TAMKANG JOURNAL OF MATHEMATICS
Volume 47, Number 3, 339-349, September 2016
doi:10.5556/j.tkjm.47.2016.1968
This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

ON CERTAIN INTEGRAL FORMULAS INVOLVING THE PRODUCT OF BESSEL FUNCTION AND JACOBI POLYNOMIAL

NABI ULLAH KHAN, MOHD GHAYASUDDIN AND TALHA USMAN

Abstract. In the present paper, we establish some interesting integrals involving the product of Bessel function of the first kind with Jacobi polynomial, which are expressed in terms of Kampé de Fériet and Srivastava and Daoust functions. Some other integrals involving the product of Bessel (sine and cosine) function with ultraspherical polynomial, Gegenbauer polynomial, Tchebicheff polynomial, and Legendre polynomial are also established as special cases of our main results. Further, we derive an interesting connection between Kampé de Fériet and Srivastava and Daoust functions.

1. Introduction

Some very interesting integrals associated with a variety of special functions have been established by many authors (see, [9], [10], [13], [14]). Very recently, Choi and Agarwal [11] gave some interesting unified integrals involving the Bessel function of the first kind, which are expressed in terms of generalized (Wright) hypergeometric functions.

Motivated by the above-mentioned works, in the present paper, we establish a new class of integral formulas involving the product of Bessel function $J_\nu(z)$ with Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$, which are expressed in terms of Kampé de Fériet and Srivastava and Daoust functions. Some other integrals involving the product of Bessel (sine and cosine) function with ultraspherical polynomial, Gegenbauer polynomial, Tchebicheff polynomial, and Legendre polynomial are also established as special cases of our main results. Next, we establish a very interesting connection between Kampé de Fériet function and Srivastava and Daoust function.

The Bessel function $J_\nu(z)$ of the first kind and order ν is defined by (see [4], [7])

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{1.1}$$

Received June 29, 2015, accepted November 23, 2015.
2010 *Mathematics Subject Classification*. 33C10, 33C45, 33C65, 33C70.
Key words and phrases. Bessel function, Jacobi polynomial, Kampé de Fériet function, Srivastava and Daoust function and integrals.
Corresponding author: M. Ghayasuddin.

It is well known that

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\sin z \tag{1.2}$$

and

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\cos z. \tag{1.3}$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ is defined by (see [4], [7])

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{array}{ccc} -n, & 1+\alpha+\beta+n; & \frac{1-z}{2} \\ & 1+\alpha; & \end{array}\right], \tag{1.4}$$

or equivalently,

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(1+\alpha+\beta)_{n+k}}{k!\,(n-k)!\,(1+\alpha)_k(1+\alpha+\beta)_n}\left(\frac{z-1}{2}\right)^k. \tag{1.5}$$

From (1.4) and (1.5) it follows that $P_n^{(\alpha,\beta)}(z)$ is a polynomial of degree n and that

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}. \tag{1.6}$$

For $\beta=\alpha$, the polynomial $P_n^{(\alpha,\alpha)}(z)$ is called the ultraspherical polynomial and

$$P_n^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(z) = \frac{(\mu+\frac{1}{2})_n}{(2\mu)_n}\,C_n^{\mu}(z), \tag{1.7}$$

where $C_n^{\mu}(z)$ is the Gegenbauer polynomial (see [4], [7]),

$$P_n^{(-\frac{1}{2},-\frac{1}{2})}(z) = \frac{(\frac{1}{2})_n}{n!}\,T_n(z) \tag{1.8}$$

and

$$P_n^{(\frac{1}{2},\frac{1}{2})}(z) = \frac{(\frac{3}{2})_n}{(n+1)!}\,U_n(z), \tag{1.9}$$

where $T_n(z)$ and $U_n(z)$ are the Tchebicheff polynomials of the first and second kind, (see [4], [7]) and

$$P_n^{(0,0)}(z) = P_n(z), \tag{1.10}$$

where $P_n(z)$ is the Legendre polynomial (see [4], [7]).

