cap X'}^X)^0$. Thus, $(U \cap X') \cup (U^+ \cap U^-) \subseteq (\overline{U \cap X'}^X)^0$. Conversely let, $(x,y) \in (\overline{U \cap X'}^X)^0$. If $y = 0, \exists$ an open nbd. $W \cap X'$ of $(x,0)$ such that $(x,0) \in W \cap X' \subseteq (\overline{U \cap X'}^X) \implies (x,0) \in W \cap X' \subseteq \overline{U} \cap X'$. If $y \neq 0$, $\exists$ an open nbd. $B((x,y);W_1,W_2)$ of (x,y) such that $B((x,y);W_1,W_2) \subseteq \overline{U \cap X'}^X \implies (x,y) \in (\overline{U}^+ \cup \overline{U}^-)$ and $W_1 \subseteq \overline{U}$, $W_2 \subseteq \overline{U}$. But, $(x,y) \in W_1^+ \cap W_2^- \implies x - \frac{y}{\alpha} \in W_1 \subseteq \overline{U}$ and $x + \frac{y}{\alpha} \in W_2 \subseteq \overline{U}$. Since W_1, W_2 are open intervals it follows that, $x - \frac{y}{\alpha} \in U$, $x + \frac{y}{\alpha} \in U$ so that $(x,y) \in U^+ \cap U^-$.
Thus, $(\overline{U \cap X'}^X)^0 \subseteq (U \cap X') \cup (U^+ \cap U^-)$. This completes the proof.

(ii) In a similar way as in (i) we have, $(U \cap X') \cup (V \cap X') \cup (U^+ \cap U^-) \cup (V^+ \cap V^-) \subseteq (\overline{B(z;U,V)}^X)^0$. Now, $(x,y) \in U^+ \cap V^- \implies B((x,y);U,V) \subseteq \overline{B(z;U,V)}^X \implies (x,y) \in (\overline{B(z;U,V)}^X)^0$. Similarly, $(x,y) \in U^- \cap V^+ \implies B((x,y);U,V) \subseteq \overline{B(z;U,V)}^X \implies (x,y) \in (\overline{B(z;U,V)}^X)^0$. Thus, $(U \cap X') \cup (V \cap X') \cup (U^+ \cap U^-) \cup (V^+ \cap V^-) \cup (U^+ \cap V^-) \cup (U^- \cap V^+) \subseteq (\overline{B(z;U,V)}^X)^0$. Conversely let, $(x,y) \in (\overline{B(z;U,V)}^X)^0$. If $y = 0$, then arguing similarly as in (i) we get, $(x,0) \in (U \cap X') \cup (V \cap X')$. If $y \neq 0$, $\exists$ an open nbd. $B((x,y);W_1,W_2)$ of (x,y) such that $B((x,y);W_1,W_2) \subseteq \overline{B(z;U,V)}^X \implies (x,y) \in (\overline{U}^+ \cup \overline{U}^-) \cup (\overline{V}^+ \cup \overline{V}^-)$ and $W_1 \subseteq \overline{U} \cup \overline{V} = \overline{U \cup V}$, $W_2 \subseteq \overline{U \cup V}$. But, $(x,y) \in W_1^+ \cap W_2^- \implies x + \frac{y}{\alpha} \in W_2 \subseteq \overline{U \cup V}$ and $x - \frac{y}{\alpha} \in W_1 \subseteq \overline{U \cup V}$. Since, W_1, W_2 are open intervals it follows that, $x + \frac{y}{\alpha} \in U \cup V$ and $x - \frac{y}{\alpha} \in U \cup V$ so that, $(x,y) \in (U^+ \cap U^-) \cup (V^+ \cap V^-) \cup (U^+ \cap V^-) \cup (U^- \cap V^+)$. Therefore, $(\overline{B(z;U,V)}^X)^0 \subseteq (U \cap X') \cup (V \cap X') \cup (U^+ \cap U^-) \cup (V^+ \cap V^-) \cup (U^+ \cap V^-) \cup (U^- \cap V^+)$. This completes the proof.

Since $U^+ \cap U^- \neq \Phi$ for any $U \in \mathcal{E}$, it follows from proposition 3.4 that, no basic open set is regular open. Also the sets in (i) and (ii) of this proposition are regular open.

