INEQUALITIES FOR SOME CLASSICAL INTEGRAL TRANSFORMS

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Abstract. By using a form of the Cauchy-Bunyakovsky-Schwarz inequality, we establish new inequalities for some classical integral transforms such as Laplace transform, Fourier transform, Fourier cosine transform, Fourier sine transform, Mellin transform and Hankel transform.

1. Introduction

The well-known Cauchy-Bunyakovsky-Schwarz (CBS) inequality [1] states that

$$\left(\int_a^b u^{\frac{1}{2}}(x) v^{\frac{1}{2}}(x) dx \right)^2 \le \left(\int_a^b u(x) dx \right) \left(\int_a^b v(x) dx \right), \tag{1}$$

for every function $u, v : [a, b] \to (-\infty, \infty)$ such that the integrals exist.

As we know, the Cauchy Schwarz and Cauchy-Bunyakovsky-Schwarz inequalities play an important role in different branches of modern mathematics such as Hilbert space theory, classical real and complex analysis, numerical analysis, probability and statistics, qualitative theory of differential equations and their applications. To date, a large number of generalizations and refinements of these inequalities have been investigated in the literature (see, e.g., [2]−[8]).

A. Laforgia and P. Natalini [9] used the following form of the CBS inequality (1):

$$\left(\int_a^b g(x) f^{\frac{(m+n)}{2}}(x) dx \right)^2 \le \left(\int_a^b g(x) f^m(x) dx \right) \left(\int_a^b g(x) f^n(x) dx \right) \tag{2}$$

to establish some new Turán-type inequalities involving the special functions as gamma, or Polygamma functions and Riemann zeta function. Here f and g are two nonnegative functions of a real variable and m and n belong to a set S of real numbers, such that the involved integrals in (2) exist.

Motivated by this remark, we have the idea to replace $u(x)$ and $v(x)$ in (1) by $g^\gamma(x)h^{\alpha+\beta}(x)$ and $g^{2-\gamma}(x)h^{\alpha-\beta}(x)$ respectively, to introduce the following new inequality:

$$\left(\int_a^b g(x)h^\alpha(x)dx\right)^2 \le \left(\int_a^b g^\gamma(x)h^{\alpha+\beta}(x)dx\right)\left(\int_a^b g^{2-\gamma}(x)h^{\alpha-\beta}(x)dx\right) \tag{3}$$

in which $\alpha, \beta, \gamma \in \mathbb{R}$ and g, h are real integrable functions such that the involved integrals in (3) exist.

The aim of this paper is to apply the continuous inequality (3) for some classical integral transforms in order to get new inequalities.

2. The Results

In this section, we apply the inequality (3) to establish new inequalities for some classical integral transforms.

2.1. An inequality for the Laplace transform

Theorem 1. *Let $L\{f(x); p\}$ or $\overline{f}(p)$ be the Laplace transform of a function $f(x)$ defined for $x > 0$ with parameter p such that $Re(p) > 0$. Then, for every real numbers α, β and γ such that $\alpha > 0$ and $\alpha > |\beta|$, we have*

$$L^2\left\{f\left(\frac{x}{\alpha}\right)\right\} \le \frac{\alpha^2}{\alpha^2 - \beta^2}\left[L\left\{f^\gamma\left(\frac{x}{\alpha+\beta}\right)\right\}L\left\{f^{2-\gamma}\left(\frac{x}{\alpha-\beta}\right)\right\}\right].$$

Proof. Let $f(x)$ be a function defined for $x > 0$, then the Laplace transform [10] of $f(x)$, denoted by $L\{f(x); p\}$ or $\overline{f}(p)$, is defined by

$$L\{f(x); p\} = \overline{f}(p) = \int_0^\infty e^{-px}f(x)dx \tag{4}$$

provided that the integral exists for some real or complex value of parameter p. In general $Re(p) > 0$, where $Re(p)$ means real part of p.