In 1921, the four Appell functions were unified and generalized by Kampé de Fériet, who defined a general hypergeometric function of two variables. The notation introduced by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy. We recall here the definition of a more general double hypergeometric function in a slightly modified notation (see [7]):

$$F^{p:\ q;\ k}_{l:\ m;\ n}\left[\begin{array}{l}(a_p):\ (b_q);\ (c_k);\\ (\alpha_l):\ (\beta_m);\ (\gamma_n);\end{array} x, y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.11}$$

where, for convergence,

(i) $p+q<l+m+1,\ p+k<l+n+1,\ |x|<\infty,\ |y|<\infty$, or

(ii) $p+q=l+m+1,\ p+k=l+n+1$, and

$$\begin{cases} |x|^{1/(p-l)}+|y|^{1/(p-l)}<1, & \text{if } p>l,\\ \max\{|x|,\ |y|\}<1, & \text{if } p\le l.\end{cases}$$

Srivastava and Daoust [7] multivariable hypergeometric function is given as follows:

$$F^{p:\ q_1;\dots.\ q_r}_{l:\ m_1;\dots.\ m_r}\left[\begin{array}{l}(a_j:\alpha^1_j,..\alpha^{(r)}_j)_{1,p}:\ (c^1_j,r^1_j)_{1,q_1};\dots.;(c^{(r)}_j,r^{(r)}_j)_{1,q_r};\\ (b_j:\beta^1_j,..\beta^{(r)}_j)_{1,l}:\ (d^1_j,\delta^1_j)_{1,m_1};\dots.;(d^{(r)}_j,\delta^{(r)}_j)_{1,m_r};\end{array} x_1,x_2,\dots.x_r\right]$$

$$= \sum_{n_1,n_2,\dots n_r=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n_1\alpha^1_j+\dots n_r\alpha^{(r)}_j}\prod_{j=1}^{q_1}(c^1_j)_{n_1r^1_j}\dots\dots\prod_{j=1}^{q_r}(c^{(r)}_j)_{n_rr^{(r)}_j}}{\prod_{j=1}^{l}(b_j)_{n_1\beta^1_j+\dots n_r\beta^{(r)}_j}\prod_{j=1}^{m_1}(d^1_j)_{n_1\delta^1_j}\dots\dots\prod_{j=1}^{m_r}(d_j)^{(r)}_{n_r\delta^{(r)}_j}}\frac{x_1^{n_1}}{n_1!}\dots.\frac{x_r^{n_r}}{n_r!}, \tag{1.12}$$

where the multiple hypergeometric series converges absolutely under the parametric variable constraints, and $(\lambda)_\nu$ denotes the well known Pochhammer symbol.

In our present investigation, we also need to recall the following Obhettinger's integral formula [5]:

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{1.13}$$

provided $0<\Re(\mu)<\Re(\lambda)$.

2. Main results

In this section, we establish two interesting integrals involving the product of Bessel function with Jacobi polynomial, which are expressed in terms of Kampé de Fériet and Srivastava and Daoust functions.

Theorem 2.1. *The following integral formula (in terms of Kampé de Fériet) holds true: For* $\Re(\nu)>-1,\ 0<\Re(\mu)<\Re(\lambda+\nu)$ *and* $x>0$,

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}J_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= y^{\nu}\, 2^{1-\nu-\mu}\, a^{\mu-\lambda-\nu}\, \frac{(1+\alpha)_n \Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{n!\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{array}{l} \Delta(2;\lambda+\nu+1), \quad \Delta(2;\lambda+\nu-\mu): \quad \_; \quad \Delta(2;-n), \\ \\ \Delta(2;\lambda+\nu), \quad \Delta(2;1+\lambda+\nu+\mu): \ \nu+1; \ \Delta(2;1+\alpha), \end{array}\right.$$

$$\left.\begin{array}{l} \Delta(2;1+\alpha+\beta+n); \\ \frac{1}{2}; \end{array} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]$$

$$+\frac{by}{2a}\, \frac{n(1+\alpha+\beta+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{(\lambda+\nu)(1+\lambda+\nu+\mu)(1+\alpha)}$$

$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{array}{l} \Delta(2;\lambda+\nu+2), \ \Delta(2;\lambda+\nu-\mu+1): \quad \_; \\ \\ \Delta(2;\lambda+\nu+1), \ \Delta(2;2+\lambda+\nu+\mu): \ \nu+1; \end{array}\right.$$

$$\left.\begin{array}{ll} \Delta(2;-n+1), \ \Delta(2;2+\alpha+\beta+n); & \\ & -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2} \\ \Delta(2;2+\alpha), \qquad \frac{3}{2}; & \end{array}\right], \tag{2.1}$$

where $\Delta(m;l)$ *abbreviates the array of* m *parameters* $\frac{l}{m}, \frac{l+1}{m}, \ldots, \frac{l+m-1}{m}$, $m \geq 1$.