Proposition 3.5. *The space* (X, τ') *is not almost regular.*

Proof . Let $U = \{x \in \mathcal{R} : 0 < x < 1\}$. Then by above discussion $(\overline{U \cap X'}^X)^0$ is a regular open set. We denote, $G = (\overline{U \cap X'}^X)^0$. We show that, G does not contain the closure of any basic open set contained in G. Let, B be an arbitrary basic open set such

that $B \subseteq G$.
Case-I: $B = \{(x, 0) \in Q^2 : a < x < b\}, B = (a, b) \cap X'(\subset \mathcal{R})$. Then, $\overline{B}^X = ([a, b] \cap X') \cup (\overline{V}^+ \cup \overline{V}^-)$ [by proposition 3.3], where $V = (a, b)(\subset \mathcal{R})$. Since, $B \subseteq G$ it follows that $(a, b) \subset (0, 1)$ i.e. $0 < a < b < 1$. We choose a rational $x' < 0$ and another rational y' satisfying $\alpha(x'-b) < y' < \alpha(x'-a)$. $\Longrightarrow (x', y') \in V^+$ and $(x', y') \notin U^- \Longrightarrow (x'y') \in \overline{B}^X$ but $(x', y') \notin G$ [since $y' \neq 0$ and by proposition 3.4]. Thus, $\overline{B}^X \not\subseteq G$.
Case-II: $B = B(z; U, V)$. Since by case-I, $U^+ \not\subseteq G$ so from proposition 3.2 it follows that, $\overline{B}^X \not\subseteq G$. Since any regular open set contained in G must contain basic open sets, it follows from above discussion that G does not contain the closure of any regular open set. Therefore by result 2.2.1, the space is not almost regular.

Proposition 3.6. *The space (X, τ') is not semi-regular.*

Proof. We took the point $(1, 0) \in X$ and its open nbd. $U \cap X'$, where $U = \{x \in \mathcal{R} : 0 < x < 2\}$. We denote $U \cap X' = G$. Any open nbd. of $(1, 0)$ contained in G must be of the form $V \cap X'$, where $V \in \mathcal{E}$ and $V \subseteq U$. But we have seen earlier that no open set of the form $V \cap X'$ is regular open. Consequently, G does not contain any regular-open nbd. of $(1, 0)$. This completes the proof.

4. N-Continuum

In this section we introduce the concept of N-continuum and study its several properties.

Definition 4.1. Let, X be a T_2-space. A δ-connected (relative to X) N-closed set in X is called an N-continuum.

Theorem 4.2. *The union of two N-continua of a T_2-space X, which have a point in common, is an N-continuum of X.*

Proof. Let A, B be two N-continua of X with $A \cap B \neq \Phi$. Let (P, Q) be a δ-separation relative to X and $A \cup B \subseteq P \cup Q$. Since A is δ-connected relative to X so either $A \subseteq P$ or $A \subseteq Q$ [by 2.2.12]. Now, $A \subseteq P \Longrightarrow B \subseteq P$ or $A \subseteq Q \Longrightarrow B \subseteq Q$ [since $A \cap B \neq \Phi$ and B is δ-connected relative to X]. Thus, $A \cup B \subseteq P$ or $A \cup B \subseteq Q$. Consequently $A \cup B$ is δ-connected relative to X [by 2.2.12]. Also, $A \cup B$ is an N-closed set in X, since A, B are so.

Lemma 4.3. *Let A, B be two δ-closed sets in X. If the sets $A \cup B$ and $A \cap B$ are δ-connected relative to X, then the sets A, B are also δ-connected relative to X.*

Proof. If possible let, A be not δ-connected relative to X. Then $\exists$ a δ-separation (P, Q) of A relative to X such that $A = P \cup Q$. Then $A \cup B = P \cup (Q \cup B)$ and $A \cap B = (P \cap B) \cup (Q \cap B)$. If $P \cap B \neq \Phi \neq Q \cap B$ then $(P \cap B, Q \cap B)$ will form a δ-separation of $A \cap B$ relative to X——— contradicting that $A \cap B$ is δ-connected relative to X.

