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HERMITE-HADAMARD TYPE INEQUALITIES FOR (p_1,h_1)-(p_2,h_2)-CONVEX FUNCTIONS ON THE CO-ORDINATES

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Abstract. In this paper, we establish some Hermite-Hadamard type inequalities for (p_1,h_1)-(p_2,h_2)-convex function on the co-ordinates. Furthermore, some inequalities of Hermite-Hadamard type involving product of two convex functions on the co-ordinates are also considered. The results presented here would provide extensions of those given in earlier works.

1. Introduction

The following double integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality, see [1, 2]. Since Hermite-Hadamard's inequality for convex functions has been considered the most useful inequality in mathematical analysis, it has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found, we would like to refer the reader to the papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references cited therein. For example, Dragomir and Fitzpatrick [13] proved the variant of the Hermite-Hadamard inequality holds for s-convex functions in the second sense. Sarikaya et al. [14] obtained that variant of the Hadamard inequality holds for an h-convex function.

Fang and Shi [15] introduced the definition of (p,h)-convex function.

Definition 1.1. Let $h:J\to\mathbb{R}$ be a nonnegative and non-zero function. We say that $f:I\to\mathbb{R}$ is a (p,h)-convex function or that f belongs to the class $ghx(h,p,I)$, if f is nonnegative and

$$f([\alpha x^p+(1-\alpha)y^p]^{\frac{1}{p}}) \leq h(\alpha)f(x)+h(1-\alpha)f(y), \tag{1.2}$$

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for all $x, y \in I$ and $\alpha \in (0,1)$. Similarly, if the inequality sign in (1.2) is reversed, then f is said to be a (p,h)-concave function or belong to the class $ghv(h,p,I)$.

In [15], Fang and Shi obtained the following Hermite-Hadamard type inequalities of (p,h)-convex function.

Theorem 1.1. *If $f \in ghx(h,p,I) \cap L_1([a,b])$ for $a, b \in I$ with $a < b$, then we have*

$$\frac{1}{2h(\frac{1}{2})} f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big) \le \frac{p}{b^p-a^p}\int_a^b x^{p-1} f(x)dx \le [f(a)+f(b)]\int_0^1 h(t)dt. \tag{1.3}$$

Furthermore, they established the following inequality of Hermite-Hadamard type involving product of two convex functions.

Theorem 1.2. *Suppose that f and g are functions such that $f \in ghx(h,p,I), \in ghx(k,p,I)$, $fg \cap L_1([a,b])$, and $hk \cap L_1([0,1])$, with $a, b \in I$ and $a < b$, then we have*

$$\frac{p}{b^p-b^p}\int_a^b x^{p-1} f(x)g(x)dx \le M(a,b)\int_0^1 h(t)k(t)dt + N(a,b)\int_0^1 h(t)k(1-t)dt,$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Let us consider now a bidimensional interval $\Delta = [a,b] \times [c,d] \in \mathbb{R}^2$ with $a < b$ and $c < d$. In [16], Dragomir introduced co-ordinated convex function in the following way:

Definition 1.2. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality

$$f(tx+(1-t)y, ru+(1-r)w) \le tr f(x,u) + t(1-r)f(x,w) + r(1-t)f(y,u) + (1-t)(1-r)f(y,w),$$

holds for all $t, r \in [0,1]$ and $(x,u), (y,w) \in \Delta$.

For such a mapping Dragomir [16] proved the following Hermite-Hadamard type inequalities:

Theorem 1.3. *Suppose that $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned}
f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) &\le \frac{1}{2}\Big(\frac{1}{b-a}\int_a^b f\Big(x, \frac{c+d}{2}\Big)dx + \frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2}, y\Big)dy\Big) \\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
&\le \frac{1}{4}\Big(\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \\
&\quad + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\Big) \\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

This idea of Dragomir inspired many researchers to extend various generalizations of convex functions on co-ordinates. In 2008, Alomari and Darus [17] defined the co-ordinated s-convexity in the second sense as follows:

Definition 1.3. A mapping $f:\Delta\to\mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ if the inequality

$$\begin{aligned}&f(tx+(1-t)y,ru+(1-r)w)\\&\quad\le t^sr^sf(x,u)+t^s(1-r)^sf(x,w)+r^s(1-t)^sf(y,u)+(1-t)^s(1-r)^sf(y,w),\end{aligned}$$

holds for all $t,r\in[0,1]$, $(x,u),(y,w)\in\Delta$ and for some fixed $s\in[0,1]$.

For such a mapping they proved the following Hermite-Hadamard type inequalities:

Theorem 1.4. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated s-convex on Δ. Then one has the inequalities:*

$$\begin{aligned}4^{s-1}f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)&\le 2^{s-2}\Big(\frac{1}{b-a}\int_a^b f\Big(x,\frac{c+d}{2}\Big)dx+\frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2},y\Big)dy\Big)\\&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy\\&\le\frac{1}{2(s+1)}\Big(\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\\&\quad+\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\Big)\\&\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{(s+1)^2}.\end{aligned}$$

For refinements and counterparts of other type convex functions on the co-ordinates, see [18, 19, 20, 21, 22, 23, 24, 25].

In [26], Latif and Alomari established Hadamard-type inequalities for product of two convex functions on the co-ordinates as follow:

Theorem 1.5. *Suppose that $f,g:\Delta\to[0,\infty)$ are convex functions on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)\,dx\,dy\\&\qquad\le\frac{1}{9}L(a,b,c,d)+\frac{1}{18}M(a,b,c,d)+\frac{1}{36}N(a,b,c,d),\end{aligned}$$

and

$$4f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)\,dx\,dy$$

$$+\frac{5}{36}L(a,b,c,d)+\frac{7}{36}M(a,b,c,d)+\frac{2}{9}N(a,b,c,d),$$

where

$$\begin{aligned} L(a,b,c,d) &= f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d), \\ M(a,b,c,d) &= f(a,c)g(a,d)+f(a,d)g(a,c)+f(b,c)g(b,d)+f(b,d)g(b,c) \\ &\quad +f(b,c)g(a,c)+f(b,d)g(a,d)+f(a,c)g(b,c)+f(a,d)g(b,d), \\ N(a,b,c,d) &= f(b,c)g(a,d)+f(b,d)g(a,c)+f(a,c)g(b,d)+f(a,d)g(b,c). \end{aligned}$$

In [27], Ödemir and Akdemir established Hermite-Hadamard-type inequalities for product of convex functions and s-convex functions of 2-variables on the co-ordinates as follow:

Theorem 1.6. *$f:\Delta\to[0,\infty)$ be s_1-convex function on the co-ordinates and $g:\Delta\to[0,\infty)$ be s_2-convex function on the co-ordinates with $a<b$, $c<d$ for some fixed $s_1,s_2\in(0,1]$. Then one has the inequalities:*

$$\begin{aligned} &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\ &\le \frac{1}{(s_1+s_2+1)^2}L(a,b,c,d)+\frac{B(s_1+1,s_2+2)}{s_1+s_2+1}M(a,b,c,d)+[B(s_1+1,s_2+2)]^2N(a,b,c,d), \end{aligned}$$

and let $f:\Delta\to[0,\infty)$ be convex function on the co-ordinates and $g:\Delta\to[0,\infty)$ be s-convex function on the co-ordinates with $a<b$, $c<d$ for some fixed $s\in(0,1]$. Then one has the inequality:

$$\begin{aligned} 2^{2s}f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\ &\quad +\frac{2s+3}{(s+1)^2(s+2)^2}L(a,b,c,d)+\frac{s^2+3s+3}{(s+1)^2(s+2)^2}M(a,b,c,d) \\ &\quad +\frac{s^2+4s+3}{(s+1)^2(s+2)^2}N(a,b,c,d), \end{aligned}$$

where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ are defined in Theorem 1.5, and $B(\star,*)$ is Beta function.

Now we first give a definition of (p,h)-convex functions on Δ.

Definition 1.4. Let $h:J\to\mathbb{R}$ be a nonnegative and non-zero function. A mapping $f:\Delta\to\mathbb{R}$ is said to be (p,h)-convex on the co-ordinates on Δ if the inequality

$$f\big([\lambda x^p+(1-\lambda)y^p]^{\frac{1}{p}},[\lambda u^p+(1-\lambda)w^p]^{\frac{1}{p}}\big)\le h(\lambda)f(x,u)+h(1-\lambda)f(y,w),$$

holds for all $\lambda\in[0,1]$ and $(x,u),(y,w)\in\Delta$.

Furthermore, we introduce the definition of (p_1,h_1)-(p_2,h_2)-convex functions on the co-ordinates on Δ.

Definition 1.5. Let $h_1,h_2: J\to\mathbb{R}$ be two nonnegative and non-zero functions. A mapping $f:\Delta\to\mathbb{R}$ is said to be (p_1,h_1)-(p_2,h_2)-convex on the co-ordinates on Δ if if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$, are (p_1,h_1)-convex with respect to u and (p_2,h_2)-convex with respect to v, respectively, for all $y\in[c,d]$ and $x\in[a,b]$.

From the above definition it follows that if f is a co-ordinated (p_1,h_1)-(p_2,h_2)-convex function, then

$$\begin{aligned} &f\big([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\big)\\ &\quad\le h_1(t)h_2(r)f(x,u)+h_1(t)h_2(1-r)f(x,w)+h_1(1-t)h_2(r)f(y,u)\\ &\qquad+h_1(1-t)h_2(1-r)f(y,w). \end{aligned} \tag{1.4}$$

Similar to the proof of [16, 17], it is easily to see that every (p,h)-convex mapping $f:\Delta\to\mathbb{R}$ is (p,h)-(p,h)-convex on the co-ordinates, but converse is not general true.

Motivated by the results mentioned above, the main aim of this paper is to establish some new Hermite-Hadamard type inequalities for (p_1,h_1)-(p_2,h_2)-convex function on the co-ordinates on Δ. Furthermore, some inequalities of Hermite-Hadamard type involving product of two convex functions on the co-ordinates are also considered. The results presented here would provide extensions of those given in earlier works.

2. Main results

In this section, we will give the Hermite-Hadamard type inequalities by using (p_1,h_1)-(p_2,h_2)-convex functions of two variables on the co-ordinates on Δ.

