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ASYMPTOTICALLY POWER SOLUTIONS OF HIGHER ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Necessary and sufficient conditions are derived for the existence of asymptotically polynomial solutions of a class of neutral functional differential equations.

In [1, 2], n-th order neutral functional differential equations of the form

$$(x(t)+c(t)x(t-d))^{(n)}+f(t,x(g_1(t)),x(g_2(t)),\cdots,x(g_m(t)))=0, t\geq t_0, \tag{1}$$

are investigated and necessary and sufficient conditions for the existence of nonoscillatory solutions are derived. We observed that in these papers, the condition that $c(t)\geq -1$ is assumed. The question then arises as to whether existence criteria can be established in case $c(t)<-1$.

In this note, we will show that under the basic assumptions that f is either superlinear or sublinear and that $\lim_{t\to\infty} c(t)=c_0<-1$, necessary and sufficient conditions can be found for the existence of "asymptotically power" solutions.

More precisely, let $d>0$, $h>0, m\in\{1,2,\ldots\}$ and $n\in\{2,3,\ldots\}$. Let $c\in C([t_0,+\infty),R)$ and $\lim_{t\to+\infty} c(t)=c_0$. We assume that $g_1,g_2,\ldots,g_m\in C([t_0,+\infty),R)$ and satisfy $g_1(t),\ldots,g_m(t)\geq t-h$ for some constant $h>0$, $f(t,x_1,x_2,\ldots,x_m)\in C([t_0,+\infty)\times R^m,R)$, and,

$$x_1 f(t,x_1,x_2,\ldots,x_m)>0,\ t\geq t_0,$$

if $x_1x_i>0$ for $i=1,2,\ldots,m$.

A function $f(t,x_1,x_2,\ldots,x_m)$ is said to be superlinear if there exist continuous functions $p_i(t)\geq 0$ for $i=1,2,\ldots,m$, satisfying $\sum_{i=0}^m p_i(t)>0$ and

$$\frac{f(t,y_1,y_2,\ldots,y_m)}{\sum_{i=0}^m p_i(t)y_i}\geq\frac{f(t,x_1,x_2,\ldots,x_m)}{\sum_{i=0}^m p_i(t)x_i} \tag{2}$$

when $y_i\geq x_i>0$ or $y_i\leq x_i<0$ for $i=1,2,\ldots,m$. If the inequality $\geq$ in (2) is changed into $\leq$, then the function $f(t,x_1,x_2,\ldots,x_m)$ is said to be sublinear.

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The following is shown in [1] which can also be shown easily.

Lemma 1. *Suppose that* $0 < a \le x_i \le b$ *for* $i = 1, 2, \ldots, m$. *If* f *is superlinear, then*

$$f(t, a, \ldots, a) \le f(t, x_1, \ldots, x_m) \le f(t, b, \ldots, b),\ t \ge t_0.$$

If f *is sublinear, then*

$$\frac{a}{b} f(t, b, \ldots, b) \le f(t, x_1, \ldots, x_m) \le \frac{b}{a} f(t, a, \ldots, a),\ t \ge t_0.$$

Lemma 2. *Suppose that* $\lim_{t\to+\infty} c(t) = c_0 \ne \pm 1$ *and that* $x(t)/t^i$ *is bounded and eventually positive or eventually negative, where* i *is a nonnegative integer. Let* $z(t) = x(t) + c(t)x(t-d)$. *If* $\lim_{t\to+\infty} z(t)/t^i = b$ *exists, then* $\lim_{t\to+\infty} x(t)/t^i = b/(1+c_0)$.

The proof is similar to that of Lemma 2 in [1] and is ommited.

Lemma 3.(Krasnoselskii [1]) *Suppose* B *is a Banach space and* X *is a bounded, convex and closed subset of* B. *Let* $U, S : X \to B$ *satisfy the following conditions:*

(i) $Ux + Sy$ *for any* $x, y \in X$,
(ii) U *is a contraction mapping, and*
(iii) S *is completely continuous.*

Then $U + S$ *has a fixed point in* X.