If at least one of $P \cap B$, $Q \cap B$ be empty, say $P \cap B = \Phi$ then, we show that $(P, Q \cup B)$ will form a δ-separation of $A \cup B$ relative to X ——— contradicting that $A \cup B$ is δ-connected relative to X.

$$\begin{aligned}(\overline{Q \cup B}^{\delta}) \cap P &= (\overline{Q}^{\delta} \cup \overline{B}^{\delta}) \cap P = (\overline{Q}^{\delta} \cup B) \cap P \text{ [since } B \text{ is } \delta\text{-closed]}\\ &= (\overline{Q}^{\delta} \cap P) \cup (B \cap P) = \Phi\end{aligned}$$

and

$$\overline{P}^{\delta} \cap (Q \cup B) = (\overline{P}^{\delta} \cap Q) \cup (B \cap \overline{P}^{\delta}) \text{ ——— } (\star)$$

Now, $P \subseteq A \Longrightarrow \overline{P}^{\delta} \subseteq \overline{A}^{\delta} = A$ [since A is δ-closed]
$\Longrightarrow \overline{P}^{\delta} \cap B \subseteq A = P \cup Q$.
But since $\overline{P}^{\delta} \cap Q = \Phi$ so, $\overline{P}^{\delta} \cap B \subseteq P$. Again since $B \cap P = \Phi$ it follows that $\overline{P}^{\delta} \cap B = \Phi$.
Therefore from $(\star)$ $\overline{P}^{\delta} \cap (Q \cup B) = \Phi$.

Thus, the assertion is proved and the lemma is complete.

Theorem 4.4. *If A, B be two N-closed sets in a T_2-space X such that $A \cup B$ and $A \cap B$ are N-continua of X, then A, B are also N-continua of X.*

Proof. Since N-closed sets in a T_2-space are δ-closed [by 2.2.7], the result follows from the lemma 4.3.

Theorem 4.5. *If $f : X \longrightarrow Y (X, Y$ both are Hausdorff) is δ-continuous and A is an N-continuum of X then $f(A)$ is an N-continuum of Y.*

Proof. The theorem follows from the results 2.2.14 and 2.2.15.

Theorem 4.6. *If $\{C_i\}_{i=1}^{\infty}$ be a decreasing sequence of N-continua of a locally nearly compact Hausdorff space X then $\bigcap_{i=1}^{\infty} C_i$ is also an N-continuum of X.*

Proof. Let $C = \bigcap_{i=1}^{\infty} C_i$. Each C_i being N-closed of the T_2-space X, is δ-closed [by result 2.2.7] and so C is δ-closed [by note 2.2.3]. Thus C being a δ-closed subset of an N-closed set C_1, is N-closed [by result 2.2.10]. We claim that $C \neq \Phi$. For, otherwise $C = \Phi \Longrightarrow X \setminus \bigcap_{i=1}^{\infty} C_i = X \Longrightarrow \bigcup_{i=1}^{\infty}(X \setminus C_i) = X \supseteq C_1$. Now, $\{(X \setminus C_i) : i = 1, \ldots\}$ is a δ-open cover of C_1 and C_1 is N-closed. So it has a finite subcover, say, $\{(X \setminus C_{i_n}) : n = 1, \ldots, p\}$ [by result 2.2.9]. Let $k = \max\{i_1, \ldots, i_p\}$. Then $C_1 \subseteq \bigcup_{n=1}^{p}(X \setminus C_{i_n}) = X \setminus \bigcap_{n=1}^{p} C_{i_n} = X \setminus C_k$ [since $\{C_i\}$ is a decreasing sequence] $\Longrightarrow C_1 \cap C_k = \Phi$ ——— a contradiction.

We now prove that C is δ-connected relative to X.

We assume the contrary. Then $\exists$ a δ-separation (P, Q) relative to X such that $C = P \cup Q$. Now C being δ-closed, so are P, Q in X [by result 2.2.6]. Therefore P, Q must be N-closed, since C is so [by result 2.2.10]. Also P, Q are disjoint. Hence by result 2.2.16, $\exists$ two disjoint regular open sets U, V of X such that $P \subseteq U$, $Q \subseteq V$. Therefore, $\bigcap_{i=1}^{\infty} C_i = C = P \cup Q \subseteq U \cup V = T$ (say). Then T is δ-open [by note 2.2.3]. Let $x \in C$. Then $x \in T$.