Proof. In order to derive (2.1), we denote the left-hand side of (2.1) by I, expanding the Bessel function J_ν and Jacobi polynomial $P_n^{(\alpha,\beta)}$ with the help of (1.1) and (1.4) and interchanging the order of integration and summation (which is verified by uniform convergence of the involved series under the given conditions), we get

$$\mathrm{I} = y^{\nu}\, 2^{-\nu} \frac{(1+\alpha)_n}{n!} \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+n)_k}{m!\Gamma(\nu+m+1)(1+\alpha)_k}\left(\frac{-y^2}{4}\right)^m\left(\frac{-by}{2}\right)^k$$
$$\times \int_0^{\infty} x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-(\lambda+\nu+2m+k)}\,dx. \tag{2.2}$$

Using (1.13) in the above expression and after a little simplification, we get

$$\mathrm{I} = \mathrm{y}^{\nu}\, 2^{1-\nu-\mu}\, \mathrm{a}^{\mu-\lambda-\nu} \frac{(1+\alpha)_{\mathrm{n}}\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\mathrm{n}!\,\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$
$$\times \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda+\nu+1)_{2m+k}(\lambda+\nu-\mu)_{2m+k}(-n)_k(1+\alpha+\beta+n)_k}{m!k!(\lambda+\nu)_{2m+k}(1+\lambda+\nu+\mu)_{2m+k}(\nu+1)_m\Gamma(\nu+m+1)(1+\alpha)_k}$$
$$\times \left(\frac{-y^2}{4a^2}\right)^m \left(\frac{-by}{2a}\right)^k. \tag{2.3}$$

Now separating the k-series into its even and odd terms and then using the result $(A)_{m+n} = (A)_m(A+m)_n$ in the second term of the given expression, we get

$$\mathrm{I} = \mathrm{y}^{\nu}\, 2^{1-\nu-\mu}\, \mathrm{a}^{\mu-\lambda-\nu} \frac{(1+\alpha)_{\mathrm{n}}\ \Gamma(2\mu)\ \Gamma(\lambda+\nu+1)\ \Gamma(\lambda+\nu-\mu)}{\mathrm{n}!\ \Gamma(\nu+1)\ \Gamma(\lambda+\nu)\ \Gamma(1+\lambda+\nu+\mu)}$$

$$\times \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda+\nu+1)_{2(m+k)}\,(\lambda+\nu-\mu)_{2(m+k)}\,(-n)_{2k}\,(1+\alpha+\beta+n)_{2k}}{m!\,k!\,(\lambda+\nu)_{2(m+k)}\,(1+\lambda+\nu+\mu)_{2(m+k)}\,(\nu+1)_m\,(1+\alpha)_{2k}\,2^{2k}\,(\frac{1}{2})_k}$$
$$\times \left(\frac{-y^2}{4a^2}\right)^m \left(\frac{b^2y^2}{4a^2}\right)^k + \frac{byn\,(1+\alpha+\beta+n)\,(\lambda+\nu+1)\,(\lambda+\nu-\mu)}{2a\,(\lambda+\nu)\,(1+\lambda+\nu+\mu)\,(1+\alpha)}$$
$$\times \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda+\nu+2)_{2(m+k)}(\lambda+\nu-\mu+1)_{2(m+k)}(-n+1)_{2k}(2+\alpha+\beta+n)_{2k}}{m!k!(\lambda+\nu+1)_{2(m+k)}(2+\lambda+\nu+\mu)_{2(m+k)}(\nu+1)_m(2+\alpha)_{2k}\,2^{2k}(\frac{3}{2})_k}$$
$$\times \left(\frac{-y^2}{4a^2}\right)^m \left(\frac{b^2y^2}{4a^2}\right)^k. \tag{2.4}$$

Finally after a little simplification, summing up the above series with the help of (1.11), we arrive at the right-hand side of (2.1). This completes the proof. □