Theorem 2.1. *Suppose that $f:\Delta\to\mathbb{R}$ is (p_1,h_1)-(p_2,h_2)-convex function on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} &\frac{1}{4h_1\big(\frac12\big)h_2\big(\frac12\big)}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\\ &\quad\le\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)\,dx\,dy\\ &\quad\le[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\int_0^1 h_1(t)\,dt\int_0^1 h_2(t)\,dt. \end{aligned} \tag{2.1}$$

Proof. According to (1.4) with $x^{p_1}=sa^{p_1}+(1-s)b^{p_1}$, $y^{p_1}=(1-s)a^{p_1}+sb^{p_1}$, $u^{p_2}=zc^{p_2}+(1-z)d^{p_2}$, $w^{p_2}=(1-z)c^{p_2}+zd^{p_2}$ and $t=r=\frac12$, we find that

$$f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$

$$\leq h_1\Big(\frac{1}{2}\Big)h_2\Big(\frac{1}{2}\Big)\Big(f([sa^{p_1}+(1-s)b^{p_1}]^{\frac{1}{p_1}},[zc^{p_2}+(1-z)d^{p_2}]^{\frac{1}{p_2}})$$
$$+f([sa^{p_1}+(1-s)b^{p_1}]^{\frac{1}{p_1}},[(1-z)c^{p_2}+zd^{p_2}]^{\frac{1}{p_2}})$$
$$+f([(1-s)a^{p_1}+sb^{p_1}]^{\frac{1}{p_1}},[zc^{p_2}+(1-z)d^{p_2}]^{\frac{1}{p_2}})$$
$$+f([(1-s)a^{p_1}+sb^{p_1}]^{\frac{1}{p_1}},[(1-z)c^{p_2}+zd^{p_2}]^{\frac{1}{p_2}})\Big). \tag{2.2}$$

Integrating (2.2) with respect to (s,z) on $[0,1]\times[0,1]$, we obtain

$$f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$\leq h_1\Big(\frac{1}{2}\Big)h_2\Big(\frac{1}{2}\Big)\Big(\int_0^1\int_0^1 f([sa^{p_1}+(1-s)b^{p_1}]^{\frac{1}{p_1}},[zc^{p_2}+(1-z)d^{p_2}]^{\frac{1}{p_2}})dsdz$$
$$+\int_0^1\int_0^1 f([sa^{p_1}+(1-s)b^{p_1}]^{\frac{1}{p_1}},[(1-z)c^{p_2}+zd^{p_2}]^{\frac{1}{p_2}})dsdz$$
$$+\int_0^1\int_0^1 f([(1-s)a^{p_1}+sb^{p_1}]^{\frac{1}{p_1}},zc^{p_2}+(1-z)d^{p_2}]^{\frac{1}{p_2}})dsdz$$
$$+\int_0^1\int_0^1 f([(1-s)a^{p_1}+sb^{p_1}]^{\frac{1}{p_1}},(1-z)c^{p_2}+zd^{p_2}]^{\frac{1}{p_2}})dsdz\Big). \tag{2.3}$$

Using the change of the variable in (2.3), we get

$$f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\leq\frac{4p_1p_2h_1(\frac{1}{2})h_2(\frac{1}{2})}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy,$$

which the first inequality is proved. For the proof of the second inequality in (2.1), we first note that since f is a (p_1,h_1)-(p_2,h_2)-convex function on the co-ordinates on Δ, then, by using (1.4) with $x=a$, $y=b$, $u=c$ and $w=d$, it yields that

$$f\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)$$
$$\leq h_1(t)h_2(r)f(a,c)+h_1(t)h_2(1-r)f(a,d)+h_1(1-t)h_2(r)f(b,c)$$
$$+h_1(1-t)h_2(1-r)f(b,d). \tag{2.4}$$

Similarly, we have

$$f\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[(1-r)c^{p_2}+rd^{p_2}]^{\frac{1}{p_2}}\big)$$
$$\leq h_1(t)h_2(1-r)f(a,c)+h_1(t)h_2(r)f(a,d)+h_1(1-t)h_2(1-r)f(b,c)$$
$$+h_1(1-t)h_2(r)f(b,d), \tag{2.5}$$

$$f\big([(1-t)a^{p_1}+tb^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)$$
$$\leq h_1(1-t)h_2(r)f(a,c)+h_1(1-t)h_2(1-r)f(a,d)+h_1(t)h_2(r)f(b,c)$$
$$+h_1(t)h_2(1-r)f(b,d), \tag{2.6}$$

and

$$f\Big([(1-t)a^{p_1}+tb^{p_1}]^{\frac{1}{p_1}},[(1-r)c^{p_2}+rd^{p_2}]^{\frac{1}{p_2}}\Big)$$
$$\le h_1(1-t)h_2(1-r)f(a,c)+h_1(1-t)h_2(r)f(a,d)+h_1(t)h_2(1-r)f(b,c)$$
$$+h_1(t)h_2(r)f(b,d). \tag{2.7}$$

Adding the inequalities (2.4)-(2.7), we have

$$f\Big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\Big)+f\Big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[(1-r)c^{p_2}+rd^{p_2}]^{\frac{1}{p_2}}\Big)$$
$$+f\Big([(1-t)a^{p_1}+tb^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\Big)+f\Big([(1-t)a^{p_1}+tb^{p_1}]^{\frac{1}{p_1}},[(1-r)c^{p_2}+rd^{p_2}]^{\frac{1}{p_2}}\Big)$$
$$\le [h_1(t)h_2(r)+h_1(t)h_2(1-r)+h_1(1-t)h_2(r)+h_1(1-t)h_2(1-r)][f(a,c)+f(a,d)$$
$$+f(b,c)+f(b,d)]. \tag{2.8}$$

Integrating (2.8) with respect to (t,r) on $[0,1]\times[0,1]$ and using the change of the variable, we get

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy$$
$$\le [f(a,c)+f(a,d)+f(b,c)+f(b,d)]\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt.$$

The proof is completed. □

By using different method, the following inequalities will be obtained.

Theorem 2.2. *Suppose that $f:\Delta\to\mathbb{R}$ is (p_1,h_1)-(p_2,h_2)-convex function on the co-ordinates on Δ. Then one has the inequalities:*

$$\frac{1}{4h_1\big(\frac12\big)h_2\big(\frac12\big)}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$\le \frac{p_1}{4h_2(\frac12)(b^{p_1}-a^{p_1})}\int_a^b x^{p_1-1}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)dx$$
$$+\frac{p_2}{4h_1(\frac12)(d^{p_2}-d^{p_2})}\int_c^d y^{p_2-1}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)dx$$
$$\le \frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy$$
$$\le \frac{p_1}{2(b^{p_1}-a^{p_1})}\Big(\int_a^b x^{p_1-1}f(x,c)dx+\int_a^b x^{p_1-1}f(x,d)dx\Big)\int_0^1 h_2(t)dt$$
$$+\frac{p_2}{2(d^{p_2}-c^{p_2})}\Big(\int_c^d y^{p_2-1}f(a,y)dy+\int_c^d y^{p_2-1}f(b,y)dy\Big)\int_0^1 h_1(t)dt$$
$$\le [f(a,c)+f(a,d)+f(b,c)+f(b,d)]\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt. \tag{2.9}$$

Proof. Since $f:\Delta\to\mathbb{R}$ is (p_1,h_1)-(p_2,h_2)-convex function on the co-ordinates on Δ, it follows that the mapping $f_x:[c,d]\to\mathbb{R}$, $f_x(y)=f(x,y)$, is (p_2,h_2)-convex on $[c,d]$ for all $x\in[a,b]$. Then by using inequalities (1.3), we can write

$$\frac{1}{2h_2(\frac{1}{2})}f_x\Big(\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\le\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f_x(y)dy\le[f_x(c)+f_x(d)]\int_0^1 h_2(t)dt,\ \forall x\in[a,b].$$

That is,

$$\begin{aligned}\frac{1}{2h_2(\frac{1}{2})}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)&\le\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(x,y)dy\\&\le[f(x,c)+f(x,d)]\int_0^1 h_2(t)dt,\ \forall x\in[a,b].\end{aligned}\tag{2.10}$$

Multiplying both sides of (2.10) by $\frac{p_1x^{p_1-1}}{b^{p_1}-a^{p_1}}$ and integrating with respect to x over $[a,b]$, respectively, we have

$$\begin{aligned}&\frac{p_1}{2h_2(\frac{1}{2})(b^{p_1}-a^{p_1})}\int_a^b x^{p_1-1}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)dx\\&\le\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy\\&\le\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)dx+\int_a^b x^{p_1-1}f(x,d)dx\Big)\int_0^1 h_2(t)dt.\end{aligned}\tag{2.11}$$

A similar arguments applied for the mapping $f_y:[a,b]\to\mathbb{R}$, $f_y(x)=f(x,y)$, we get

$$\begin{aligned}&\frac{p_2}{2h_1(\frac{1}{2})(d^{p_2}-d^{p_2})}\int_c^d y^{p_2-1}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)dx\\&\le\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy\\&\le\frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)dy+\int_c^d y^{p_2-1}f(b,y)dy\Big)\int_0^1 h_1(t)dt.\end{aligned}\tag{2.12}$$

Summing the inequalities (2.11) and (2.12), we get the second and the third inequalities in (2.9).

Now, by using the first inequality in (1.3), we also have

$$\frac{1}{2h_1(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\le\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)dx,\tag{2.13}$$

and

$$\frac{1}{2h_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\le\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)dy.\tag{2.14}$$

Multiplying both sides of (2.13) and (2.14) by $\frac{1}{4h_2(\frac{1}{2})}$ and $\frac{1}{4h_1(\frac{1}{2})}$, respectively, and adding the obtained results, we get the first inequality in (2.9).

Finally, by using the second inequality in (1.3), we can also state that

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,c)dx \le [f(a,c)+f(b,c)]\int_0^1 h_1(t)dt,$$
$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,d)dx \le [f(a,d)+f(b,d)]\int_0^1 h_1(t)dt,$$
$$\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(a,y)dy \le [f(a,c)+f(a,d)]\int_0^1 h_2(t)dt,$$

and

$$\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(b,y)dy \le [f(b,c)+f(b,d)]\int_0^1 h_2(t)dt,$$

which give, by addition, the last inequality in (2.9). The proof is completed. □

Remark 1. In Theorem 2.2, letting $p_1 = p_2 = 1$ and $h_1(t) = h_2(t) = t$, Theorem 2.2 reduces to Theorem 1.3. In Theorem 2.2, letting $p_1 = p_2 = 1$ and $h_1(t) = h_2(t) = t^s$, Theorem 2.2 reduces to Theorem 1.4.

Similarly, from the proof of Theorem 2.1, we can obtain the following theorem.

Theorem 2.3. *Suppose that $f : \Delta \to \mathbb{R}$ is (p_1,h_1)-(p_2,h_2)-convex function on the co-ordinates on Δ. Then one has the inequalities:*

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\left[\frac{a^{p_1}+b^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}+d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}w(x,y)dxdy$$
$$\le \int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)w(x,y)dxdy$$
$$\le [f(a,c)+f(a,d)+f(b,c)+f(b,d)]\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}h_1\left(\frac{x^{p_1}-a^{p_1}}{b^{p_1}-a^{p_1}}\right)h_2\left(\frac{y^{p_2}-c^{p_2}}{d^{p_2}-c^{p_2}}\right)w(x,y)dxdy.$$

where $w : \Delta \to [0,\infty)$ is a symmetric function with respect to $\left(\left[\frac{a^{p_1}+b^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{c^{p_2}+d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$, that is, $w\left(\left[ta^{p_1}+(1-t)b^{p_1}\right]^{\frac{1}{p_1}},\left[rc^{p_2}+(1-r)d^{p_2}\right]^{\frac{1}{p_2}}\right) = w\left(\left[(1-t)a^{p_1}+tb^{p_1}\right]^{\frac{1}{p_1}},\left[rc^{p_2}+(1-r)d^{p_2}\right]^{\frac{1}{p_2}}\right) = w\left(\left[ta^{p_1}+(1-t)b^{p_1}\right]^{\frac{1}{p_1}},\left[(1-r)c^{p_2}+rd^{p_2}\right]^{\frac{1}{p_2}}\right) = w\left(\left[(1-t)a^{p_1}+tb^{p_1}\right]^{\frac{1}{p_1}},\left[(1-r)c^{p_2}+rd^{p_2}\right]^{\frac{1}{p_2}}\right)$.

3. Some inequalities involving product of two convex functions

In this section, we will consider some inequalities of Hermite-Hadamard type involving product of two convex functions on the co-ordinates on Δ.

