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DIOPHANTINE QUADRUPLES OF NUMBERS WHOSE ELEMENTS ARE IN PROPORTION

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Abstract. In this paper certain non-F-type $P_{3,k}$ sequences which contain Diophantine quadruples of numbers in proportion are presented. It is proved that there exist an infinite number of non-F-type $P_{3,k}$ sequences which possess Diophantine quadruples of numbers in proportion.

1. Introduction

The Greek mathematician Diophantus raised the question as to four numbers such that the product of any two increased by a given number shall be a square. M. Gardner [6] asked for a fifth number that can be added to the set $\{1,3,8,120\}$ without destroying the property that the product of any two integers is one less than a perfect square. For historical details of the problem, one may refer to J. Roberts [14] and the author [12]. Various studies have been conducted on the quadruples with Diophantine property through which several interesting results have been established. Pell's equation has been applied in [4], [7] and [13] to bring out the nature of Diophantine triples and quadruples. The purpose of this paper is to present certain non-F-type $P_{3,k}$ sequences which contain Diophantine quadruples of numbers in proportion.

2. Definitions

Let $\mathbb{N}$ denote the set of all natural numbers. We recall the definitions furnished in [8].

Definition 2.1. Let k be a given element of $\mathbb{N}$. Two integers α and β are said to have the property p_k (resp. p_{-k}) if $\alpha\beta + k$ (resp. $\alpha\beta - k$) is a perfect square.

Definition 2.2. Let k be a given element of $\mathbb{N}$. A set S of elements of $\mathbb{N}$ is said to be a P_k set or a Diophantine set with property p_k if every pair of distinct elements in S has the property p_k.

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Definition 2.3. A P_k set S is called extendable if, for some integer d, $d \notin S$, the set $S \cup \{d\}$ is a P_k set.

Definition 2.4. A sequence of integers is said to be a $P_{r,k}$ sequence if every r consecutive terms of the sequence constitute a P_k set.

Example 1. The sequence $\{1,3,8,21,55,144,\dots\}$, obtained using Fibonacci numbers, is a $P_{3,1}$ sequence. The sets $\{1,3,8\}$, $\{3,8,21\}$, $\{8,21,55\}$, ... have property p_1. However, the numbers 1 and 21 do not have property p_1.

Example 2. The sequence $\{1,6,17,45,118,309,\dots\}$ is a $P_{3,19}$ sequence wherein the first and fourth elements viz. 1 and 45 also have property p_{19}.

The method of constructing a $P_{3,k}$ sequence was furnished in [8].

Definition 2.5. Let $\{a_n\}$ be a $P_{3,k}$ sequence together with the associated sequences $\{b_n\}$ and $\{c_n\}$. The sequence $\{a_n\}$ is said to be of F-type if the sequence $\{f_n\} = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$ obtained by juxtaposing the two sequences $\{a_n\}$ and $\{b_n\}$ is of Fibonacci type.

i.e., $f_1 = a_1$, $f_2 = b_1$, and $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$.
Polynomial expressions for F-type $P_{3,k}$ sequences have been provided in [11].

3. Background of the problem

A. Baker and H. Davenport [1] proved that the triple $\{1,3,8\}$ can be extended into the quadruple $\{1,3,8,120\}$ with property p_1 and the quadruple cannot be extended further. From example 2.5 it is observed that the fourth number of the quadruple, namely 120 is not an element of the $P_{3,1}$ sequence. It was proved in [9] that the triple $\{1,5,12\}$ with property p_4 can be extended into the quadruple $\{1,5,12,96\}$ and the quadruple cannot be extended further. Here again it is seen that while the first three numbers of the quadruple form a $P_{3,4}$ sequence, the fourth number 96 is not an element of the sequence. In fact, the $P_{3,4}$ sequence is obtained as $\{1,12,33,85,224\}$. A question that naturally arises from these examples is about a quadruple with property p_k such that the fourth number is also an element of the $P_{3,k}$ sequence constituted by the first three numbers. As regards this question, it was proved in [8] that if $k \equiv 2 \ (mod\ 4)$, then there is no $P_{r,k}$ sequence with $r \geq 4$. Several interesting results on Diophantine quadruples and quintuples have been obtained by A.Dujella (see for e.g. [2], [3]). In [5], A. Dujella and N. Saradha considered m-tuples possessing Diophantine property p_1 with elements in arithmetic progressions. The present paper addresses the problem of Diophantine quadruples of numbers whose elements are in proportion.