Theorem 2.2. *The following integral formula (in terms of Srivastava and Daoust function) holds true: For* $\Re(\nu) > -1$, $0 < \Re(\mu) < \Re(\lambda+\nu)$ *and* $x > 0$,

$$\int_0^{\infty} x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) P_n^{(\alpha,\beta)}\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= y^\nu\, 2^{1-\nu-\mu}\, a^{\mu-\lambda-\nu}\, \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$
$$\times F^{4:\,0;\,0}_{5:\,0;\,1}\left[\begin{array}{l} (\lambda+\nu+1:2,3),\ (\lambda+\nu-\mu:2,3),\ (1+\alpha+\beta:1,2),\ (1+\alpha:1,1):\_;\ \_; \\ (\lambda+\nu:2,3),\ (1+\lambda+\nu+\mu:2,3),\ (\nu+1:1,1),\ (1+\alpha+\beta:1,1),\ (1:1,1):\ \_;\ (1+\alpha,1); \end{array} -\frac{y^2}{4a^2}, \frac{by^3}{8a^3}\right]. \tag{2.5}$$

Proof. In order to derive (2.5), we denote the left-hand side of (2.5) by I', expanding J_ν and $P_n^{(\alpha,\beta)}$ in their series form and then using the lemma (see [4]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k),$$

we get

$$\mathrm{I}' = \mathrm{y}^\nu\, 2^{-\nu} \sum_{\mathrm{n}=0}^{\infty}\sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^{\mathrm{n+k}}\,(1+\alpha)_{\mathrm{n+k}}\,(1+\alpha+\beta)_{\mathrm{n+2k}}\,(-\mathrm{b})^{\mathrm{k}}\,(\frac{\mathrm{y}}{2})^{\mathrm{2n+3k}}}{(\mathrm{n+k})!\,\Gamma(\nu+\mathrm{n+k}+1)\,(1+\alpha+\beta)_{\mathrm{n+k}}\,\mathrm{n}!\,\mathrm{k}!}$$
$$\times \int_0^{\infty} x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-(\lambda+\nu+2n+3k)}dx. \tag{2.6}$$

On using (1.13) in the above expression and after a little simplifications, we arrive at

$$\mathrm{I}' = \mathrm{y}^\nu\, 2^{1-\nu-\mu}\, \mathrm{a}^{\mu-\lambda-\nu} \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times \sum_{n,k=0}^{\infty} \frac{(\lambda+\nu+1)_{2n+3k}(\lambda+\nu-\mu)_{2n+3k}}{n!k!(\lambda+\nu)_{2n+3k}(1+\lambda+\nu+\mu)_{2n+3k}(\nu+1)_{n+k}}$$
$$\times \frac{(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}}{(1+\alpha+\beta)_{n+k}(1)_{n+k}(1+\alpha)_k}\left(\frac{-y^2}{4a^2}\right)^n\left(\frac{by^3}{8a^3}\right)^k. \tag{2.7}$$

Finally, summing up the above series with the help of (1.12), we arrive at the right-hand side of (2.5). This completes the proof.

Remark 1. On setting $\alpha=\beta=b=0$ and using $P_n^{(0,0)}(1)=1$ in (2.1) and (2.5), respectively, the resulting identities reduce to Theorem 1 of Choi and Agarwal [11].

3. Special cases

In this section, first, we give some integral formulas involving the product of Bessel function with ultraspherical polynomial, Gegenbauer polynomial, Tchebicheff polynomial and Legendre polynomial and then we derive some other integrals involving the product of sine (cosine) function with Jacobi polynomial, ultraspherical polynomial, Gegenbauer polynomial, Tchebicheff polynomial and Legendre polynomial as special cases of our main results.