Theorem 3.1. *Suppose that $f, g : \Delta \to [0,\infty)$ are (p_1,h_1)-(p_2,h_2)-convex and (p_1,k_1)-(p_2,k_2)-convex functions on the co-ordinates on Δ, respectively. Then one has the inequality:*

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)g(x,y)dxdy$$

$$
\begin{aligned}
&\le M_1(a,b,c,d)\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(t)dt\\
&\quad+M_2(a,b,c,d)\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt\\
&\quad+M_3(a,b,c,d)\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(t)dt\\
&\quad+M_4(a,b,c,d)\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(1-t)dt, \qquad (3.1)
\end{aligned}
$$

where

$$
\begin{aligned}
M_1(a,b,c,d) &= f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d),\\
M_2(a,b,c,d) &= f(a,c)g(a,d)+f(a,d)g(a,c)+f(b,c)g(b,d)+f(b,d)g(b,c),\\
M_3(a,b,c,d) &= f(a,c)g(b,c)+f(a,d)g(b,d)+f(b,c)g(a,c)+f(b,d)g(a,d),\\
M_4(a,b,c,d) &= f(a,c)g(b,d)+f(a,d)g(b,c)+f(b,c)g(a,d)+f(b,d)g(a,c).
\end{aligned}
$$

Proof. Since f is (p_1,h_1)-(p_2,h_2)-convex on the co-ordinates and g is (p_1,k_1)-(p_2,k_2)-convex on the coordinates on Δ, it follows that

$$
\begin{aligned}
&f\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)\\
&\le h_1(t)h_2(r)f(a,c)+h_1(t)h_2(1-r)f(a,d)+h_1(1-t)h_2(r)f(b,c)+h_1(1-t)h_2(1-r)f(b,d), \quad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
&g\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)\\
&\le k_1(t)k_2(r)g(a,c)+k_1(t)k_2(1-r)g(a,d)+k_1(1-t)k_2(r)g(b,c)+k_1(1-t)k_2(1-r)g(b,d). \quad (3.3)
\end{aligned}
$$

Multiplying (3.2) and (3.3) and integrating the obtained result with respect to (t,r) on $[0,1]\times[0,1]$, we obtain our inequality (3.1). □

Theorem 3.2. *Suppose that $f,g:\Delta\to[0,\infty)$ are (p_1,h_1)-(p_2,h_2)-convex and (p_1,k_1)-(p_2,k_2)-convex functions on the co-ordinates on Δ, respectively. Then one has the inequality:*

$$
\begin{aligned}
&\frac{1}{4h_1(\frac12)h_2(\frac12)k_1(\frac12)k_2(\frac12)}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\\
&\quad-\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy\\
&\le \Omega_1(h_1,h_2,k_1,k_2)M_1(a,b,c,d)+\Omega_2(h_1,h_2,k_1,k_2)M_2(a,b,c,d)\\
&\quad+\Omega_3(h_1,h_2,k_1,k_2)M_3(a,b,c,d)+\Omega_4(h_1,h_2,k_1,k_2)M_4(a,b,c,d), \qquad (3.4)
\end{aligned}
$$

where $M_1(a,b,c,d)$, $M_2(a,b,c,d)$, $M_3(a,b,c,d)$ and $M_4(a,b,c,d)$ are defined in Theorem 3.1, and

$$
\Omega_1(h_1,h_2,k_1,k_2)=\int_0^1 h_1(t)h_2(t)dt\int_0^1 k_1(t)k_2(1-t)dt+\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(t)dt
$$

$$+\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(1-t)dt,$$

$$\Omega_2(h_1,h_2,k_1,k_2) = \int_0^1 h_1(t)h_2(t)dt\int_0^1 k_1(t)k_2(t)dt+\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(t)dt$$
$$+\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(1-t)dt,$$

$$\Omega_3(h_1,h_2,k_1,k_2) = \int_0^1 h_1(t)h_2(t)dt\int_0^1 k_1(t)k_2(t)dt+\int_0^1 h_1(t)h_2(t)dt\int_0^1 k_1(t)k_2(1-t)dt$$
$$+\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(1-t)dt,$$

and

$$\Omega_4(h_1,h_2,k_1,k_2) = \int_0^1 h_1(t)h_2(t)dt\int_0^1 k_1(t)k_2(t)dt+\int_0^1 h_1(t)h_2(t)dt\int_0^1 k_1(t)k_2(1-t)dt$$
$$+\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(t)dt.$$

Proof. Since f is (p_1,h_1)-(p_2,h_2)-convex on the co-ordinates and g is (p_1,k_1)-(p_2,k_2)-convex on the coordinates on Δ, it follows that $f_y:[a,b]\to[0,\infty)$, $f_y(x)=f(x,y)$ and $f_x:[c,d]\to[0,\infty)$, $f_x(y)=f(x,y)$ are are (p_1,h_1)-convex and (p_2,h_2)-convex on $[a,b]$ and $[c,d]$, respectively, where $x\in[a,b]$, $y\in[c,d]$. Similarly, $g_y:[a,b]\to[0,\infty)$, $g_y(x)=g(x,y)$ and $g_x:[c,d]\to[0,\infty)$, $g_x(y)=g(x,y)$ are are (p_1,k_1)-convex and (p_2,k_2)-convex on $[a,b]$ and $[c,d]$, respectively, where $x\in[a,b]$, $y\in[c,d]$.

Using Theorem 7 in [15] and multiplying both sides of the inequalities by $\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}$, we get

$$\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$-\frac{p_1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})(b^{p_1}-a^{p_1})}\int_a^b x^{p_1-1}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)dx$$
$$\le\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}\Big[f\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$+f\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\Big]\int_0^1 h_1(t)h_2(1-t)dt$$
$$+\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}\Big[f\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$+f\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\Big]\int_0^1 h_1(t)h_2(t)dt. \tag{3.5}$$

and

$$\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$

$$-\frac{p_2}{2h_1(\frac{1}{2})h_2(\frac{1}{2})(d^{p_2}-c^{p_2})}\int_c^d y^{p_2-1}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)dy$$
$$\leq \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\Big[f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)$$
$$+f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)\Big]\int_0^1 k_1(t)k_2(1-t)dt$$
$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\Big[f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)$$
$$+f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)\Big]\int_0^1 k_1(t)k_2(t)dt. \tag{3.6}$$

Now, by adding (3.5) and (3.6), we obtain

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$-\frac{p_1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})(b^{p_1}-a^{p_1})}\int_a^b x^{p_1-1}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)dx$$
$$-\frac{p_2}{2h_1(\frac{1}{2})h_2(\frac{1}{2})(d^{p_2}-c^{p_2})}\int_c^d y^{p_2-1}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)dy$$
$$\leq \frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}\Big[f\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$+f\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\Big]\cdot\int_0^1 h_1(t)h_2(1-t)dt$$
$$+\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}\Big[f\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$+f\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\Big]\int_0^1 h_1(t)h_2(t)dt$$
$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\Big[f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)$$
$$+f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)\Big]\cdot\int_0^1 k_1(t)k_2(1-t)dt$$
$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\Big[f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)$$
$$+f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)\Big]\int_0^1 k_1(t)k_2(t)dt. \tag{3.7}$$

Applying Theorem 1.2 to each term of right hand side of the above inequality (3.7), we have

$$\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\leq\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(a,y)g(a,y)dy$$

$$+[f(a,c)g(a,c)+f(a,d)g(a,d)]\int_0^1 k_1(t)k_2(1-t)dt$$

$$+[f(a,c)g(a,d)+f(a,d)g(a,c)]\int_0^1 k_1(t)k_2(t)dt, \tag{3.8}$$

$$\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\leq\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(b,y)g(b,y)dy$$

$$+[f(b,c)g(b,c)+f(b,d)g(b,d)]\int_0^1 k_1(t)k_2(1-t)dt+[f(b,c)g(b,d)$$

$$+f(b,d)g(b,c)]\int_0^1 k_1(t)k_2(t)dt, \tag{3.9}$$

$$\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\leq\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(a,y)g(b,y)dy$$

$$+[f(a,c)g(b,c)+f(a,d)g(b,d)]\int_0^1 k_1(t)k_2(1-t)dt$$

$$+[f(a,c)g(b,d)+f(a,d)g(b,c)]\int_0^1 k_1(t)k_2(t)dt, \tag{3.10}$$

$$\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(b,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(a,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\leq\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(b,y)g(a,y)dy$$

$$+[f(b,c)g(a,c)+f(b,d)g(a,d)]\int_0^1 k_1(t)k_2(1-t)dt$$

$$+[f(b,c)g(a,d)+f(b,d)g(a,c)]\int_0^1 k_1(t)k_2(t)dt, \tag{3.11}$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)\leq\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,c)g(x,c)dx$$

$$+[f(a,c)g(a,c)+f(b,c)g(b,c)]\int_0^1 h_1(t)h_2(1-t)dt$$

$$+[f(a,c)g(b,c)+f(b,c)g(a,c)]\int_0^1 h_1(t)h_2(t)dt, \tag{3.12}$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},d\Big)\leq\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,d)g(x,d)dx$$

$$+[f(a,d)g(a,d)+f(b,d)g(b,d)]\int_0^1 h_1(t)h_2(1-t)dt+[f(a,d)g(b,d)$$

$$+f(b,d)g(a,d)]\int_0^1 h_1(t)h_2(t)dt, \tag{3.13}$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)\leq\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,c)g(x,d)dx$$

$$+[f(a,c)g(a,d)+f(b,c)g(b,d)]\int_0^1 h_1(t)h_2(1-t)dt$$

$$+[f(a,c)g(b,d)+f(b,c)g(a,d)]\int_0^1 h_1(t)h_2(t)dt, \tag{3.14}$$

and

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},c\Big)\le\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,d)g(x,c)dx$$
$$+[f(a,d)g(a,c)+f(b,d)g(b,c)]\int_0^1 h_1(t)h_2(1-t)dt$$
$$+[f(a,d)g(b,c)+f(b,d)g(a,c)]\int_0^1 h_1(t)h_2(t)dt. \tag{3.15}$$

Using the inequalities (3.8)-(3.15) in (3.7), we get

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$-\frac{p_1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})(b^{p_1}-a^{p_1})}\int_a^b x^{p_1-1}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)dx$$
$$-\frac{p_2}{2h_1(\frac{1}{2})h_2(\frac{1}{2})(d^{p_2}-c^{p_2})}\int_c^d y^{p_2-1}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)dy$$
$$\le\frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)g(a,y)dy+\int_c^d y^{p_2-1}f(b,y)g(b,y)dy\Big)\int_0^1 h_1(t)h_2(1-t)dt$$
$$+\frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)g(b,y)dy+\int_c^d y^{p_2-1}f(b,y)g(a,y)dy\Big)\int_0^1 h_1(t)h_2(t)dt$$
$$+\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)g(x,c)dx+\int_a^b x^{p_1-1}f(x,d)g(x,d)dx\Big)\int_0^1 k_1(t)k_2(1-t)dt$$
$$+\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)g(x,d)dx+\int_a^b x^{p_1-1}f(x,d)g(x,c)dx\Big)\int_0^1 k_1(t)k_2(t)dt$$
$$+2M_1(a,b,c,d)\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(1-t)dt$$
$$+2M_2(a,b,c,d)\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(t)dt$$
$$+2M_3(a,b,c,d)\int_0^1 h_1(t)h_2(t)dt\int_0^1 k_1(t)k_2(1-t)dt$$
$$+2M_4(a,b,c,d)\int_0^1 h_1(t)h_2(t)dt\int_0^1 k_2(t)k_2(t)dt, \tag{3.16}$$

and

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$-\frac{p_1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})(b^{p_1}-a^{p_1})}\int_a^b x^{p_1-1}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)dx$$
$$-\frac{p_2}{2h_1(\frac{1}{2})h_2(\frac{1}{2})(d^{p_2}-c^{p_2})}\int_c^d y^{p_2-1}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)dy$$
$$\le\frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)g(a,y)dy+\int_c^d y^{p_2-1}f(b,y)g(b,y)dy\Big)\int_0^1 h_1(t)h_2(1-t)dt$$

$$+\frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)g(b,y)dy+\int_c^d y^{p_2-1}f(b,y)g(a,y)dy\Big)\int_0^1 h_1(t)h_2(t)dt$$
$$+\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)g(x,c)dx+\int_a^b x^{p_1-1}f(x,d)g(x,d)dx\Big)\int_0^1 k_1(t)k_2(1-t)dt$$
$$+\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)g(x,d)dx+\int_a^b x^{p_1-1}f(x,d)g(x,c)dx\Big)\int_0^1 k_1(t)k_2(t)dt$$
$$+2M_1(a,b,c,d)\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(1-t)dt$$
$$+2M_2(a,b,c,d)\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 k_1(t)k_2(t)dt$$
$$+2M_3(a,b,c,d)\int_0^1 h_1(t)h_2(t)dt\int_0^1 k_1(t)k_2(1-t)dt$$
$$+2M_4(a,b,c,d)\int_0^1 h_1(t)h_2(t)dt\int_0^1 k_2(t)k_2(t)dt. \tag{3.17}$$