4. Diophantine Quadruples from a $P_{3,k}$ sequence

4.1. Construction of $P_{3,k}$ sequence

We construct a $P_{3,k}$ sequence as follows:
Suppose $a_1, a_2, b \in \mathbb{N}$ with $a_1 < a_2$, such that $a_1 a_2 + k = b^2$ for some integer k. We extend the set $\{a_1, a_2\}$ into a $P_{3,k}$ sequence $\{a_1, a_2, a_3, \ldots\}$ by employing the method specified in [8].
We construct three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ as follows:

Take $b_1 = b$, $c_1 = a_1 + b_1$, $b_2 = a_2 + b_1$, $a_3 = b_2 + c_1$, $b_3 = a_3 + b_2$, $c_2 = a_2 + b_2$, $a_4 = b_3 + c_2$, etc. We see that the elements of the three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ have the following recurrence relations:

$$\begin{aligned} a_{n+1} &= 2(a_n + a_{n-1}) - a_{n-2}, \\ b_{n+1} &= 2(b_n + b_{n-1}) - b_{n-2}, \\ c_{n+1} &= 2(c_n + c_{n-1}) - c_{n-2}, \end{aligned}$$

One can check that $a_1 a_3 + k = c_1^2$, $a_2 a_4 + k = c_2^2$, $a_3 a_4 + k = b_3^2$, etc.

It follows that every triple of three consecutive terms of the sequence $\{a_n\}$ constitutes a P_k set. Therefore $\{a_1, a_2, a_3, \ldots\}$ is a $P_{3,k}$ sequence.

4.2. Quadruples in which the elements are in proportion

Four numbers a,b,c,d are said to be in proportion if they have the property $ad = bc$.

There have been various studies to identify quadruples $\{\alpha, \beta, \gamma, \delta\}$ with Diophantine property p_k. If α, β, γ and δ are chosen as four consecutive terms of a $P_{3,k}$ sequence, then $\{\alpha, \beta, \gamma\}$ and $\{\beta, \gamma, \delta\}$ are triples with Diophantine property p_k. In order that the quadruple $\{\alpha, \beta, \gamma, \delta\}$ has Diophantine property p_k, we have to ensure that α and δ have property p_k. However, if we can identify a set of four consecutive terms of a $P_{3,k}$ sequence in which the elements are in proportion, it has wider implications; indeed, the condition for the Diophantine property of the quadruple will automatically be satisfied. This arouses the interest to think of quadruples in which the elements are in proportion. We investigate whether there exists one such quadruple in a $P_{3,k}$ sequence and if so whether there exist infinite number of such quadruples.

4.3. Construction of $P_{3,k}$ sequences with a condition

Henceforth we impose the condition

$$a_3 = 3a_2 - a_1 - 2 \tag{1}$$

for the sequence $\{a_n\}$ so that we have

$$k = a_1^2 - 3a_1a_2 + a_2^2 + 2a_1 - 2a_2 + 1 \tag{2}$$

It is ascertained that $k \equiv 0$ or 1 (mod 4).
We have the recurrence relations

$$a_{2m+1} = 3a_{2m} - a_{2m-1} - 2, \forall\ m \geq 1 \quad \text{and} \quad a_{2m} = 3a_{2m-1} - a_{2m-2} + 2, \forall\ m \geq 2. \tag{3}$$

We see that

$$a_4 = 8a_2 - 3a_1 - 4, \quad a_5 = 21a_2 - 8a_1 - 12, \quad a_6 = 55a_2 - 21a_1 - 30, \quad \text{etc.}$$

We observe that

$$c_1 = a_2 - 1, c_2 = a_3 + 1, c_3 = a_4 - 1, \text{etc. So we have}$$

$$c_{2m+1} = a_{2m+2} - 1, \forall m \geq 0 \text{ and } c_{2m} = a_{2m+1} + 1, \forall\ m \geq 1.$$

Hence the sequence $\{a_n\}$ is of non-F-type.