Corollary 3.1. *The following integral formula holds true under the same condition of Theorem 2.1:*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) P_n^{(\alpha,\alpha)}\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= y^\nu\, 2^{1-\nu-\mu}\, a^{\mu-\lambda-\nu}\,\frac{(1+\alpha)_n\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{n!\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix}\Delta(2;\lambda+\nu+1),\ \ \Delta(2;\lambda+\nu-\mu): \ \ \_\ ; \ \ \Delta(2;-n),\ \Delta(2;1+2\alpha+n);\\ \Delta(2;\lambda+\nu),\ \ \Delta(2;1+\lambda+\nu+\mu): \nu+1;\ \Delta(2;1+\alpha),\ \ \tfrac{1}{2};\end{matrix}\ -\frac{y^2}{4a^2},\ \frac{b^2y^2}{4a^2}\right]$$
$$+\frac{by}{2a}\,\frac{n(1+2\alpha+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{(\lambda+\nu)(1+\lambda+\nu+\mu)(1+\alpha)}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix}\Delta(2;\lambda+\nu+2),\ \Delta(2;\lambda+\nu-\mu+1): \ \ \_\ ; \ \ \Delta(2;-n+1),\ \Delta(2;2+2\alpha+n);\\ \Delta(2;\lambda+\nu+1),\ \Delta(2;2+\lambda+\nu+\mu): \nu+1;\ \ \Delta(2;2+\alpha),\ \ \tfrac{3}{2};\end{matrix}\ -\frac{y^2}{4a^2},\ \frac{b^2y^2}{4a^2}\right], \tag{3.1}$$

where $P_n^{(\alpha,\alpha)}(z)$ is the ultraspherical polynomial (see [4], [7]).

This corollary can be established with the help of Theorem 2.1 by putting $\beta=\alpha$.

Corollary 3.2. *The following integral formula holds true under the same condition of Theorem 2.1:*

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) C_n^l\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right) dx \\
&= y^\nu\, 2^{1-\nu-\mu}\, a^{\mu-\lambda-\nu}\, \frac{(l)_n\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{n!\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)} \\
&\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix} \Delta(2;\lambda+\nu+1), \ \Delta(2;\lambda+\nu-\mu): \ _\,; \ \Delta(2;-n), \ \Delta(2;2l+n); \\ \Delta(2;\lambda+\nu), \ \Delta(2;1+\lambda+\nu+\mu): \nu+1; \ \Delta(2;l+\tfrac{1}{2}), \ \tfrac{1}{2}; \end{matrix} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right] \\
&+\frac{by}{2a}\,\frac{n(2l+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{(\lambda+\nu)(1+\lambda+\nu+\mu)(l+\frac{1}{2})} \\
&\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix} \Delta(2;\lambda+\nu+2), \Delta(2;\lambda+\nu-\mu+1): \ _\,; \ \Delta(2;-n+1), \Delta(2;1+2l+n); \\ \Delta(2;\lambda+\nu+1), \Delta(2;2+\lambda+\nu+\mu): \nu+1; \ \Delta(2;l+\tfrac{3}{2}), \ \tfrac{3}{2}; \end{matrix} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right],
\end{aligned} \tag{3.2}$$

where $C_n^l(z)$ is the Gegenbauer polynomial (*see [4], [7]*).

The above corollary can be established with the help of Theorem 2.1 by putting $\beta=\alpha=l-\frac{1}{2}$ and then using (1.7).

Corollary 3.3. *The following integral formula holds true under the same condition of Theorem 2.1:*

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) T_n\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right) dx \\
&= y^\nu\, 2^{1-\nu-\mu}\, a^{\mu-\lambda-\nu}\, \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)} \\
&\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix} \Delta(2;\lambda+\nu+1), \ \Delta(2;\lambda+\nu-\mu): \ _\,; \ \Delta(2;-n), \Delta(2;n); \\ \Delta(2;\lambda+\nu), \ \Delta(2;1+\lambda+\nu+\mu): \nu+1; \ \Delta(2;\tfrac{1}{2}), \ \tfrac{1}{2}; \end{matrix} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right] \\
&+\frac{by}{a}\,\frac{n^2(\lambda+\nu+1)(\lambda+\nu-\mu)}{(\lambda+\nu)(1+\lambda+\nu+\mu)} \\
&\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix} \Delta(2;\lambda+\nu+2), \Delta(2;\lambda+\nu-\mu+1): \ _\,; \ \Delta(2;-n+1), \Delta(2;n+1); \\ \Delta(2;\lambda+\nu+1), \Delta(2;2+\lambda+\nu+\mu): \nu+1; \ \Delta(2;\tfrac{3}{2}), \ \tfrac{3}{2}; \end{matrix} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right],
\end{aligned} \tag{3.3}$$

where $T_n(z)$ is the Tchebicheff polynomial of the first kind (*see [4], [7]*).