By applying Theorem 7 in [15] to $\frac{1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$, multiplying both sides by $\frac{p_1x^{p_1-1}}{b^{p_1}-a^{p_1}}$ and integrating over $[a,b]$, we have

$$\frac{p_1}{2k_1(\frac{1}{2})k_2(\frac{1}{2})(b^{p_1}-a^{p_1})}\int_a^b x^{p_1-1}f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)dx$$
$$-\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy$$
$$\leq\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)g(x,c)dx+\int_a^b x^{p_1-1}f(x,d)g(x,d)dx\Big)\int_0^1 k_1(t)k_2(1-t)dt$$
$$+\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)g(x,d)dx+\int_a^b x^{p_1-1}f(x,d)g(x,c)dx\Big)\int_0^1 k_1(t)k_2(t)dt. \tag{3.18}$$

Similarly by applying Theorem 7 in [15] to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)$, multiplying both sides by $\frac{p_2y^{p_2-1}}{d^{p_2}-c^{p_2}}$ and integrating over $[c,d]$, we have

$$\frac{p_2}{2h_1(\frac{1}{2})h_2(\frac{1}{2})(d^{p_2}-c^{p_2})}\int_c^d y^{p_2-1}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)dy$$
$$-\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy$$
$$\leq\frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)g(a,y)dy+\int_c^d y^{p_2-1}f(b,y)g(b,y)dy\Big)\int_0^1 h_1(t)h_2(1-t)dt$$
$$+\frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)g(b,y)dy+\int_c^d y^{p_2-1}f(b,y)g(a,y)dy\Big)\int_0^1 h_1(t)h_2(t)dt. \tag{3.19}$$

Adding (3.16)-(3.19), we get

$$\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$

$$
\begin{aligned}
&-\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)\,dx\,dy\\
\le{}& \frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)g(a,y)\,dy+\int_c^d y^{p_2-1}f(b,y)g(b,y)\,dy\Big)\int_0^1 h_1(t)h_2(1-t)\,dt\\
&+\frac{p_2}{d^{p_2}-c^{p_2}}\Big(\int_c^d y^{p_2-1}f(a,y)g(b,y)\,dy+\int_c^d y^{p_2-1}f(b,y)g(a,y)\,dy\Big)\int_0^1 h_1(t)h_2(t)\,dt\\
&+\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)g(x,c)\,dx+\int_a^b x^{p_1-1}f(x,d)g(x,d)\,dx\Big)\int_0^1 k_1(t)k_2(1-t)\,dt\\
&+\frac{p_1}{b^{p_1}-a^{p_1}}\Big(\int_a^b x^{p_1-1}f(x,c)g(x,d)\,dx+\int_a^b x^{p_1-1}f(x,d)g(x,c)\,dx\Big)\int_0^1 k_1(t)k_2(t)\,dt\\
&+M_1(a,b,c,d)\int_0^1 h_1(t)h_2(1-t)\,dt\int_0^1 k_1(t)k_2(1-t)\,dt\\
&+M_2(a,b,c,d)\int_0^1 h_1(t)h_2(1-t)\,dt\int_0^1 k_1(t)k_2(t)\,dt\\
&+M_3(a,b,c,d)\int_0^1 h_1(t)h_2(t)\,dt\int_0^1 k_1(t)k_2(1-t)\,dt\\
&+M_4(a,b,c,d)\int_0^1 h_1(t)h_2(t)\,dt\int_0^1 k_2(t)k_2(t)\,dt. \qquad (3.20)
\end{aligned}
$$

Applying Theorem 1.2 to each term of right hand side of the above inequality (3.19), we have

$$
\begin{aligned}
&\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(a,y)g(a,y)\,dy\\
&\le [f(a,c)g(a,c)+f(a,d)g(a,d)]\int_0^1 k_1(t)k_2(t)\,dt\\
&\quad +[f(a,c)g(a,d)+f(a,d)g(a,c)]\int_0^1 k_1(t)k_2(1-t)\,dt, \qquad (3.21)\\
&\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(b,y)g(b,y)\,dy\\
&\le [f(b,c)g(b,c)+f(b,d)g(b,d)]\int_0^1 k_1(t)k_2(t)\,dt\\
&\quad +[f(b,c)g(b,d)+f(b,d)g(b,c)]\int_0^1 k_1(t)k_2(1-t)\,dt, \qquad (3.22)\\
&\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(a,y)g(b,y)\,dy\\
&\le [f(a,c)g(b,c)+f(a,d)g(b,d)]\int_0^1 k_1(t)k_2(t)\,dt\\
&\quad +[f(a,c)g(b,d)+f(a,d)g(b,c)]\int_0^1 k_1(t)k_2(1-t)\,dt, \qquad (3.23)\\
&\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d y^{p_2-1}f(b,y)g(a,y)\,dy
\end{aligned}
$$

$$\leq [f(b,c)g(a,c)+f(b,d)g(a,d)]\int_0^1 k_1(t)k_2(t)dt$$
$$+[f(b,c)g(a,d)+f(b,d)g(a,c)]\int_0^1 k_1(t)k_2(1-t)dt, \tag{3.24}$$

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,c)g(x,c)dx$$
$$\leq [f(a,c)g(a,c)+f(b,c)g(b,c)]\int_0^1 h_1(t)h_2(t)dt$$
$$+[f(a,c)g(b,c)+f(b,c)g(a,c)]\int_0^1 h_1(t)h_2(1-t)dt, \tag{3.25}$$

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,d)g(x,d)dx$$
$$\leq [f(a,d)g(a,d)+f(b,d)g(b,d)]\int_0^1 h_1(t)h_2(t)dt$$
$$+[f(a,d)g(b,d)+f(b,d)g(a,d)]\int_0^1 h_1(t)h_2(1-t)dt, \tag{3.26}$$

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,c)g(x,d)dx$$
$$\leq [f(a,c)g(a,d)+f(b,c)g(b,d)]\int_0^1 h_1(t)h_2(t)dt$$
$$+[f(a,c)g(b,d)+f(b,c)g(a,d)]\int_0^1 h_1(t)h_2(1-t)dt, \tag{3.27}$$

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,d)g(x,c)dx$$
$$\leq [f(a,d)g(a,c)+f(b,d)g(b,c)]\int_0^1 h_1(t)h_2(t)dt$$
$$+[f(a,d)g(b,c)+f(b,d)g(a,c)]\int_0^1 h_1(t)h_2(1-t)dt. \tag{3.28}$$

Using the inequalities (3.20)-(3.28) in (3.19), we get

$$\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})}f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)g\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)$$
$$-\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)dxdy$$
$$\leq \Omega_1(h_1,h_2,k_1,k_2)M_1(a,b,c,d)+\Omega_2(h_1,h_2,k_1,k_2)M_2(a,b,c,d)$$
$$+\Omega_3(h_1,h_2,k_1,k_2)M_3(a,b,c,d)+\Omega_4(h_1,h_2,k_1,k_2)M_4(a,b,c,d),$$

which give the desired result (3.4). This completes the proof. □

Remark 2. In Theorems 3.1 and 3.2, letting $p_1 = p_2 = 1$ and $h_1(t) = h_2(t) = k_1(t) = k_2(t) = t$, Theorems 3.1 and 3.2 reduce to Theorem 1.5. In Theorems 3.1 and 3.2, letting $p_1 = p_2 = 1$, and $h_1(t)=h_2(t)=t^{s_1}$, $k_1(t)=k_2(t)=t^{s_2}$ or $h_1(t)=h_2(t)=k_1(t)=k_2(t)=t^s$, Theorems 3.1 and 3.2 reduces to Theorem 1.6.

Theorem 3.3. *Suppose that $f,g:\Delta\to[0,\infty)$ are (p_1,h_1)-(p_2,h_2)-convex and (p_1,k_1)-(p_2,k_2)-convex functions on the co-ordinates on Δ, respectively. Then one has the inequality:*

$$\begin{aligned}
&\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\Big(g(a,c)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}k_1\Big(\frac{b^{p_1}-x^{p_1}}{b^{p_1}-a^{p_1}}\Big)k_2\Big(\frac{d^{p_2}-y^{p_2}}{d^{p_2}-c^{p_2}}\Big)f(x,y)\,dx\,dy\\
&\quad+g(a,d)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}k_1\Big(\frac{b^{p_1}-x^{p_1}}{b^{p_1}-a^{p_1}}\Big)k_2\Big(\frac{y^{p_2}-c^{p_2}}{d^{p_2}-c^{p_2}}\Big)f(x,y)\,dx\,dy\\
&\quad+g(b,c)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}k_1\Big(\frac{x^{p_1}-a^{p_1}}{b^{p_1}-a^{p_1}}\Big)k_2\Big(\frac{d^{p_2}-y^{p_2}}{d^{p_2}-c^{p_2}}\Big)f(x,y)\,dx\,dy\\
&\quad+g(b,d)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}k_1\Big(\frac{x^{p_1}-a^{p_1}}{b^{p_1}-a^{p_1}}\Big)k_2\Big(\frac{y^{p_2}-c^{p_2}}{d^{p_2}-c^{p_2}}\Big)f(x,y)\,dx\,dy\Big)\\
&\quad+\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\Big(f(a,c)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}h_1\Big(\frac{b^{p_1}-x^{p_1}}{b^{p_1}-a^{p_1}}\Big)h_2\Big(\frac{d^{p_2}-y^{p_2}}{d^{p_2}-c^{p_2}}\Big)g(x,y)\,dx\,dy\\
&\quad+f(a,d)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}h_1\Big(\frac{b^{p_1}-x^{p_1}}{b^{p_1}-a^{p_1}}\Big)h_2\Big(\frac{y^{p_2}-c^{p_2}}{d^{p_2}-c^{p_2}}\Big)g(x,y)\,dx\,dy\\
&\quad+f(b,c)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}h_1\Big(\frac{x^{p_1}-a^{p_1}}{b^{p_1}-a^{p_1}}\Big)h_2\Big(\frac{d^{p_2}-y^{p_2}}{d^{p_2}-c^{p_2}}\Big)g(x,y)\,dx\,dy\\
&\quad+f(b,d)\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}h_1\Big(\frac{x^{p_1}-a^{p_1}}{b^{p_1}-a^{p_1}}\Big)h_2\Big(\frac{y^{p_2}-c^{p_2}}{d^{p_2}-c^{p_2}}\Big)g(x,y)\,dx\,dy\Big)\\
&\le\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)\,dx\,dy\\
&\quad+M_1(a,b,c,d)\int_0^1 h_1(t)k_1(t)\,dt\int_0^1 h_2(t)k_2(t)\,dt\\
&\quad+M_2(a,b,c,d)\int_0^1 h_1(t)k_1(t)\,dt\int_0^1 h_2(t)k_2(1-t)\,dt\\
&\quad+M_3(a,b,c,d)\int_0^1 h_1(t)k_1(1-t)\,dt\int_0^1 h_2(t)k_2(t)\,dt\\
&\quad+M_4(a,b,c,d)\int_0^1 h_1(t)k_1(1-t)\,dt\int_0^1 h_2(t)k_2(1-t)\,dt,
\end{aligned}$$

where $M_1(a,b,c,d)$, $M_2(a,b,c,d)$, $M_3(a,b,c,d)$ and $M_4(a,b,c,d)$ are defined in Theorem 3.1.