4.4. Condition for the elements of a Diophantine quadruple to be in proportion

Suppose we require a $P_{3,k}$ sequence $\{a_n\}$ constructed by the above method which contains the quadruple $\{a_{2m}, a_{2m+1}, a_{2m+2}, a_{2m+3}\}$ of non-zero numbers such that

$$a_{2m} : a_{2m+1} = a_{2m+2} : a_{2m+3}\ (m > 0). \tag{4}$$

Substituting for a_{2m+2} and a_{2m+3} using (3), we are led to the relation

$$a_{2m+1}(3a_{2m+1} - a_{2m} + 2) - a_{2m}(3a_{2m+2} - a_{2m+1} - 2) = 0.$$

This gives the relation

$$3a_{2m+1}^2 + 2a_{2m+1} - 3a_{2m}a_{2m+2} + 2a_{2m} = 0.$$

Again substituting for a_{2m+2} from (3), we get

$$3a_{2m+1}^2 - 9a_{2m}a_{2m+1} + 2a_{2m+1} + 3a_{2m}^2 - 4a_{2m} = 0. \tag{5}$$

Treating this as a quadratic equation in a_{2m+1}, we obtain

$$a_{2m+1} = \frac{9a_{2m} - 2 \pm \sqrt{45a_{2m}^2 + 12a_{2m} + 4}}{6} \tag{6}$$

This implies that the expression $45a_{2m}^2+12a_{2m}+4$ shall be a square and

$$6 \mid 9a_{2m}-2 \pm \sqrt{45a_{2m}^2+12a_{2m}+4}. \tag{7}$$

We search for some $\lambda \in \mathbb{N}$ such that $45a_{2m}^2+12a_{2m}+4=\lambda^2$. This leads to the relation

$$(15a_{2m}+2)^2-5\lambda^2=-16. \tag{8}$$

Thus we obtain the Pell's equation

$$U^2-5V^2=-16 \tag{9}$$

where $U=15a_{2m}+2$ and $V=\lambda$. One may refer to T.Nagell [10] for a theory of the general Pell's equation

$$U^2-DV^2=N$$

where D is a square-free natural number.

Let us consider the Pell's equation

$$A^2-5B^2=1. \tag{10}$$

It is well-known that equation (10) has an infinite number of integral solutions. Using the continued fraction expansion of $\sqrt{5}$, we obtain the fundamental solution of (10) as $9+4\sqrt{5}$.

Given a positive integer k, it was proved in [8] that the number of distinct classes of solutions of the equation $x^2-5y^2=4k$ is divisible by 3. The equation (9) possesses three non-associated classes of solutions. Upon computation, it is found that the solutions $U_n+V_n\sqrt{5}$ of the equation (9) in the three classes are provided by the expressions

$$(-22+10\sqrt{5})(9+4\sqrt{5})^n,$$

$$(-8+4\sqrt{5})(9+4\sqrt{5})^n$$

and

$$(-2+2\sqrt{5})(9+4\sqrt{5})^n$$

respectively, where $n=0,1,2,\dots$

In class I, we have $U_0=-22$, $V_0=10$; $U_1=2$, $V_1=2$; $U_2=58$, $V_2=26$; $U_3=1042$, $V_3=466$; $U_4=18698$, $V_4=8362,\dots$.

In class II, $U_0=-8$, $V_0=4$; $U_1=8$, $V_1=4$; $U_2=152$, $V_2=68$; $U_3=2728$, $V_3=1220$, $U_4=48952$, $V_4=21892,\dots$.

In class III, $U_0=-2$, $V_0=2$; $U_1=22$, $V_1=10$; $U_2=398$, $V_2=178$; $U_3=7142$, $V_3=3194$, $U_4=128158$, $V_4=57314,\dots$

Since we require $U=15a_{2m}+2$ in (7), we have to consider $U_n \pmod{15}$ in the concerned class.

In each class, we observe that $U_n \pmod{15}$ is periodic with a period of 4. In view of (7), we have to consider $V_n \pmod 6$. We observe that in each class, $V_n \pmod 6$ is periodic with a period of 4.