The above corollary can be established with the help of Theorem 2.1 by putting $\beta = \alpha = -\frac{1}{2}$ and then using (1.8).

Corollary 3.4. *The following integral formula holds true under the same condition of Theorem 2.1:*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) U_n\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= y^\nu\, 2^{1-\nu-\mu}\, a^{\mu-\lambda-\nu} \frac{(n+1)\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{array}{l} \Delta(2;\lambda+\nu+1),\ \ \Delta(2;\lambda+\nu-\mu):\ \_\,;\ \Delta(2;-n),\, \Delta(2;2+n); \\ \\ \Delta(2;\lambda+\nu),\ \ \Delta(2;1+\lambda+\nu+\mu):\nu+1;\ \Delta(2;\tfrac{3}{2}),\ \ \tfrac{1}{2}; \end{array} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]$$
$$+\frac{by}{3a}\,\frac{n(2+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{(\lambda+\nu)(1+\lambda+\nu+\mu)}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{array}{l} \Delta(2;\lambda+\nu+2),\, \Delta(2;\lambda+\nu-\mu+1):\ \_\,;\ \Delta(2;-n+1),\, \Delta(2;3+n); \\ \\ \Delta(2;\lambda+\nu+1),\, \Delta(2;2+\lambda+\nu+\mu):\nu+1;\ \ \Delta(2;\tfrac{5}{2}),\ \ \tfrac{3}{2}; \end{array} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right], \tag{3.4}$$

where $U_n(z)$ is the Tchebicheff polynomial of the second kind (see [4], [7]).

The above corollary can be established with the help of Theorem 2.1 by putting $\beta = \alpha = \frac{1}{2}$ and then using (1.9).

Corollary 3.5. *The following integral formula holds true under the same condition of Theorem 2.1:*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) P_n\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= y^\nu\, 2^{1-\nu-\mu}\, a^{\mu-\lambda-\nu} \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{array}{l} \Delta(2;\lambda+\nu+1),\ \ \Delta(2;\lambda+\nu-\mu):\ \_\,;\ \Delta(2;-n),\, \Delta(2;1+n); \\ \\ \Delta(2;\lambda+\nu),\ \ \Delta(2;1+\lambda+\nu+\mu):\nu+1;\ \Delta(2;1),\ \ \tfrac{1}{2}; \end{array} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]$$
$$+\frac{by}{2a}\,\frac{n(1+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{(\lambda+\nu)(1+\lambda+\nu+\mu)}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{array}{l} \Delta(2;\lambda+\nu+2),\, \Delta(2;\lambda+\nu-\mu+1):\ \_\,;\ \Delta(2;-n+1),\, \Delta(2;2+n); \\ \\ \Delta(2;\lambda+\nu+1),\, \Delta(2;2+\lambda+\nu+\mu):\nu+1;\ \ \Delta(2;2),\ \ \tfrac{3}{2}; \end{array} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right], \tag{3.5}$$

where $P_n(z)$ is the Legendre polynomial (see [4], [7]).

The above corollary can be established with the help of Theorem 2.1 by putting $\beta = \alpha = 0$ and then using (1.10).

Corollary 3.6. *The following integral formula holds true: For* $0 < \Re(\mu) < \Re(\lambda + \frac{1}{2})$ *and* $x > 0$,