As usual, a solution $x(t)$ of (1) is said to be nonoscillatory if it eventually positive or eventually negative. A nonoscillatory solution $x(t)$ of (1) is said to be asymptotically constant or belong to $T_0(a)$ if

$$\lim_{t\to+\infty} x(t) = a \ne 0.$$

Two natural extensions of the concept of an asymptotically constant solution can be stated as follows. A nonoscillatory solution $x(t)$ of (1) is said to belong to $T_r(\infty, a)$, where $r \in \{1, 2, \ldots, n-1\}$, if

$$\limsup_{t\to+\infty} \frac{|x(t)|}{t^{r-1}} = +\infty \text{ and } \limsup_{t\to+\infty} \frac{|x(t)|}{t^r} = a \ne 0,$$

and said to belong to $T_r(a, 0)$, where $r \in \{1, 2, \ldots, n-1\}$, if

$$\limsup_{t\to+\infty} \frac{|x(t)|}{t^{r-1}} = a \ne 0 \text{ and } \limsup_{t\to+\infty} \frac{|x(t)|}{t^r} = 0.$$

We will be interested in finding necessary and sufficient conditions for the existence of solutions in $T_0(a)$, $T_r(\infty, a)$ and $T_r(a, 0)$.

Theorem 1. *Suppose* $\lim_{t\to+\infty} c(t) = c_0 < -1$ *and* f *is either superlinear or sublinear. Then (1) has a nonoscillatory solution* $x(t) \in T_r(\infty, a)$ *if, and only if, there is some* $K \ne 0$ *such that*

$$\int_{t_0}^{+\infty} s^{n-r-1} |f(s, Kg_1^r(s), Kg_2^r(s), \ldots, Kg_m^r(s))| ds < +\infty. \tag{3}$$

If $r = 0$, $T_r(\infty, a)$ *can be replaced by* $T_0(a)$ *in the above statement.*

Proof. Let

$$G_x(t) = f(t, x(g_1(t)), x(g_2(t)), \dots, x(g_m(t))), \tag{4}$$

and

$$F_r(t) = f(t, Kg_1^r(t), Kg_2^r(t), \dots, Kg_m^r(t)). \tag{5}$$

Since

$$\int_\alpha^x (x-y)^n g(y)dy = n! \int_\alpha^x dy_n \int_\alpha^{y_n} dy_{n-1} \cdots \int_\alpha^{y_2} dy_1 \int_\alpha^{y_1} g(y)dy,$$

(3) is equivalent to

$$\int_{t_0}^{+\infty} \int_{s_{n-r-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} |F_r(s)| ds ds_1 \cdots ds_{n-r-1} < +\infty. \tag{6}$$

Let $x(t)$ be an eventually positive solution of (1) in $T_r(\infty, a)$. Then without loss of generality, we may suppose there exists $T > t_0$ such that $x(t) > 0$, $x(t-d) > 0$ and $x(g_i(t)) > 0$ for $t \geq T$ and $i = 1, 2, \dots, m$. Let

$$z(t) = x(t) + c(t)x(t-d).$$

Then, by (1),we have

$$z^{(n)}(t) = -f(t, x(g_1(t)), x(g_2(t)), \dots, x(g_m(t))). \tag{7}$$

In view of (1), we have $z^{(n)}(t) < 0$ for $t \geq T$. Therefore $z^{(i)}(t)$ is eventually monotonic for all $i \in \{0, 1, 2, \dots, n-1\}$. Since $\lim_{t\to+\infty} \frac{x(t)}{t^r} = a > 0$, there exists $T_1 \geq T$ such that

$$\frac{1}{2}at^r \leq x(t) \leq \frac{3}{2}at^r, t \geq T_1. \tag{8}$$

Noticing $\lim_{t\to+\infty} \frac{z(t)}{t^r} = (1+c_0)a$, we have

$$\lim_{t\to+\infty} z^{(r)}(t) = (1+c_0)ar!. \tag{9}$$

Invoking the monotonicity of $z^{(i)}(t)$ and (9), we have

$$\lim_{t\to+\infty} z^{(i)}(t) = 0, \ i = r+1, r+2, \dots, n-1. \tag{10}$$

After integrating (7) $n-r-1$ times, we obtain

$$z^{(r+1)}(t) = (-1)^{n-r} \int_t^{+\infty} \int_{s_{n-r-2}}^{+\infty} \cdots \int_{s_1}^{+\infty} G_x(s) ds ds_1 \cdots ds_{n-r-2}, \ t \geq T_1.$$