Proof. Since f is (p_1,h_1)-(p_2,h_2)-convex on the co-ordinates and g is (p_1,k_1)-(p_2,k_2)-convex on the coordinates on Δ, it follows that

$$\begin{aligned}
f\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)&\le h_1(t)h_2(r)f(a,c)+h_1(t)h_2(1-r)f(a,d)\\
&\quad+h_1(1-t)h_2(r)f(b,c)+h_1(1-t)h_2(1-r)f(b,d),
\end{aligned}\tag{3.29}$$

and

$$g\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)\le k_1(t)k_2(r)g(a,c)+k_1(t)k_2(1-r)g(a,d)$$
$$+k_1(1-t)k_2(r)g(b,c)+k_1(1-t)k_2(1-r)g(b,d). \quad (3.30)$$

By (3.29)-(3.30) and using the elementary inequality, if $e\le f$ and $p\le r$, then $er+fp\le ep+fr$ for all $e,f,p,r\in\mathbb{R}$, we get the following ienquality

$$\begin{aligned}
&f\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)\Big(k_1(t)k_2(r)g(a,c)+k_1(t)k_2(1-r)g(a,d)\\
&\quad+k_1(1-t)k_2(r)g(b,c)+k_1(1-t)k_2(1-r)g(b,d)\Big)\\
&\quad+g\big([ta^{p_1}+(1-t)d^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)\Big(h_1(t)h_2(r)f(a,c)\\
&\quad+h_1(t)h_2(1-r)f(a,d)+h_1(1-t)h_2(r)f(b,c)+h_1(1-t)h_2(1-r)f(b,d)\Big)\\
&\le f\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)g\big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\big)\\
&\quad+\Big(h_1(t)h_2(r)f(a,c)+h_1(t)h_2(1-r)f(a,d)+h_1(1-t)h_2(r)f(b,c)+h_1(1-t)h_2(1-r)f(b,d)\Big)\\
&\quad\times\Big(k_1(t)k_2(r)g(a,c)+k_1(t)k_2(1-r)g(a,d)+k_1(1-t)k_2(r)g(b,c)\\
&\quad+k_1(1-t)k_2(1-r)g(b,d)\Big).
\end{aligned} \quad (3.31)$$

By integrating the above inequality (3.31) on $[0,1]\times[0,1]$ with respect to t,r and by taking into account the change of variables $ta^{p_1}+(1-t)b^{p_1}=x^{p_1}$ and $rc^{p_2}+(1-r)d^{p_2}=y^{p_2}$, we obtain the desired result. □

Remark 3. In Theorem 3.3, letting $p_1=p_2=1$ and $h_1(t)=h_2(t)=k_1(t)=k_2(t)=t$, Theorem 3.3 reduces to Theorem 10 obtained by Ödemir and Akdemir [27].

Theorem 3.4. *Suppose that $f,g:\Delta\to[0,\infty)$ are (p_1,h_1)-(p_2,h_2)-convex and (p_1,k_1)-(p_2,k_2)-convex functions on the co-ordinates on Δ, respectively. Then one has the inequality:*

$$\begin{aligned}
&\frac{p_1^2p_2^2}{4(b^{p_1}-a^{p_1})^2(d^{p_2}-c^{p_2})^2}\int_a^b\int_a^b\int_c^d\int_c^d\int_0^1\int_0^1 x^{p_1-1}y^{p_1-1}u^{p_2-1}w^{p_2-1}\\
&\quad\times f\big([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\big)\\
&\quad\times g\big([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\big)dtdrdxdydudw\\
&\le\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)g(x,y)dxdy\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(t)dt\\
&\quad+(\Theta_1+\Theta_3+\Theta_5)M_1(a,b,c,d)+(\Theta_1+\Theta_4+\Theta_5)M_2(a,b,c,d)\\
&\quad+(\Theta_2+\Theta_3+\Theta_5)M_3(a,b,c,d)+(\Theta_2+\Theta_4+\Theta_5)M_4(a,b,c,d),
\end{aligned} \quad (3.32)$$

where $M_1(a,b,c,d)$, $M_2(a,b,c,d)$, $M_3(a,b,c,d)$ and $M_4(a,b,c,d)$ are defined in Theorem 3.1, and

$$\Theta_1=\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\Big(\int_0^1 h_1(t)k_1(t)dt\Big)^2\int_0^1 h_2(t)k_2(1-t)dt,$$

$$\Theta_2 = \int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(1-t)dt,$$
$$\Theta_3 = \int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\Big(\int_0^1 h_2(t)k_2(t)dt\Big)^2\int_0^1 h_1(t)k_1(1-t)dt,$$
$$\Theta_4 = \int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dr\int_0^1 h_2(t)k_2(1-t)dt\int_0^1 h_1(t)k_1(1-t)dt,$$
$$\Theta_5 = \int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(1-t)dt.$$

Proof. Since f is (p_1,h_1)-(p_2,h_2)-convex on the co-ordinates and g is (p_1,k_1)-(p_2,k_2)-convex on the coordinates on Δ, it follows that

$$\begin{aligned}&f\big([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\big)\\&\le h_1(t)h_2(r)f(x,u)+h_1(t)h_2(1-r)f(x,w)+h_1(1-t)h_2(r)f(y,u)\\&\quad+h_1(1-t)h_2(1-r)f(y,w),\end{aligned}\tag{3.33}$$

and

$$\begin{aligned}&g\big([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\big)\\&\le k_1(t)k_2(r)g(x,u)+k_1(t)k_2(1-r)g(x,w)+k_1(1-t)k_2(r)g(y,u)\\&\quad+k_1(1-t)k_2(1-r)g(y,w),\end{aligned}\tag{3.34}$$

for $t,r\in[0,1]$, $x,y\in[a,b]$ and $u,w\in[c,d]$. Because f and g are nonnegative, from (3.33) and (3.34), we get the inequality

$$\begin{aligned}&f\big([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\big)g\big([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\big)\\
&\le\Big(h_1(t)h_2(r)f(x,u)+h_1(t)h_2(1-r)f(x,w)+h_1(1-t)h_2(r)f(y,u)+h_1(1-t)h_2(1-r)f(y,w)\Big)\\
&\quad\times\Big(k_1(t)k_2(r)g(x,u)+k_1(t)k_2(1-r)g(x,w)+k_1(1-t)k_2(r)g(y,u)+k_1(1-t)k_2(1-r)g(y,w)\Big)\\
&=h_1(t)h_2(r)k_1(t)k_2(r)f(x,u)g(x,u)+h_1(t)h_2(1-r)k_1(t)k_2(1-r)f(x,w)g(x,w)\\
&\quad+h_1(1-t)h_2(r)k_1(1-t)k_2(r)f(y,u)g(y,u)+h_1(1-t)h_2(1-r)k_1(1-t)k_2(1-r)f(y,w)g(y,w)\\
&\quad+h_1(t)h_2(1-r)k_1(t)k_2(r)f(x,w)g(x,u)+h_1(t)h_2(r)k_1(t)k_2(1-r)f(x,u)g(x,w)\\
&\quad+h_1(1-t)h_2(1-r)k_1(1-t)k_2(r)f(y,w)g(y,u)+h_1(1-t)h_2(r)k_1(1-t)k_2(1-r)f(y,u)g(y,w\\
&\quad+h_1(t)h_2(r)k_1(1-t)k_2(r)f(x,u)g(y,u)+h_1(1-t)h_2(r)k_1(t)k_2(r)f(y,u)g(x,u)\\
&\quad+h_1(1-t)h_2(1-r)k_1(t)k_2(1-r)f(y,w)g(x,w)+h_1(t)h_2(1-r)k_1(1-t)k_2(1-r)f(x,w)g(y,w)\\
&\quad+h_1(1-t)h_2(1-r)k_1(t)k_2(r)f(y,w)g(x,u)+h_1(1-t)h_2(r)k_1(t)k_2(1-r)f(y,u)g(x,w)\\
&\quad+h_1(t)h_2(1-r)k_1(1-t)k_2(r)f(x,w)g(y,u)+h_1(t)h_2(r)k_1(1-t)k_2(1-r)f(x,u)g(y,w).\end{aligned}\tag{3.35}$$

Multiplying both sides of (3.35) by $\frac{p_1^2p_2^2x^{p_1-1}y^{p_1-1}u^{p_2-1}w^{p_2-1}}{(b^{p_1}-a^{p_1})^2(d^{p_2}-c^{p_2})^2}$ and integrating over $[a,b]^2\times[c,d]^2\times[0,1]^2$, we have

$$\frac{p_1^2p_2^2}{(b^{p_1}-a^{p_1})^2(d^{p_2}-c^{p_2})^2}\int_a^b\int_a^b\int_c^d\int_c^d\int_0^1\int_0^1 x^{p_1-1}y^{p_1-1}u^{p_2-1}w^{p_2-1}$$

$$
\begin{aligned}
&\times f\left([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\right)\\
&\times g\left([tx^{p_1}+(1-t)y^{p_1}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)w^{p_2}]^{\frac{1}{p_2}}\right)dtdrdxdydudw\\
\le&\ \frac{4p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}y^{p_2-1}f(x,y)g(x,y)dxdy\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(t)dt\\
&+\frac{4p_1p_2^2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})^2}\int_a^b\int_c^d\int_c^d x^{p_1-1}u^{p_2-1}w^{p_2-1}f(x,w)g(x,u)dxdudw\\
&\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt\\
&+\frac{4p_1^2p_2}{(b^{p_1}-a^{p_1})^2(d^{p_2}-c^{p_2})}\int_a^b\int_a^b\int_c^d x^{p_1-1}y^{p_1-1}u^{p_2-1}f(x,u)g(y,u)dxdydu\\
&\times\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(t)dt\\
&+\frac{4p_1^2p_2^2}{(b^{p_1}-a^{p_1})^2(d^{p_2}-c^{p_2})^2}\int_a^b\int_c^d x^{p_1-1}w^{p_2-1}f(x,w)dxdw\int_a^b\int_c^d y^{p_1-1}u^{p_2-1}g(y,u)dydu\\
&\times\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(1-t)dt. \qquad (3.36)
\end{aligned}
$$