Since T is δ-open, by note 2.2.3, $\exists$ a regular open set T_x such that $x \in T_x \subseteq T$. Since X is a locally nearly compact Hausdorff space, $\exists$ an open set W_x such that $x \in W_x \subseteq \overline{W_x} \subseteq T_x$ and $\overline{W_x}$ is N-closed [by result 2.2.11 (ii)]. Here W_x can be taken as a regular open (and hence δ-open) set [taking $(\overline{W_x})^0$ instead of W_x]. Thus $\{W_x : x \in C\}$ is a regular open cover of the N-closed set C. So it has a finite subcover $\{W_{x_i} : i = 1, \ldots, n\}$(say). Let $W = \bigcup_{i=1}^{n} W_{x_i}$. Then $C \subseteq W \subseteq \overline{W} \subseteq T$ and $\overline{W}$ is N-closed. Also W is δ-open [by note 2.2.3]. Therefore, $T \backslash \bigcap_{i=1}^{\infty} C_i \supseteq \overline{W} \backslash W \Longrightarrow \overline{W} \backslash W \subseteq \bigcup_{i=1}^{\infty} (T \backslash C_i)$. Now, $\overline{W} \backslash W$ is δ-closed and $\overline{W} \backslash W \subseteq \overline{W}$. So $\overline{W} \backslash W$ is N-closed. Also, $T \backslash C_i$ is δ-open $\forall i$ (since T is δ-open and C_i is δ-closed $\forall i$). So, $\exists\ i_1, \ldots, i_p$ such that $\overline{W} \setminus W \subseteq \bigcup_{t=1}^{p} (T \setminus C_{i_t}) = T \setminus \bigcap_{t=1}^{p} C_{i_t} = T \setminus C_n$, where $n = \max\{i_1, \ldots, i_p\} \Longrightarrow (\overline{W} \setminus W) \cap C_n = \Phi$ ——— $(\star)$

Now, $C_n = (C_n \setminus \overline{W}) \cup (C_n \cap \overline{W}) \subseteq (C_n \setminus W) \cup (C_n \cap \overline{W})$. We note that, $(C_n \setminus W)$ and $(C_n \cap \overline{W})$ both are δ-closed and $(C_n \setminus W) \cap (C_n \cap \overline{W}) = C_n \cap (\overline{W} \setminus W) = \Phi$ [by $(\star)$]. So $(C_n \setminus W, C_n \cap \overline{W})$ forms a δ-separation relative to X. Since C_n is δ-connected relative to X and $C_n \nsubseteq C_n \setminus W$ (for, $W \supseteq C \neq \Phi$) so $C_n \subseteq C_n \cap \overline{W} \subseteq \overline{W} \subseteq T = U \cup V$ [by result 2.2.12] [Infact: $C_n \subseteq W$ since $(\overline{W} \setminus W) \cap C_n = \Phi$].

Now, U, V being disjoint regular open, (U, V) forms a δ-separation relative to X [by result 2.2.5]. Since C_n is δ-connected relative to X, either $C_n \subseteq U$ or $C_n \subseteq V$ [by result 2.2.12] $\Longrightarrow C \subseteq U$ or $C \subseteq V \Longrightarrow$ either $C \cap Q = \Phi$ or $C \cap P = \Phi$ ——— a contradiction.

Thus, $C = \bigcap_{i=1}^{\infty} C_i$ is an N-continuum of X.

5. δ-component and δ-quasicomponent

In this article we introduce the concept of δ-component and δ-quasicomponent and see when these two concepts become identical.

Definition 5.1. Let $A \subseteq X$. A subset C of A is said to be a δ-component of A relative to X if C is δ-connected relative to X and is not contained properly in any other δ-connected relative to X subset of A.

Definition 5.2. A subset $C \subseteq X$ is said to be δ-connected between A and B (where $A \cup B \subseteq C$) if there is no δ-separation (P, Q) of C relative to X such that $A \subseteq P,\ B \subseteq Q,\ C = P \cup Q$.

Definition 5.3. We define a relation ρ on $A \subseteq X$ as follows :- $(x, y) \in \rho$ iff A is δ-connected between x and y.

It is easy to verify that ρ is an equivalence relation and hence induces a partition on A. The equivalence classes of A are called δ-quasicomponents of A. We denote the δ-quasicomponents of A containing $x(\in A)$ as $A[x]$.

Theorem 5.4. *A set $C(\subseteq X)$ is δ-connected between A and B (when $A \cup B \subseteq C$) iff $C \subseteq P \cup Q$ for any δ-separation (P, Q) relative to X implies if $A \subseteq P$ then $B \cap P \neq \Phi$.*

Proof. Follows immediately from definition.