Then integrating the above formula from T_1 to t, we obtain

$$z^{(r)}(t) = z^{(r)}(T) + (-1)^{n-r} \int_{T_1}^t \int_{s_{n-r-1}}^{+\infty} \int_{s_{n-r-2}}^{+\infty} \dots \int_{s_1}^{+\infty} G_x(s) ds ds_1 \dots ds_{n-r-1}.$$

In view of (9), we have

$$\int_{T_1}^{+\infty}\int_{s_{n-r-1}}^{+\infty}\int_{s_{n-r-2}}^{+\infty}\cdots\int_{s_1}^{+\infty} G_x(s)dsds_1\ldots ds_{n-r-1} < +\infty. \tag{11}$$

In view of Lemma 1 and (8), we see that $F_r(t) \le G_x(t)$ where we set $K = a/2$ if f is superlinear, and $F_r(t) \le 3G_x(t)$ where we set $K = 3a/2$ if f is sublinear. In view of (11), we see that (7) holds when $x(t)$ is eventually positive.

The case that $x(t)$ is eventually negative can be proved in a similar manner.

Conversely, suppose that $K > 0$. Let $e = K/2$ if f is superlinear and $e = K$ if f is sublinear. Set $R(t) = t^r$ or $R(t) \equiv 1$ when $r = 0$. Take c_1 and c_2 so that $(-8c_0 - 1)/7 > c_2 > |c_0| > c_1 > 1$. Then $c_0 < -(7c_2 + 1)/8$. Since

$$\lim_{t\to+\infty}\frac{7-7c_2-8c(t+d)}{8} = \frac{7-7c_2-8c_0}{8} > 1$$

and

$$\lim_{t\to+\infty}(c(t)+c_2) = c_0 + c_2 < \frac{1}{8}(c_2-1),$$

there exists a sufficiently large $T > t_0 + h + d$ such that when $t \ge T$, we have

$$\frac{1}{|c(t+d)|}\frac{R^2(t+d)}{R^2(t-d-h)} \le \frac{1}{c_1}, \tag{12}$$

$$|c(t)| \ge c_1, |c(t)| \le c_2, \tag{13}$$

$$c(t) + c_2 \le \frac{1}{8}(c_2 - 1), \tag{14}$$

$$\frac{R(t+d)}{R(t)} \le \frac{7-7c_2-8c(t+d)}{8}, \tag{15}$$

and

$$\int_{T}^{+\infty}\int_{s_{n-r-1}}^{+\infty}\int_{s_{n-r-2}}^{+\infty}\cdots\int_{s_1}^{+\infty} F_r(s)dsds_1\ldots ds_{n-r-1} < \frac{c_2-1}{8}e. \tag{16}$$

Take $\bar{T} = T - d - h$ and the linear space

$$C_R[\bar{T},+\infty) = \left\{x \in C([\bar{T},+\infty),R) : \sup_{t\ge\bar{T}}\frac{|x(t)|}{R^2(t)} < +\infty\right\}$$

with norm $||x||_R = \sup_{t\ge\bar{T}}\frac{|x(t)|}{R^2(t)}$. Then $C_R[\bar{T},+\infty)$ is a Banach space. Set

$$X = \{x \in [\bar{T},+\infty) : eR(t) \le x(t) \le 2eR(t)\}.$$

Then it is obvious that X is a bounded convex and closed subset of $C_R[\bar{T},+\infty)$ and for any $x \in X$ and $t \ge \bar{T} + h$,

$$G_x(t) \le 2F_r(t). \tag{17}$$

Define two operators on X as follows:

$$(Ux)(t) = \begin{cases} \dfrac{3eR(t)}{2c(T+d)} - \dfrac{1}{c(T+d)}\dfrac{x(T+d)}{R(T)}R(t) & \bar{T} \leq t < T \\ \dfrac{3eR(t)}{2c(t+d)} - \dfrac{1}{c(t+d)}x(t+d) & t \geq T \end{cases},$$

and

$$(Sx)(t) = \begin{cases} -\dfrac{3c_2eR(t)}{2c(T+d)} & \bar{T} \leq t < T \\ -\dfrac{3c_2eR(t)}{2c(t+d)} + \dfrac{(-1)^{n-r-1}}{c(t+d)}H(t) & t \geq T \end{cases}$$

where

$$H(t) = \int_{T+d}^{t+d} \int_{T}^{s_{n-1}} \cdots \int_{T}^{s_{n-r+1}} \int_{s_{n-r}}^{+\infty} \cdots \int_{s_1}^{+\infty} G_x(s)dsds_1 \ldots ds_{n-1}.$$

We will show that the operator U and S satisfy the conditions of the Krasnoselskii fixed point theorem.

(i) First we assert that $Ux + Sy \in X$ for any $x, y \in X$. Indeed, for $t \in [\bar{T}, T)$, in view of (13) and (15), we have

$$\begin{aligned} (Ux)(t) + (Sy)(t) &= \left(\frac{3(1-c_2)}{2c(T+d)}eR(T) - \frac{x(T+d)}{c(T+d)} \right) \frac{R(t)}{R(T)} \\ &\geq \left(\frac{3(1-c_2)}{2c(T+d)}eR(T) - \frac{eR(T+d)}{c(T+d)} \right) \frac{R(t)}{R(T)} \\ &\geq eR(t), \end{aligned}$$

and

$$\begin{aligned} (Ux)(t) + (Sy)(t) &\leq \left(\frac{3(1-c_2)}{2c(T+d)}eR(T) - \frac{2eR(T+d)}{c(T+d)} \right) \frac{R(t)}{R(T)} \\ &= \left(\frac{3(1-c_2)}{2c(T+d)} - \frac{2}{c(T+d)}\frac{R(T+d)}{R(T)} \right) eR(t) \\ &\leq \left(\frac{3(1-c_2)}{2c(T+d)} - \frac{1}{c(T+d)}\frac{7-7c_2-8c(T+d)}{4} \right) \frac{R(t)}{R(T)} \\ &\leq 2eR(t). \end{aligned}$$

When $t \in [T, +\infty)$, in view of (16) and (17), we have

$$\int_{T}^{t} \int_{T}^{s_{n-1}} \cdots \int_{T}^{s_{n-r+1}} \int_{s_{n-r}}^{+\infty} \cdots \int_{s_1}^{+\infty} G_x(s)dsds_1 \ldots ds_{n-1} \leq \frac{(c_2-1)eR(t)}{4}. \tag{18}$$

Hence,

$$\begin{aligned}(Ux)(t)+(Sy)(t) &\geq \frac{3(1-c_2)}{2c(t+d)}eR(t)-\frac{x(t+d)}{c(t+d)}+\frac{(c_2-1)eR(t)}{4c(t+d)}\\ &\geq \left(\frac{3(1-c_2)}{2c(t+d)}-\frac{1}{c(t+d)}+\frac{c_2-1}{4c(t+d)}\right)eR(t)\\ &\geq eR(t).\end{aligned}$$

Again, in view of (15) and (18), we have

$$\begin{aligned}&(Ux)(t)+(Sy)(t)\\ &\leq \frac{3(1-c_2)}{2c(t+d)}eR(t)-\frac{x(t+d)}{c(t+d)}-\frac{(c_2-1)eR(t)}{4c(t+d)}\\ &= \frac{3(1-c_2)}{2c(t+d)}eR(t)-\frac{2e}{c(t+d)}\frac{R(t+d)}{R(t)}R(t)-\frac{(c_2-1)eR(t)}{4c(t+d)}\\ &\leq \left(\frac{3(1-c_2)}{2c(t+d)}-\frac{7-7c_2-8c(t+d)}{4c(t+d)}-\frac{(c_2-1)}{4c(t+d)}\right)eR(t)\\ &\leq 2eR(t).\end{aligned}$$

That is, $Ux+Sy\in X$.