In class I, we have

$$U_n \equiv \begin{cases} 8 \pmod{15} & \text{for } n \equiv 0 \pmod 4, \\ 2 \pmod{15} & \text{for } n \equiv 1 \pmod 4, \\ 13 \pmod{15} & \text{for } n \equiv 2 \pmod 4, \\ 7 \pmod{15} & \text{for } n \equiv 3 \pmod 4. \end{cases} \quad V_n \equiv \begin{cases} 4 \pmod 6 & \text{for } n \equiv 0 \pmod 4, \\ 2 \pmod 6 & \text{for } n \equiv 1 \pmod 4, \\ 2 \pmod 6 & \text{for } n \equiv 2 \pmod 4, \\ 4 \pmod 6 & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

In class II,

$$U_n \equiv \begin{cases} 7 \pmod{15} & \text{for } n \equiv 0 \pmod 4, \\ 8 \pmod{15} & \text{for } n \equiv 1 \pmod 4, \\ 2 \pmod{15} & \text{for } n \equiv 2 \pmod 4, \\ 13 \pmod{15} & \text{for } n \equiv 3 \pmod 4. \end{cases} \quad V_n \equiv \begin{cases} 4 \pmod 6 & \text{for } n \equiv 0 \pmod 4, \\ 4 \pmod 6 & \text{for } n \equiv 1 \pmod 4, \\ 2 \pmod 6 & \text{for } n \equiv 2 \pmod 4, \\ 2 \pmod 6 & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

In class III,

$$U_n \equiv \begin{cases} 13 \pmod{15} & \text{for } n \equiv 0 \pmod 4, \\ 7 \pmod{15} & \text{for } n \equiv 1 \pmod 4, \\ 8 \pmod{15} & \text{for } n \equiv 2 \pmod 4, \\ 2 \pmod{15} & \text{for } n \equiv 3 \pmod 4. \end{cases} \quad V_n \equiv \begin{cases} 2 \pmod 6 & \text{for } n \equiv 0 \pmod 4, \\ 4 \pmod 6 & \text{for } n \equiv 1 \pmod 4, \\ 4 \pmod 6 & \text{for } n \equiv 2 \pmod 4, \\ 2 \pmod 6 & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

On the basis of the values of U_n provided by (9), it becomes necessary to restrict to U_n where

$$n \equiv \begin{cases} 1 \pmod 4 & \text{in class I}, \\ 2 \pmod 4 & \text{in class II}, \\ 3 \pmod 4 & \text{in class III}. \end{cases}$$

Since $a_{2m} < a_{2m+1}$, the negative sign cannot hold in (6). We have to select those a_{2m}'s in (6) such that $6 \mid 3a_{2m} - 2 + V_n$. We have $V_n \equiv 2 \pmod 6$ where

$$n \equiv \begin{cases} 1 \pmod 4 & \text{in class I}, \\ 2 \pmod 4 & \text{in class II}, \\ 3 \pmod 4 & \text{in class III}. \end{cases}$$

Since the values assumed by U in (9) are even, it is observed that the condition (7) is fulfilled for all V_n where

$$n \equiv \begin{cases} 1 \pmod 4 & \text{in class I,} \\ 2 \pmod 4 & \text{in class II,} \\ 3 \pmod 4 & \text{in class III.} \end{cases}$$

We see that the same conditions on n hold for the values of U_n and V_n in the three respective classes. Thus we have proved the following:

Theorem 4.1. *For every given natural number m, each one of the three classes of solutions of the Pell's equation $U^2 - 5V^2 = -16$ contributes an infinite number of non-F-type $P_{3,k}$ sequences $\{a_n\}$ containing the Diophantine quadruples $\{a_{2m}, a_{2m+1}, a_{2m+2}, a_{2m+3}\}$ whose elements are in proportion.*

5. Determination in specific cases

Now we determine the elements of a Diophantine quadruple in proportion for certain specific values of m.

Case 1. m = 1. Let us consider the Diophantine quadruples from the three classes of solutions of (9) separately.

Class I

Let us consider the solutions $U_n + V_n\sqrt{5}$ of the equation (9) in class I where $n \equiv 1 \pmod 4$. When $n = 1$, we have $U_1 = 2$. This gives $a_2 = 0$ which is inadmissible. Next, when $n = 5$, we have $U_5 = 335522$. From this we get $a_2 = 22368$. By means of (3) we obtain $k = 63169$, $a_1 = 8542$, $a_3 = 58560$, $a_4 = 153314$, $a_5 = 401380, \ldots$. Thus we obtain Diophantine quadruples $\{a_2, a_3, a_4, a_5\}$ with $a_2 : a_3 = a_4 : a_5$ from the solutions $U_n + V_n\sqrt{5}$ of the equation (9) for all $n \equiv 1 \pmod 4$, $n > 1$.