Applying Theorems 1.1 and 1.2 to second and third term of right hand side of the inequality (3.36), we have

$$
\begin{aligned}
&\frac{p_1p_2^2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})^2}\int_a^b\int_c^d\int_c^d x^{p_1-1}u^{p_2-1}w^{p_2-1}f(x,w)g(x,u)dxdudw\\
&=\frac{p_2^2}{(d^{p_2}-c^{p_2})^2}\int_c^d\int_c^d u^{p_2-1}w^{p_2-1}\left(\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,w)g(x,u)dx\right)dudw\\
&\le\frac{p_2^2}{(d^{p_2}-c^{p_2})^2}\int_c^d\int_c^d u^{p_2-1}w^{p_2-1}\Big([f(a,w)g(a,u)+f(b,w)g(b,u)]\int_0^1 h_1(t)k_1(t)dt\\
&\quad+[f(a,w)g(b,u)+f(b,w)g(a,u)]\int_0^1 h_1(t)k_1(1-t)dt\Big)dudw\\
&\le\frac{p_2^2}{(d^{p_2}-c^{p_2})^2}\Big(\Big[\int_c^d w^{p_2-1}f(a,w)dw\int_c^d u^{p_2-1}g(a,u)du\\
&\quad+\int_c^d w^{p_2-1}f(b,w)dw\int_c^d u^{p_2-1}g(b,u)du\Big]\int_0^1 h_1(t)k_1(t)dt\\
&\quad+\Big[\int_c^d w^{p_2-1}f(a,w)dw\int_c^d u^{p_2-1}g(b,u)du\\
&\quad+\int_c^d w^{p_2-1}f(b,w)dw\int_c^d u^{p_2-1}g(a,u)du\Big]\int_0^1 h_1(t)k_1(1-t)dt\Big)\\
&\le[f(a,c)+f(a,d)][g(a,c)+g(a,d)]\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(t)dt\\
&\quad+[f(b,c)+f(b,d)][g(b,c)+g(b,d)]\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(t)dt
\end{aligned}
$$

$$+[f(a,c)+f(a,d)][g(b,c)+g(b,d)]\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(1-t)dt$$
$$+[f(b,c)+f(b,d)][g(a,c)+g(a,d)]\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(1-t)dt$$
$$=[M_1(a,b,c,d)+M_2(a,b,c,d)]\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(t)dt$$
$$+[M_3(a,b,c,d)+M_4(a,b,c,d)]\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(1-t)dt, \tag{3.37}$$

and

$$\frac{p_1^2p_2}{(b^{p_1}-a^{p_1})^2(d^{p_2}-c^{p_2})}\int_a^b\int_a^b\int_c^d x^{p_1-1}y^{p_1-1}u^{p_2-1}f(x,u)g(y,u)dxdydu$$
$$=\frac{p_1^2}{(b^{p_1}-a^{p_1})^2}\int_a^b\int_a^b x^{p_1-1}y^{p_1-1}\Big(\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}f(x,u)g(y,u)du\Big)dxdy$$
$$\le\frac{p_1^2}{(b^{p_1}-a^{p_1})^2}\int_a^b\int_a^b x^{p_1-1}y^{p_1-1}\Big([f(x,c)g(y,c)+f(x,d)g(y,d)]\int_0^1 h_2(t)k_2(t)dt$$
$$+[f(x,c)g(y,d)+f(x,d)g(y,c)]\int_0^1 h_2(t)k_2(1-t)dt\Big)dxdy$$
$$\le\frac{p_1^2}{(b^{p_1}-a^{p_1})^2}\Big(\Big[\int_a^b x^{p_1-1}f(x,c)dx\int_a^b y^{p_1-1}g(y,c)dy$$
$$+\int_a^b x^{p_1-1}f(x,d)dx\int_a^b y^{p_1-1}g(y,d)dy\Big]\int_0^1 h_2(t)k_2(t)dt$$
$$+\Big[\int_a^b x^{p_1-1}f(x,c)dx\int_a^b y^{p_1-1}g(y,d)dy$$
$$+\int_a^b x^{p_1-1}f(x,d)dx\int_a^b y^{p_1-1}g(y,c)dt\Big]\int_0^1 h_2(t)k_2(1-t)dt\Big)$$
$$\le[f(a,c)+f(b,c)][g(a,c)+g(b,c)]\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dt$$
$$+[f(a,d)+f(b,d)][g(a,d)+g(b,d)]\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dt$$
$$+[f(a,c)+f(b,c)][g(a,d)+g(b,d)]\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt$$
$$+[f(a,d)+f(b,d)][g(a,c)+g(b,c)]\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt$$
$$=[M_1(a,b,c,d)+M_3(a,b,c,d)]\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dt$$
$$+[M_2(a,b,c,d)+M_4(a,b,c,d)]\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt. \tag{3.38}$$

Applying Theorem 2.1 to fourth term of right hand side of the inequality (3.36), we have

$$\frac{p_1^2p_2^2}{(b^{p_1}-a^{p_1})^2(d^{p_2}-c^{p_2})^2}\int_a^b\int_c^d x^{p_1-1}w^{p_2-1}f(x,w)dxdw\int_a^b\int_c^d y^{p_1-1}u^{p_2-1}g(y,u)dydu$$

$$\le \left([f(a,c)+f(a,d)+f(b,c)+f(b,d)]\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\right)$$
$$\cdot\left([g(a,c)+g(a,d)+g(b,c)+g(b,d)]\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt\right)$$
$$= [M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt. \tag{3.39}$$

Using the inequalities (3.37)-(3.39) in (3.36), we get the desired result (3.32). □

Remark 4. Suppose that $f,g:\Delta\to[0,\infty)$ are convex functions on the co-ordinates on Δ. Then one has the inequality:

$$\frac{9}{4(b-a)^2(d-c)^2}\int_a^b\int_a^b\int_c^d\int_c^d\int_0^1\int_0^1 f(tx+(1-t)y,ru+(1-r)w)g(tx+(1-t)y,$$
$$ru+(1-r)w)dtdrdxdydudw$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy+\frac{19}{192}L(a,b,c,d)+\frac{5}{64}M(a,b,c,d)+\frac{11}{192}N(a,b,c,d),$$

where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ are defined in Theorem 1.5.

Remark 5. Suppose that $f,g:\Delta\to[0,\infty)$ are s-convex functions on the co-ordinates on Δ. Then one has the inequality:

$$\frac{(1+2s)^2}{4(b-a)^2(d-c)^2}\int_a^b\int_a^b\int_c^d\int_c^d\int_0^1\int_0^1 f(tx+(1-t)y,ru+(1-r)w)$$
$$g(tx+(1-t)y,ru+(1-r)w)dtdrdxdydudw$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy$$
$$+[\Gamma(1+s)]^2\left(\frac{(1+2s)^2[\Gamma(1+s)]^2+2(1+s)^2\Gamma(2+2s)}{(1+s)^4[\Gamma(2+2s)]^2}L(a,b,c,d)\right.$$
$$+\frac{(2+8s+9s^2+2s^3)[\Gamma(1+s)]^2+(1+s)^2\Gamma(2+2s)}{(1+s)^4[\Gamma(2+2s)]^2}M(a,b,c,d)$$
$$\left.+\frac{(3+6s+2s^2)[\Gamma(1+s)]^2}{(1+s)^4[\Gamma(2+2s)]^2}N(a,b,c,d)\right),$$

where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ are defined in Theorem 1.5.

Remark 6. Suppose that $f,g:\Delta\to[0,\infty)$ are s_1-convex and s_2-convex functions on the co-ordinates on Δ, respectively. Then one has the inequality:

$$\frac{(1+s_1+s_2)^2}{4(b-a)^2(d-c)^2}\int_a^b\int_a^b\int_c^d\int_c^d\int_0^1\int_0^1 f(tx+(1-t)y,ru+(1-r)w)g(tx+(1-t)y,ru+(1-r)w)$$
$$dtdrdxdydudw$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy+\Xi_1L(a,b,c,d)+\Xi_2M(a,b,c,d)+\Xi_3N(a,b,c,d),$$

where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ are defined in Theorem 1.5, and

$$\Xi_1 = \frac{\Gamma(1+s_1)\Gamma(1+s_2)\big((1+s_1+s_2)^2\Gamma(1+s_1)\Gamma(1+s_2)+2(1+s_1)(1+s_2)\Gamma(2+s_1+s_2)\big)}{(1+s_1)^2(1+s_2)^2[\Gamma(2+s_1+s_2)]^2},$$
$$\Xi_2 = \frac{(2(1+s_2)^2+s_1^2(2+s_2)+s_1(4+5s_2+s_2^2))[\Gamma(1+s_1)]^2[\Gamma(1+s_2)]^2}{(1+s_1)^2(1+s_2)^2[\Gamma(2+s_1+s_2)]^2} + \frac{(1+s_1)(1+s_2)\Gamma(1+s_1)\Gamma(1+s_2)\Gamma(2+s_1+s_2)}{(1+s_1)^2(1+s_2)^2[\Gamma(2+s_1+s_2)]^2},$$
$$\Xi_3 = \frac{(1+s_1+s_2)(3+3s_1+3s_2+2s_1s_2)[\Gamma(1+s_1)]^2[\Gamma(1+s_2)]^2}{(1+s_1)^2(1+s_2)^2[\Gamma(2+s_1+s_2)]^2}.$$

Theorem 3.5. *Suppose that $f,g:\Delta\to[0,\infty)$ are (p_1,h_1)-(p_2,h_2)-convex and (p_1,k_1)-(p_2,k_2)-convex functions on the co-ordinates on Δ, respectively. Then one has the inequality:*

$$\begin{aligned}
&\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d\int_0^1\int_0^1 x^{p_1-1}u^{p_2-1}f\Big(\big[tx^{p_1}+(1-t)\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},\\
&\qquad\big[ru^{p_2}+(1-r)\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big)\\
&\quad\times g\Big(\big[tx^{p_1}+(1-t)\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},\big[ru^{p_2}+(1-r)\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big)dtdrdxdu\\
&\le \frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,u)g(x,u)dxdu\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(t)dt\\
&\quad+(\Lambda_1+\Lambda_3+\Lambda_5)M_1(a,b,c,d)+(\Lambda_1+\Lambda_4+\Lambda_5)M_2(a,b,c,d)\\
&\quad+(\Lambda_2+\Lambda_3+\Lambda_5)M_3(a,b,c,d)+(\Lambda_2+\Lambda_4+\Lambda_5)M_4(a,b,c,d),
\end{aligned}\tag{3.40}$$

where $M_1(a,b,c,d)$, $M_2(a,b,c,d)$, $M_3(a,b,c,d)$ and $M_4(a,b,c,d)$ are defined in Theorem 3.1, and

$$\begin{aligned}
\Lambda_1 &= 2\Big(2h_2(\tfrac{1}{2})k_2(\tfrac{1}{2})\int_0^1 h_2(t)k_2(t)dt+\big(h_2(\tfrac{1}{2})+k_2(\tfrac{1}{2})\big)\int_0^1 h_2(t)k_2(1-t)dt\Big)\\
&\quad\int_0^1 h_2(t)\int_0^1 k_2(t)dtdt\Big(\int_0^1 h_1(t)k_1(t)dt\Big)^2,\\
\Lambda_2 &= 2\Big(2h_2(\tfrac{1}{2})k_2(\tfrac{1}{2})\int_0^1 h_2(t)k_2(t)dt+\big(h_2(\tfrac{1}{2})+k_2(\tfrac{1}{2})\big)\int_0^1 h_2(t)k_2(1-t)dt\Big)\int_0^1 h_2(t)dt\\
&\quad\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_1(t)k_1(1-t)dt,\\
\Lambda_3 &= 2\Big(2h_1(\tfrac{1}{2})k_1(\tfrac{1}{2})\int_0^1 h_1(t)k_1(t)dt+\big(h_1(\tfrac{1}{2})+k_1(\tfrac{1}{2})\big)\int_0^1 h_1(t)k_1(1-t)dt\Big)\\
&\quad\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\Big(\int_0^1 h_2(t)k_2(t)dt\Big)^2,\\
\Lambda_4 &= 2\Big(2h_1(\tfrac{1}{2})k_1(\tfrac{1}{2})\int_0^1 h_1(t)k_1(t)dt+\big(h_1(\tfrac{1}{2})+k_1(\tfrac{1}{2})\big)\int_0^1 h_1(t)k_1(1-t)dt\Big)\\
&\quad\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dt\int_0^1 h_2(t)k_2(1-t)dt,
\end{aligned}$$

$$\Lambda_5 = 4\Big(4h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(t)dt$$
$$+2h_1(\frac{1}{2})k_1(\frac{1}{2})\big(h_2(\frac{1}{2})+k_2(\frac{1}{2})\big)\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt$$
$$+2h_2(\frac{1}{2})k_2(\frac{1}{2})\big(h_1(\frac{1}{2})+k_1(\frac{1}{2})\big)\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(t)dt$$
$$+\big(h_1(\frac{1}{2})k_2(\frac{1}{2})+h_2(\frac{1}{2})k_1(\frac{1}{2})+h_1(\frac{1}{2})h_2(\frac{1}{2})+k_1(\frac{1}{2})k_2(\frac{1}{2})\big)\int_0^1 h_1(t)k_1(1-t)dt$$
$$\int_0^1 h_2(t)k_2(1-t)dt\Big)\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt.$$