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-(\lambda-\frac{1}{2})} \sin\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) P_n^{(\alpha,\beta)}\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= y\, 2^{1-\mu}\, a^{\mu-\lambda-\frac{1}{2}} \frac{(1+\alpha)_n \Gamma(2\mu)\Gamma(\lambda+\frac{3}{2})\Gamma(\lambda-\mu+\frac{1}{2})}{(n!)\Gamma(\lambda+\frac{1}{2})\Gamma(\lambda+\mu+\frac{3}{2})}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix} \Delta(2;\lambda+\frac{3}{2}),\, \Delta(2;\lambda-\mu+\frac{1}{2}) : \_\,; & \Delta(2;-n), \ \Delta(2;1+\alpha+\beta+n); \\ \Delta(2;\lambda+\frac{1}{2}),\, \Delta(2;\lambda+\mu+\frac{3}{2}) : \frac{3}{2}; & \Delta(2;1+\alpha), \qquad \frac{1}{2}; \end{matrix} \ -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]$$
$$+\frac{by}{2a} \frac{n(1+\alpha+\beta+n)(\lambda+\frac{3}{2})(\lambda-\mu+\frac{1}{2})}{(\lambda+\frac{1}{2})(\frac{3}{2}+\lambda+\mu)(1+\alpha)}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix} \Delta(2;\lambda+\frac{5}{2}),\, \Delta(2;\lambda-\mu+\frac{3}{2}) : \_\,; & \Delta(2;-n+1),\, \Delta(2;2+\alpha+\beta+n); \\ \Delta(2;\lambda+\frac{3}{2}),\, \Delta(2;\lambda+\mu+\frac{5}{2}) : \frac{3}{2}; & \Delta(2;2+\alpha), \qquad \frac{3}{2}; \end{matrix} \ -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]. \tag{3.6}$$

The above corollary can be established with the help of Theorem 2.1 by putting $\nu = \frac{1}{2}$ and then using (1.2).

Corollary 3.7. *The following integral formula holds true: For* $0 < \Re(\mu) < \Re(\lambda - \frac{1}{2})$ *and* $x > 0$,

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-(\lambda-\frac{1}{2})} \cos\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) P_n^{(\alpha,\beta)}\left(1-\frac{by}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= 2^{1-\mu}\, a^{\mu-\lambda+\frac{1}{2}} \frac{(1+\alpha)_n \Gamma(2\mu)\Gamma(\lambda+\frac{1}{2})\Gamma(\lambda-\mu-\frac{1}{2})}{(n!)\Gamma(\lambda-\frac{1}{2})\Gamma(\lambda+\mu+\frac{1}{2})}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix} \Delta(2;\lambda+\frac{1}{2}),\, \Delta(2;\lambda-\mu-\frac{1}{2}) : \_\,; & \Delta(2;-n), \ \Delta(2;1+\alpha+\beta+n); \\ \Delta(2;\lambda-\frac{1}{2}),\, \Delta(2;\lambda+\mu+\frac{1}{2}) : \frac{1}{2}; & \Delta(2;1+\alpha), \qquad \frac{1}{2}; \end{matrix} \ -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]$$
$$+\frac{by}{2a} \frac{n(1+\alpha+\beta+n)(\lambda+\frac{1}{2})(\lambda-\mu-\frac{1}{2})}{(\lambda-\frac{1}{2})(\frac{1}{2}+\lambda+\mu)(1+\alpha)}$$
$$\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{matrix} \Delta(2;\lambda+\frac{3}{2}),\, \Delta(2;\lambda-\mu+\frac{1}{2}) : \_\,; & \Delta(2;-n+1),\, \Delta(2;2+\alpha+\beta+n); \\ \Delta(2;\lambda+\frac{1}{2}),\, \Delta(2;\lambda+\mu+\frac{3}{2}) : \frac{1}{2}; & \Delta(2;2+\alpha), \qquad \frac{3}{2}; \end{matrix} \ -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]. \tag{3.7}$$

The above corollary can be established with the help of Theorem 2.1 by putting $\nu = -\frac{1}{2}$ and then using (1.3).

Remark 2. In a similar way, with the help of Theorem 2.1, we can find some other integral formulas involving the product of sine (cosine) function with ultraspherical polynomial, Gegenbauer polynomial, Tchebicheff polynomial and Legendre polynomial. Also, we can establish some other interesting special cases with the help of Theorem 2.2 by choosing some suitable values of α, β and ν.