Class II

Next we take up the solutions $U_n + V_n\sqrt{5}$ of the equation (9) in class II where $n \equiv 2 \pmod 4$. When $n = 2$, we get $U_2 = 152$. From this we have $k = 29$, $a_1 = 2$, $a_2 = 10$, $a_3 = 26$, $a_4 = 70$, $a_5 = 182, \ldots$. Next, when $n = 6$, we get $U_6 = 15762392$ implying $k = 2967581$, $a_1 = 401378$, $a_2 = 1050826$, $a_3 = 2751098$, $a_4 = 7202470$, $a_5 = 18856310, \ldots$. Thus we obtain Diophantine quadruples $\{a_2, a_3, a_4, a_5\}$ with $a_2 : a_3 = a_4 : a_5$ from the solutions $U_n + V_n\sqrt{5}$ of the equation (9) for all $n \equiv 2 \pmod 4$.

Class III

Now we consider the solutions $U_n + V_n\sqrt{5}$ of the equation (9) in class III where $n \equiv 3$ (mod 4). When $n = 3$, we have $U_3 = 7142$. This gives $k = 1345$, $a_1 = 180$, $a_2 = 476$, $a_3 = 1246$,

$a_4 = 3264$, $a_5 = 8544, \ldots$. Next, when $n = 7$, we obtain $U_7 = 740496902$. This furnishes the values $k = 139413121$, $a_1 = 18856308$, $a_2 = 49366460$, $a_3 = 129243070$, $a_4 = 338362752$, $a_5 = 885845184, \ldots$. Thus we get Diophantine quadruples $\{a_2, a_3, a_4, a_5\}$ with $a_2 : a_3 = a_4 : a_5$ from the solutions $U_n + V_n\sqrt{5}$ of the equation (9) for all $n \equiv 3 \pmod 4$.

Case 2. $m > 1$. In this case, we employ (3) in (5) and obtain a relation involving a_1 and a_2. Treating this relation as a quadratic in a_2, we solve for integral values of a_2, following the same procedure as in the preceding discussion We illustrate the case of $m = 2$. When $m = 2$, the equation (5) is got as

$$3a_5^2 - 9a_4a_5 + 2a_5 + 3a_4^2 - 4a_4 = 0.$$

Using (3), this equation is transformed as

$$3a_2^2 - 9a_1a_2 - 74a_2 + 3a_1^2 + 32a_1 + 40 = 0. \tag{11}$$

Treating this as a quadratic equation in a_2, we get

$$a_2 = \frac{9a_1 + 74 \pm \sqrt{45a_1^2 + 948a_1 + 4996}}{3}. \tag{12}$$

This implies that the expression $45a_1^2 + 948a_1 + 4996$ shall be a square and
$6 \mid 9a_1 + 74 \pm \sqrt{45a_1^2 + 948a_1 + 4996}$.
Taking $45a_1^2 + 948a_1 + 4996 = \gamma^2$, we are led to the Pell's equation

$$U^2 - 5V^2 = -16 \tag{13}$$

where $U = 15a_1 + 158$ and $V = \gamma$. We have already considered this Pell's equation in Section 4. We assert that the negative sign cannot hold in (12). As in the preceding discussion, we obtain an infinite number of Diophantine quadruples $\{a_4, a_5, a_6, a_7\}$ with $a_4 : a_5 = a_6 : a_7$ from each class of solutions $U_n + V_n\sqrt{5}$ of the equation (13). For example, we have $k = 63169$, $a_4 = 22368$, $a_5 = 58560$, $a_6 = 153314$, $a_7 = 401380$ from class I, $k = 2967581$, $a_4 = 1050826$, $a_5 = 2751098$, $a_6 = 7202470$, $a_7 = 18856310$ from class II and $k = 1345$, $a_4 = 476$, $a_5 = 1246$, $a_6 = 3264$, $a_7 = 8544$ from class III.
When $m > 2$, we are led to the same Pell's equation (13) with changes in the expressions for U and V in terms of a_1.

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