Proof. Since f is (p_1,h_1)-(p_2,h_2)-convex on the co-ordinates and g is (p_1,k_1)-(p_2,k_2)-convex on the coordinates on Δ, it follows that

$$f\Big(\big[tx^{p_1}+(1-t)\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},\big[ru^{p_2}+(1-r)\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big)$$
$$\le h_1(t)h_2(r)f(x,u)+h_1(t)h_2(1-r)f\big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\big)$$
$$+h_1(1-t)h_2(r)f\big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\big)+h_1(1-t)h_2(1-r)f\big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\big), \quad (3.41)$$

and

$$g\Big(\big[tx^{p_1}+(1-t)\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},\big[ru^{p_2}+(1-r)\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big)$$
$$\le k_1(t)k_2(r)f(x,u)+k_1(t)k_2(1-r)f\big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\big)$$
$$+k_1(1-t)k_2(r)f\big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\big)+k_1(1-t)k_2(1-r)f\big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\big), \quad (3.42)$$

for $t,r\in[0,1]$, $x\in[a,b]$ and $u\in[c,d]$. Because f and g are nonnegative, from (3.41) and (3.42), we get the inequality

$$f\Big(\big[tx^{p_1}+(1-t)\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},\big[ru^{p_2}+(1-r)\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big)$$
$$\times g\Big(\big[tx^{p_1}+(1-t)\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},\big[ru^{p_2}+(1-r)\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big)$$
$$\le\Big(h_1(t)h_2(r)f(x,u)+h_1(t)h_2(1-r)f\big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\big)+h_1(1-t)h_2(r)f\big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\big)$$
$$+h_1(1-t)h_2(1-r)f\big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\big)\Big)$$
$$\Big(k_1(t)k_2(r)f(x,u)+k_1(t)k_2(1-r)f\big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\big)$$
$$+k_1(1-t)k_2(r)f\big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\big)+k_1(1-t)k_2(1-r)f\big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\big)\Big)$$
$$=h_1(t)h_2(r)k_1(t)k_2(r)f(x,u)g(x,u)+h_1(t)h_2(1-r)k_1(t)k_2(1-r)f(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$g(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})+h_1(1-t)h_2(r)k_1(1-t)k_2(r)f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)$$

$$
\begin{aligned}
&+h_1(1-t)h_2(1-r)k_1(1-t)k_2(1-r)f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\\
&g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})+h_1(t)h_2(1-r)k_1(t)k_2(r)f(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(x,u)\\
&+h_1(t)h_2(r)k_1(t)k_2(1-r)f(x,u)g(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\\
&+h_1(1-t)h_2(1-r)k_1(1-t)k_2(r)f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)\\
&+h_1(1-t)h_2(r)k_1(1-t)k_2(1-r)f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\\
&+h_1(t)h_2(r)k_1(1-t)k_2(r)f(x,u)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)\\
&+h_1(1-t)h_2(r)k_1(t)k_2(r)f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)g(x,u)\\
&+h_1(1-t)h_2(1-r)k_1(t)k_2(1-r)f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\\
&+h_1(t)h_2(1-r)k_1(1-t)k_2(1-r)f(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\\
&+h_1(1-t)h_2(1-r)k_1(t)k_2(r)f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(x,u)\\
&+h_1(1-t)h_2(r)k_1(t)k_2(1-r)f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)g(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\\
&+h_1(t)h_2(1-r)k_1(1-t)k_2(r)f(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)\\
&+h_1(t)h_2(r)k_1(1-t)k_2(1-r)f(x,u)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}). \qquad (3.43)
\end{aligned}
$$

Multiplying both sides of (3.43) by $\frac{p_1p_2x^{p_1-1}u^{p_2-1}}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}$ and integrating over $[a,b]\times[c,d]\times[0,1]^2$, we have

$$
\begin{aligned}
&\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d\int_0^1\int_0^1 x^{p_1-1}u^{p_2-1}f\Big([tx^{p_1}+(1-t)\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},\\
&\qquad\qquad [ru^{p_2}+(1-r)\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\\
&\times g\Big([tx^{p_1}+(1-t)\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[ru^{p_2}+(1-r)\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)dtdrdxdu\\
&\le \frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,u)g(x,u)dxdu\int_0^1 h_1(t)k_1(t)dt\\
&\int_0^1 h_2(t)k_2(t)dt+\Big(\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}\ f\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)g\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)dx\\
&+\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\Big)g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\Big)du+f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},\\
&[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\Big)\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)g(x,u)dxdu\\
&+\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,u)g\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)dxdu\\
&+\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}\ g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\Big)du\ f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\\
&+\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}\ f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\Big)du\ g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\Big)\\
&\int_0^1 h_1(t)k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt\\
&+\Big(\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,u)g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\Big)dxdu\\
&+\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\Big)g(x,u)dxdu\\
&+\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}g\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)dx\ f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\\
&+\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)dx\\
&g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\Big)\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(t)dt\\
&+\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\Big(\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}g(x,u)dxduf\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\\
&+\int_a^b x^{p_1-1}g\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)dx\int_c^d u^{p_2-1}f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\Big)du\\
&+\int_a^b x^{p_1-1}f\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)dx\int_c^d u^{p_2-1}g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u\Big)du\\
&+\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,u)dxdu\ g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\Big)\\
&\int_0^1 h_1(t)k_1(1-t)dt\int_0^1 h_2(t)k_2(1-t)dt. \qquad (3.44)
\end{aligned}
$$

Applying Theorems 1.1 and 1.2 to 2nd-16th term of right hand side of the inequality (3.44), we have

$$
\begin{aligned}
&\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})dx\\
&\le\Big(f(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})+f(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\Big)\\
&\times\int_0^1 h_1(t)k_1(t)dt+\Big(f(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\\
&+f(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\Big)\int_0^1 h_1(t)k_1(1-t)dt
\end{aligned}
$$

$$\leq \Big([f(a,c)+f(a,d)][g(a,c)+g(a,d)]+[f(b,c)+f(b,d)][g(b,c)+g(b,d)]\Big)4h_2(\frac{1}{2})k_2(\frac{1}{2})$$
$$\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(t)dt$$
$$+\Big([f(a,c)+f(a,d)][g(b,c)+g(b,d)]+[f(a,c)+f(a,d)][g(b,c)+g(b,d)]\Big)$$
$$\times 4h_2(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_2(t)\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(1-t)dt$$
$$=(M_1(a,b,c,d)+M_2(a,b,c,d))4h_2(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_2(t)\int_0^1 k_2(t)dtdt\int_0^1 h_1(t)k_1(t)dt$$
$$+(M_3(a,b,c,d)+M_4(a,b,c,d))4h_2(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_2(t)\int_0^1 k_2(t)dtdt\int_0^1 h_1(t)k_1(1-t)dt, \tag{3.45}$$

$$\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)du$$
$$\leq \Big(f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},c)g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},c\Big)+f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},d\Big)g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},d\Big)\Big)$$
$$\times\int_0^1 h_2(t)k_2(t)dt+\Big(f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},c)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},d)$$
$$+f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},d)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},c)\Big)\int_0^1 h_2(t)k_2(1-t)dt$$
$$\leq \Big([f(a,c)+f(b,c)][g(a,c)+g(b,c)]+[f(a,d)+f(b,d)][g(a,d)+g(b,d)]\Big)4h_1(\frac{1}{2})k_1(\frac{1}{2})$$
$$\int_0^1 h_1(t)\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dt$$
$$+\Big(f(a,c)+f(b,c)][g(a,d)+g(b,d)]+[f(a,d)+f(b,d)][g(a,c)+g(b,c)]\Big)$$
$$\times 4h_1(\frac{1}{2})k_1(\frac{1}{2})\int_0^1 h_1(t)\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt$$
$$=(M_1(a,b,c,d)+M_3(a,b,c,d))4h_1(\frac{1}{2})k_1(\frac{1}{2})\int_0^1 h_1(t)\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dt$$
$$+(M_2(a,b,c,d)+M_4(a,b,c,d))4h_1(\frac{1}{2})k_1(\frac{1}{2})\int_0^1 h_1(t)\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt, \tag{3.46}$$

$$f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$\leq [f(a,c)+f(a,d)+f(b,c)+f(b,d)]4h_1(\frac{1}{2})h_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt$$
$$\times[g(a,c)+g(a,d)+g(b,c)+g(b,d)]4k_1(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt$$
$$=[M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times 16h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt, \tag{3.47}$$

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(x,u)dxdu$$
$$\le \frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}\Big(f(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(a,u)$$
$$+f(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})g(b,u)\Big)du\int_0^1 h_1(t)k_1(t)dt$$
$$+\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}\Big(f\Big(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)g(b,u)$$
$$+f\Big(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)g(a,u)\Big)du\int_0^1 h_1(t)k_1(1-t)dt$$
$$\le \Big([f(a,c)+f(a,d)]2h_2(\frac{1}{2})\int_0^1 h_2(t)dt[g(a,c)+g(a,d)]\int_0^1 k_2(t)dt$$
$$+[f(b,c)+f(b,d)]2h_2(\frac{1}{2})\int_0^1 h_2(t)dt[g(b,c)+g(b,d)]\int_0^1 k_2(t)dt\Big)du\int_0^1 h_1(t)k_1(t)dt$$
$$+\Big([f(a,c)+f(a,d)]2h_2(\frac{1}{2})\int_0^1 h_2(t)dt[g(b,c)+g(b,d)]\int_0^1 k_2(t)dt$$
$$+[f(b,c)+f(b,d)]2h_2(\frac{1}{2})\int_0^1 h_2(t)dt[g(a,c)+g(a,d)]\int_0^1 k_2(t)dt\Big)\int_0^1 h_1(t)k_1(1-t)dt$$
$$=(M_1(a,b,c,d)+M_2(a,b,c,d))2h_2(\frac{1}{2})\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(t)dt$$
$$+(M_3(a,b,c,d)+M_4(a,b,c,d))2h_2(\frac{1}{2})\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(1-t)dt, \tag{3.48}$$