Result 5.5.
(i) *Let $x \in A$. Then $A = A[x]$ iff A is δ-connected relative to X.*
(ii) *If $\Phi \neq B \subseteq A \subseteq X$ then $B[x] \subseteq A[x]$, for each $x \in B$.*
(iii) *Let $A \subseteq X$. Then $A[x]$ is a δ-component of A relative to X for each $x \in A$ for which $A[x]$ is δ-connected relative to X.*

Proof. (i) Immediate from definition.
(ii) If B be δ-connected between x and y and $B \subseteq A$ then A will also be δ-connected between x and y.
(iii) If possible let, $\exists$ a δ-connected set C relative to X such that $A[x] \subseteq C \subseteq A \Longrightarrow C[x] \subseteq A[x]$ [by (ii) above]. Since C is δ-connected relative to X, by (i) above, $C = C[x]$. Therefore $A[x] = C$. Consequently, $A[x]$ is a δ-component of A relative to X.

Theorem 5.6. *If A is a δ-closed subset of X then $A[x]$ is δ-closed in X.*

Proof. $\overline{A[x]}^{\delta} \subseteq \overline{A}^{\delta} = A$ [since A is δ-closed]——— $(\star)$
Let $y \in A \setminus A[x]$. Then $\exists$ a δ-separation (P, Q) relative to X such that $x \in P,\ y \in Q$ and $A = P \cup Q$. Therefore $A[x] \subseteq P \Longrightarrow \overline{A[x]}^{\delta} \subseteq \overline{P}^{\delta}$. But $\overline{P}^{\delta} \cap Q = \Phi$. So $y \notin \overline{A[x]}^{\delta}$. Therefore using $(\star)$ we can write, $\overline{A[x]}^{\delta} \subseteq A[x]$. This completes the proof.

Theorem 5.7. *Let X be a locally nearly compact T_2-space and A be an N-closed subset of X. Then each δ-quasicomponent of A relative to X is a δ-component of A relative to X.*

Proof. Let $x \in A$. It now suffices to prove that $A[x]$ is a δ-component of A relative to X. For this we show that $A[x]$ is δ-connected relative to X. Then the desired conclusion will follow from result 5.5.

Let $y \in A[x]$. We construct

$$\mathcal{F} = \{\ F \subseteq A : F \text{ is } \delta\text{-closed in } X,\ x \in F \text{ and } y \in F[x]\ \}$$

Since $A \in \mathcal{F}$, $\mathcal{F} \neq \Phi$. We define a relation '$\geq$' in $\mathcal{F}$ as follows :- $F_1 \geq F_2$ $(F_1, F_2 \in \mathcal{F})$ iff $F_1 \subseteq F_2$. Clearly $(\mathcal{F}, \geq)$ is a poset. Let $\mathcal{T}$ be a chain in $\mathcal{F}$ and $C = \bigcap_{F \in \mathcal{T}} F$. Then C is a δ-closed subset of A [by note 2.2.3] and hence C is N-closed [by result 2.2.10], since A is N-closed. Also $x, y \in C$. We want to show $y \in C[x]$ i.e. C is δ-connected between x and y.

If not, $\exists$ a δ-separation (P, Q) relative to X such that $C = P \cup Q,\ x \in P,\ y \in Q$. Then P, Q are disjoint δ-closed subsets of C [by result 2.2.6], since C is δ-closed. Hence P, Q are also disjoint N-closed sets (since $P \subseteq A,\ Q \subseteq A$) [by result 2.2.10]. So, $\exists$ two regular open sets U, V in X such that $P \subseteq U,\ Q \subseteq V,\ U \cap V = \Phi$ [by result 2.2.16]. Since X is a locally nearly compact T_2-space and P, Q are N-closed so $\exists$ two open sets W_1, W_2 in X such that $P \subseteq W_1 \subseteq \overline{W_1} \subseteq U,\ Q \subseteq W_2 \subseteq \overline{W_2} \subseteq V$ and $\overline{W_1},\ \overline{W_2}$ are N-closed [by result 2.2.11]. Here we can assume that W_1, W_2 are regular open (and hence δ-open) [taking $(\overline{W_1})^0$ instead of W_1]. Therefore $\bigcap_{F \in \mathcal{T}} F = C = P \cup Q \subseteq W_1 \cup W_2 \subseteq$