(ii) In view of (12), U is a contraction mapping since it is easy to see that

$$\frac{1}{R^2(t)}|(Ux)(t)-(Uy)(t)|\leq \frac{1}{c_1}\sup_{t\geq \bar{T}}\frac{|x(t)-y(t)|}{R^2(t)}$$

for any $x,y\in X$.

(iii) The operator S is a completely continuous mapping. Indeed, we first note that (13) implies $-c_2/c(t)\geq 1$ and (14) implies $-c_2/c(t)\leq 8/7$. Hence when $t\in[\bar{T},T)$, $(Sx)(t)\geq 3eR(t)/2$ and $(Sx)(t)\leq(3/2)(8/7)eR(t)\leq 2eR(t)$. For $t\in[T,+\infty)$,

$$\begin{aligned}(Sx)(t) &\geq -\frac{3c_2}{2c(t+d)}eR(t)+\frac{(c_2-1)eR(t)}{4c(t+d)}\\ &\geq \frac{eR(t)}{4c(t+d)}(-1-5c_2)\geq eR(t),\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) &\leq -\frac{3c_2}{2c(t+d)}eR(t)-\frac{(c_2-1)eR(t)}{4c(t+d)}\\ &\leq \frac{(-7c_2+1)eR(t)}{4c(t+d)}\leq 2eR(t).\end{aligned}$$

Therefore the operator S maps X into X. The fact that S is continuous and SX is relatively compact can be proved in a manner similar to that in [1] and is omitted.

By the Krasnoselskii fixed point theorem, there then exists $x \in X$ such that $(Ux)(t)+(Sx)(t) = x(t)$. Therefore,

$$x(t) = \frac{3(1-c_2)}{2c(t+d)}eR(t) - \frac{x(t+d)}{c(t+d)} + \frac{(-1)^{n-r-1}}{c(t+d)}H(t),\ t \geq T.$$

It is now clear that $x(t)$ is a nonoscillatory solution of (1) and satisfies

$$\lim_{t\to+\infty} \frac{z(t)}{t^r} = \frac{3}{2}(1-c_2)e.$$

By Lemma 2, we have

$$\lim_{t\to+\infty} \frac{x(t)}{t^r} = \frac{3(1-c_2)e}{2(1+c_0)} \text{ and } \lim_{t\to+\infty} \frac{x(t)}{t^{r-1}} = +\infty.$$

So $x(t) \in T_r(\infty, a)$. In a similar way, we can prove the other case where $K < 0$. Our proof is complete.

Theorem 2. *Suppose* $\lim_{t\to\infty} c(t) = c_0 < -1$ *and* f *is superlinear or sublinear. Then* $x(t) \in T_r(a, 0)$ *is a nonoscillatory solution of (1) if, and only if, there is some* $K \neq 0$ *such that*

$$\int_{t_0}^{+\infty} s^{n-r}|f(s, Kg_1^{r-1}(s), Kg_2^{r-1}(s), \ldots, Kg_m^{r-1}(s))|ds < +\infty.$$

The proof is similar to that of Theorem 1, except that the operator S is taken as follows:

$$(Sx)(t) = \begin{cases} -\dfrac{3c_2eR(t)}{2c(T+d)} & \bar{T} \leq t < T \\ -\dfrac{3c_2eR(t)}{2c(t+d)} + \dfrac{(-1)^{n-r}}{c(t+d)}H(t) & t \geq T \end{cases},$$

where $H(t) = \int_{T+d}^{t+d}\int_T^{s_{n-1}} \cdots \int_T^{s_{n-r+2}} \int_{s_{n-r+1}}^{+\infty} \cdots \int_{s_1}^{+\infty} G_x(s)dsds_1 \ldots ds_{n-1}$ and $R(t) = t^{r-1}$ if $r > 1$ and $R(t) \equiv 1$ if $r = 1$.

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