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,u)g(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})dxdu$$
$$\le \frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}\Big(f(a,u)g(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$+f(b,u)g\Big(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\Big)du\int_0^1 h_1(t)k_1(t)dt$$
$$+\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}\Big(f(a,u)g(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$+f(b,u)g\Big(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)\Big)du\int_0^1 h_1(t)k_1(1-t)dt$$
$$\le \Big([f(a,c)+f(a,d)]\int_0^1 h_2(t)dt[g(a,c)+g(a,d)]2k_2(\frac{1}{2})\int_0^1 k_2(t)dt$$
$$+[f(b,c)+f(b,d)]\int_0^1 h_2(t)dt[g(b,c)+g(b,d)]2k_2(\frac{1}{2})\int_0^1 k_2(t)dt\Big)du\int_0^1 h_1(t)k_1(t)dt$$
$$+\Big([f(a,c)+f(a,d)]\int_0^1 h_2(t)dt[g(b,c)+g(b,d)]2k_2(\frac{1}{2})\int_0^1 k_2(t)dt$$
$$+[f(b,c)+f(b,d)]\int_0^1 h_2(t)dt[g(a,c)+g(a,d)]2k_2(\frac{1}{2})\int_0^1 k_2(t)dt\Big)\int_0^1 h_1(t)k_1(1-t)dt$$

$$= (M_1(a,b,c,d)+M_2(a,b,c,d))2k_2(\frac{1}{2})\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(t)dt$$
$$+(M_3(a,b,c,d)+M_4(a,b,c,d))2k_2(\frac{1}{2})\int_0^1 h_2(t)dt\int_0^1 k_2(t)dt\int_0^1 h_1(t)k_1(1-t)dt, \quad (3.49)$$

$$\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d u^{p_2-1}g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)du f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)$$
$$\le \Big(g\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},c\Big)+g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},d)\Big)\int_0^1 k_2(t)dt\, f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)$$
$$\le [g(a,c)+g(b,c)+g(a,d)+g(b,d)]2k_1(\frac{1}{2})\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt$$
$$\times[f(a,c)+f(b,c)+f(a,d)+f(b,d)]4h_1(\frac{1}{2})h_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt$$
$$= [M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times 8h_1(\frac{1}{2})h_2(\frac{1}{2})k_1(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt, \quad (3.50)$$

$$\frac{p_2}{d^{p_2}-c^{p_2}}\int_a^b x^{p_1-1}f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)dx g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$\le [M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times 8h_1(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt, \quad (3.51)$$

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,u)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)dxdu$$
$$\le \frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}\Big(f(x,c)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},c)$$
$$+f(x,d)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},d)\Big)dx\int_0^1 h_2(t)k_2(t)dt$$
$$+\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}\Big(f(x,c)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},d)$$
$$+f(x,d)g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},c)\Big)dx\int_0^1 h_2(t)k_2(1-t)dt$$
$$\le \Big([f(a,c)+f(b,c)]\int_0^1 h_1(t)dt[g(a,c)+g(b,c)]2k_1(\frac{1}{2})\int_0^1 k_1(t)dt$$
$$+[f(a,d)+f(b,d)]\int_0^1 h_1(t)dt[g(a,d)+g(b,d)]2k_1(\frac{1}{2})\int_0^1 k_1(t)dt\Big)dx\int_0^1 h_2(t)k_2(t)dt$$
$$+\Big([f(a,c)+f(b,c)]\int_0^1 h_1(t)dt[g(a,d)+g(b,d)]2k_1(\frac{1}{2})\int_0^1 k_1(t)dt$$
$$+[f(a,d)+f(b,d)]\int_0^1 h_1(t)dt[g(a,c)+g(b,c)]2k_1(\frac{1}{2})\int_0^1 k_1(t)dt\Big)dx\int_0^1 h_2(t)k_2(1-t)dt$$
$$= (M_1(a,b,c,d)+M_3(a,b,c,d))2k_1(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dt$$

$$+(M_2(a,b,c,d)+M_4(a,b,c,d)2k_1(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt, \quad (3.52)$$

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)g(x,u)dxdu$$
$$\leq (M_1(a,b,c,d)+M_3(a,b,c,d))2h_1(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(t)dt$$
$$+(M_2(a,b,c,d)+M_4(a,b,c,d)2h_1(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 k_1(t)dt\int_0^1 h_2(t)k_2(1-t)dt, \quad (3.53)$$

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}g(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})dx\, f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$\leq \Big(g(a,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})+g(b,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})\Big)\int_0^1 k_1(t)dtf([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$\leq [g(a,c)+g(b,c)+g(a,d)+g(b,d)]2k_2(\frac{1}{2})\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt$$
$$\times[f(a,c)+f(b,c)+f(a,d)+f(b,d)]4h_1(\frac{1}{2})h_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt$$
$$=[M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times 8h_1(\frac{1}{2})h_2(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt, \quad (3.54)$$

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b x^{p_1-1}f(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})dxg([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$\leq [M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times 8h_2(\frac{1}{2})k_1(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt, \quad (3.55)$$

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b x^{p_1-1}g(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})dx\int_c^d u^{p_2-1}f([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)du$$
$$\leq [M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times 4h_1(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt, \quad (3.56)$$

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b x^{p_1-1}f(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})dx\int_c^d u^{p_2-1}g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},u)du$$
$$\leq [M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times 4h_2(\frac{1}{2})k_1(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt, \quad (3.57)$$

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}g(x,u)dxduf([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$\leq [M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]$$
$$\times 4h_1(\frac{1}{2})h_2(\frac{1}{2})\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt, \quad (3.58)$$

$$\frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d x^{p_1-1}u^{p_2-1}f(x,u)dxdu\, g([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}})$$
$$\le [M_1(a,b,c,d)+M_2(a,b,c,d)+M_3(a,b,c,d)+M_4(a,b,c,d)]4k_1(\frac{1}{2})k_2(\frac{1}{2})\int_0^1 h_1(t)dt$$
$$\int_0^1 h_2(t)dt\int_0^1 k_1(t)dt\int_0^1 k_2(t)dt. \tag{3.59}$$

Using the inequalities (3.45)-(3.59) in (3.44), we get the desired result (3.40). □

Remark 7. Suppose that $f,g:\Delta\to[0,\infty)$ are convex functions on the co-ordinates on Δ. Then one has the inequality:

$$\frac{9}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1 f(tx+(1-t)\frac{a+b}{2},ru+(1-r)\frac{c+d}{2})$$
$$\times g(tx+(1-t)\frac{a+b}{2},ru+(1-r)\frac{c+d}{2})dtdrdxdu$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy+\frac{37}{48}L(a,b,c,d)+\frac{11}{16}M(a,b,c,d)+\frac{29}{48}N(a,b,c,d),$$

where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ are defined in Theorem 1.5.

Remark 8. Suppose that $f,g:\Delta\to[0,\infty)$ are s-convex functions on the co-ordinates on Δ. Then one has the inequality:

$$\frac{(1+2s)^2}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1 f(tx+(1-t)\frac{a+b}{2},ru+(1-r)\frac{c+d}{2})$$
$$\times g(tx+(1-t)\frac{a+b}{2},ru+(1-r)\frac{c+d}{2})dtdrdxdu$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy+\Pi_1L(a,b,c,d)+\Pi_2M(a,b,c,d)+\Pi_3N(a,b,c,d),$$

where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ are defined in Theorem 1.5, and

$$\Pi_1=\frac{2^{3-2s}(2^s[\Gamma(1+s)]^2+\Gamma(1+2s))}{(1+s)^2(1+2s)\Gamma(1+2s)}+\frac{2^{4-4s}(2^{1+s}[\Gamma(1+s)]^2+(1+4^s)\Gamma(1+2s))}{(1+s)^4\Gamma(1+2s)},$$
$$\Pi_2=\frac{2^{2-2s}(1+[\Gamma(1+s)]^2)(2^s[\Gamma(1+s)]^2+\Gamma(1+2s))}{(1+s)^2(1+2s)\Gamma(1+2s)}+\frac{2^{4-4s}(2^{1+s}[\Gamma(1+s)]^2+(1+4^s)\Gamma(1+2s))}{(1+s)^4\Gamma(1+2s)},$$
$$\Pi_3=\frac{2^{3-2s}[\Gamma(1+s)]^2(2^s[\Gamma(1+s)]^2+\Gamma(1+2s))}{(1+s)^2(1+2s)[\Gamma(1+2s)]^2}+\frac{2^{4-4s}(2^{1+s}[\Gamma(1+s)]^2+(1+4^s)\Gamma(1+2s))}{(1+s)^4\Gamma(1+2s)}.$$

Remark 9. Suppose that $f,g:\Delta\to[0,\infty)$ are s_1-convex and s_2-convex functions on the co-ordinates on Δ, respectively. Then one has the inequality:

$$\frac{(1+s_1+s_2)^2}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1 f(tx+(1-t)\frac{a+b}{2},ru+(1-r)\frac{c+d}{2})$$
$$\times g(tx+(1-t)\frac{a+b}{2},ru+(1-r)\frac{c+d}{2})dtdrdxdu$$

$$\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)\,dx\,dy+\Sigma_1 L(a,b,c,d)+\Sigma_2 M(a,b,c,d)+\Sigma_3 N(a,b,c,d),$$

where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ are defined in Theorem 1.5, and

$$\Sigma_1=\frac{2^{3-s_1-s_2}\Gamma(1+s_1+s_2)+(2^{2-s_1}+2^{2-s_2})\Gamma(1+s_1)\Gamma(1+s_1)}{(1+s_1)(1+s_2)(1+s_1+s_2)\Gamma(1+s_1+s_2)}$$
$$+\frac{(2^{4-2s_1-2s_2}+2^{2-2s_1}+2^{3-s_1-s_2}+2^{2-2s_2})\Gamma(1+s_1+s_2)+2^{3-s_1-s_2}(2^{-s_1}+2^{-s_2})\Gamma(1+s_1)\Gamma(1+s_1)}{(1+s_1)^2(1+s_2)^2\Gamma(1+s_1+s_2)},$$
$$\Sigma_2=\frac{(1+\Gamma(1+s_1)\Gamma(1+s_2))\big(2^{2-s_1-s_2}\Gamma(1+s_1+s_2)+(2^{1-s_1}+2^{1-s_2})\Gamma(1+s_1)\Gamma(1+s_1)\big)}{(1+s_1)(1+s_2)(1+s_1+s_2)\Gamma(1+s_1+s_2)}$$
$$+\frac{(2^{4-2s_1-2s_2}+2^{2-2s_1}+2^{3-s_1-s_2}+2^{2-2s_2})\Gamma(1+s_1+s_2)+2^{3-s_1-s_2}(2^{-s_1}+2^{-s_2})\Gamma(1+s_1)\Gamma(1+s_1)}{(1+s_1)^2(1+s_2)^2\Gamma(1+s_1+s_2)},$$
$$\Sigma_3=\frac{\Gamma(1+s_1)\Gamma(1+s_2)\big(2^{3-s_1-s_2}\Gamma(1+s_1+s_2)+(2^{2-s_1}+2^{2-s_2})\Gamma(1+s_1)\Gamma(1+s_1)\big)}{(1+s_1)(1+s_2)(1+s_1+s_2)\Gamma(1+s_1+s_2)}$$
$$+\frac{(2^{4-2s_1-2s_2}+2^{2-2s_1}+2^{3-s_1-s_2}+2^{2-2s_2})\Gamma(1+s_1+s_2)+2^{3-s_1-s_2}(2^{-s_1}+2^{-s_2})\Gamma(1+s_1)\Gamma(1+s_1)}{(1+s_1)^2(1+s_2)^2\Gamma(1+s_1+s_2)}.$$

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