$\overline{W_1 \cup W_2} \subseteq U \cup V = T$ (say). Therefore $T \setminus \bigcap_{F \in \mathcal{T}} F \supseteq \overline{W_1 \cup W_2} \setminus W_1 \cup W_2 \implies \overline{W_1 \cup W_2} \setminus W_1 \cup W_2 \subseteq \bigcup_{F \in \mathcal{T}} (T \setminus F)$. Now, $\overline{W_1 \cup W_2} \setminus W_1 \cup W_2$ is a δ-closed subset of $\overline{W_1 \cup W_2}$ which is N-closed. So $\overline{W_1 \cup W_2} \setminus W_1 \cup W_2$ is N-closed. Also $T \setminus F$ is δ-open, $\forall F \in \mathcal{T}$ (since T is δ-open and F is δ-closed $\forall F$). So $\exists$ a finite subset $\mathcal{T}_0$ of $\mathcal{T}$ such that $\overline{W_1 \cup W_2} \setminus W_1 \cup W_2 \subseteq \bigcup_{F \in \mathcal{T}_0} (T \setminus F) = T \setminus \bigcap_{F \in \mathcal{T}_0} F = T \setminus F_0$, for some $F_0 \in \mathcal{T}_0$ (since $\mathcal{T}_0$ is a finite chain) $\implies F_0 \cap (\overline{W_1 \cup W_2} \setminus W_1 \cup W_2) = \Phi$ ——— $(\star)$

Now, $F_0 = (F_0 \setminus \overline{W_1 \cup W_2}) \cup (F_0 \cap \overline{W_1}) \cup (F_0 \cap \overline{W_2}) \subseteq (F_0 \setminus W_1 \cup W_2) \cup (F_0 \cap \overline{W_1}) \cup (F_0 \cap \overline{W_2})$. We note that, $(F_0 \setminus W_1 \cup W_2)$, $F_0 \cap \overline{W_1}$, $F_0 \cap \overline{W_2}$ all are δ-closed and $(F_0 \cap \overline{W_1}) \cap (F_0 \cap \overline{W_2}) = F_0 \cap (\overline{W_1} \cap \overline{W_2}) = \Phi$ [since $\overline{W_1} \subseteq U$, $\overline{W_2} \subseteq V$, $U \cap V = \Phi$]. $(F_0 \cap \overline{W_2}) \cap (F_0 \setminus W_1 \cup W_2) = F_0 \cap (\overline{W_2} \setminus W_1 \cup W_2) = \Phi$ [by $(\star)$]

Therefore, $((F_0 \setminus W_1 \cup W_2) \cup (F_0 \cap \overline{W_1}), F_0 \cap \overline{W_2})$ forms a δ- separation relative to X. Also, $x \in (F_0 \setminus W_1 \cup W_2) \cup (F_0 \cap \overline{W_1})$ and $y \in F_0 \cap \overline{W_2}$ [since $x \in P \subseteq \overline{W_1}$, $y \in Q \subseteq \overline{W_2}$, $x, y \in F_0$]. This contradicts that F_0 is δ-connected between x and y [by Theorem 5.4]. Therefore, C is δ-connected between x and y i.e. $y \in C[x]$. Consequently, $C \in \mathcal{F}$. Also C is an upper bound of $\mathcal{T}$. Then by Zorn's lemma $\mathcal{F}$ has a maximal element C_0 (say). Since $C_0 \in \mathcal{F}$ so $x, y \in C_0$.

We now show that, C_0 is δ-connected relative to X and $C_0 \subseteq A[x]$. Then by result 2.2.12, it follows that $A[x]$ is δ-connected relative to X. If possible let (M, N) be a δ-separation relative to X with $C_0 = M \cup N$. Since $y \in C_0[x]$ i.e. C_0 is δ-connected between x and y so without loss of generality we assume that $x, y \in M$. Since M is δ-closed in X with $M \subset C_0$ and C_0 is a maximal element of $\mathcal{F}$, so M cannot be δ-connected between x and y. Consequently, $\exists$ a δ-separation (M^*, M^{**}) relative to X such that $M = M^* \cup M^{**}$, $x \in M^*$, $y \in M^{**}$. Then, $C_0 = M \cup N = M^* \cup (N \cup M^{**})$. But clearly $(M^*, M^{**} \cup N)$ is a δ-separation of C_0 relative to X with $x \in M^*, y \in M^{**} \cup N$ ——— contradicting that $y \in C_0[x]$.

Thus, C_0 is δ-connected relative to X. Therefore, $C_0 = C_0[x] \subseteq A[x]$— [by result 5.5]. This completes the proof.

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