4. Connection between the Kampé de Fériet and Srivastava and Daoust functions

In this section, we gives an interesting relation between Kampé de Fériet and Srivastava and Daoust functions as follow:

$$
\begin{aligned}
&F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{array}{llll} \Delta(2;\lambda+\nu+1), & \Delta(2;\lambda+\nu-\mu): & \_; & \Delta(2;-n),\ \Delta(2;1+\alpha+\beta+n); \\ \Delta(2;\lambda+\nu), & \Delta(2;1+\lambda+\nu+\mu): & \nu+1; & \Delta(2;1+\alpha),\ \tfrac{1}{2}; \end{array} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right] \\
&+\frac{by}{2a}\,\frac{n(1+\alpha+\beta+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{(\lambda+\nu)(1+\lambda+\nu+\mu)(1+\alpha)} \\
&\times F^{4:\,0;\,4}_{4:\,1;\,3}\left[\begin{array}{llll} \Delta(2;\lambda+\nu+2), & \Delta(2;\lambda+\nu-\mu+1): & \_; & \Delta(2;-n+1),\ \Delta(2;2+\alpha+\beta+n); \\ \Delta(2;\lambda+\nu+1), & \Delta(2;2+\lambda+\nu+\mu): & \nu+1; & \Delta(2;2+\alpha),\ \tfrac{3}{2}; \end{array} -\frac{y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right] = \frac{n!}{(1+\alpha)_n} \\
&\times F^{4:\,0;\,0}_{5:\,0;\,1}\left[\begin{array}{l} (\lambda+\nu+1:2,3),\ (\lambda+\nu-\mu:2,3),\ (1+\alpha+\beta:1,2),\ (1+\alpha:1,1):\ \_;\ \_; \\ (\lambda+\nu:2,3),\ (1+\lambda+\nu+\mu:2,3),\ (\nu+1:1,1),\ (1+\alpha+\beta:1,1),\ (1:1,1):\ \_;\ (1+\alpha,1); \end{array} -\frac{y^2}{4a^2}, \frac{by^3}{8a^3}\right].
\end{aligned}
\tag{4.1}
$$

The above relation can be established by comparing (2.1) and (2.5).

5. Concluding remarks

In the present investigation, we have established some unified integral formulas, which are expressed in terms of Kampé de Fériet and Srivastava and Daoust functions. Further, we have derived a connection between Kampé de Fériet and Srivastava and Daoust functions from our main results. Also, it is noticed that, the Bessel function can be expressed in terms of the Fox H-function. Therefore, the results presented in this paper are easily converted in terms of the Fox H-function after some suitable parametric replacement.

References

[1] E. M. Wright, The asymptotic expansion of the generalized hypergeometric functions, *J. London Math. Soc.*, **10** (1935), 286–293.

[2] E. M. Wright, The asymptotic expansion of integral functions defined by Taylor series, *Philos. Trans. Roy. Soc. London, A* **238** (1940), 423–451.

[3] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function II, *Proc. Lond. Math. Soc.* (2), **46** (1940), 389–408.

[4] E. D. Rainville, Special functions, The Macmillan Company, New York, 2013.

[5] F. Oberhettinger, Tables of Mellin Transforms, Springer, New York, 1974.

[6] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals,* Elsevier Science Publisher, Amsterdam, London and New York, 2012.

[7] H. M. Srivastava and H. L. Manocha, *A Treatise on generating functions,* Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.

[8] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1985.

[9] J. Choi, P. Agarwal, S. Mathur and S. D. Purohit, Certain new integral formulas involving the generalized Bessel functions, *Bull. Korean Math. Soc.*, **51** (2014), 995–1003.

[10] J. Choi and P. Agarwal, Certain unified integrals involving a product of Bessel functions of first kind, *Honam Mathematical J.* (4), **35** (2013), 667–677.

[11] J. Choi and P. Agarwal, *Certain Unified Integrals Associated with Bessel Functions,* Boundary Value Problems, **2013** (2013), no. 95, 9 pp.

[12] J. Choi, A. Hasnove, H. M. Srivastava and M. Turaev, Integral representations for Srivastava's triple hypergeometric functions, *Taiwanese J. Math.*, **15** (2011), 2751–2762.

[13] P. Agarwal, S. Jain, S. Agarwal and M. Nagpal, On a new class of integrals involving Bessel functions of the first kind, *Communication in Numerical Analysis*, **2014** (2014), 1–7.

[14] S. Ali, On some new unified integrals, *Adv. Comput. Math. Appl.*, **1** (2012), 151–153.

[15] Y. A. Brychkov, *Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas,* CRC Press, Taylor and Francis Group, Boca Raton, London, and New York, 2008.

Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India.

E-mail: nukhanmath@gmail.com

E-mail: ghayas.maths@gmail.com

E-mail: talhausman.maths@gmail.com