Now if $g(x) = f(x)$, $h(x) = e^{-px}$ are substituted in inequality (3) for $[a, b] \to [0, \infty)$ the following inequality is derived

$$\left(\int_0^\infty e^{-\alpha px}f(x)dx\right)^2 \le \left(\int_0^\infty e^{-(\alpha+\beta)px}f^\gamma(x)dx\right)\left(\int_0^\infty e^{-(\alpha-\beta)px}f^{2-\gamma}(x)dx\right)$$

$$\Rightarrow \left(\int_0^\infty \frac{1}{\alpha}e^{-pu}f\left(\frac{u}{\alpha}\right)du\right)^2 \le \left(\int_0^\infty \frac{1}{(\alpha+\beta)}e^{-pv}f^\gamma\left(\frac{v}{\alpha+\beta}\right)dv\right)\left(\int_0^\infty \frac{1}{(\alpha-\beta)}e^{-pw}f^{2-\gamma}\left(\frac{w}{\alpha-\beta}\right)dw\right).$$

By (4), above inequality is equivalent to

$$\frac{1}{\alpha^2}L^2\left\{f\left(\frac{x}{\alpha}\right)\right\} \le \frac{1}{\alpha^2 - \beta^2}\left[L\left\{f^\gamma\left(\frac{x}{\alpha+\beta}\right)\right\} L\left\{f^{2-\gamma}\left(\frac{x}{\alpha-\beta}\right)\right\}\right]$$

$$\Rightarrow L^2\Big\{f\Big(\frac{x}{\alpha}\Big)\Big\} \le \frac{\alpha^2}{\alpha^2 - \beta^2}\Big[L\Big\{f^\gamma\Big(\frac{x}{\alpha + \beta}\Big)\Big\} L\Big\{f^{2-\gamma}\Big(\frac{x}{\alpha - \beta}\Big)\Big\}\Big],$$
$$\alpha > 0, \ \alpha > |\beta| \text{ and } \alpha, \beta, \gamma \in \mathbb{R}. \tag{5}$$

if and only if the corresponding integrals exist.

Corollary 1. *Let* $L\{f(x); p\}$ *or* $\overline{f}(p)$ *be the Laplace transform of a function* $f(x)$ *defined for* $x > 0$ *with parameter* p *such that* $Re(p) > 0$. *Then, for every real numbers* α, β *and* γ *such that* $\alpha > 0$ *and* $\alpha > |\beta|$, *we have*

$$L^2\Big\{f\Big(\frac{x}{\alpha}\Big)\Big\} \le \frac{\alpha^2}{\alpha^2 - \beta^2}L\Big\{\int_0^x f^\gamma\Big(\frac{u}{\alpha + \beta}\Big)f^{2-\gamma}\Big(\frac{x-u}{\alpha - \beta}\Big)du\Big\}$$

Proof. According to the convolution theorem of Laplace transform, i.e.

$$L\{f(x)\}L\{g(x)\} = L\Big\{\int_0^x f(u)g(x-u)du\Big\} \tag{6}$$

Using (6), inequality (5) is in fact equivalent to:

$$L^2\Big\{f\Big(\frac{x}{\alpha}\Big)\Big\} \le \frac{\alpha^2}{\alpha^2 - \beta^2}L\Big\{\int_0^x f^\gamma\Big(\frac{u}{\alpha + \beta}\Big)f^{2-\gamma}\Big(\frac{x-u}{\alpha - \beta}\Big)du\Big\}, \ \alpha > 0, \ \alpha > |\beta| \text{ and } \alpha, \beta, \gamma \in \mathbb{R}. \tag{7}$$

2.2. An inequality for the Fourier transform

Theorem 2. *Let* $\mathscr{F}\{f(x); p\}$ *or* $\mathscr{F}(p)$ *be the Fourier transform of a function* $f(x)$ *defined on* $(-\infty, \infty)$ *with parameter* p. *Then, for every real numbers* α, β *and* γ *such that* $\alpha > 0$ *and* $\alpha > |\beta|$, *we have*

$$\mathscr{F}^2\Big\{f\Big(\frac{x}{\alpha}\Big)\Big\} \le \frac{\alpha^2}{\alpha^2 - \beta^2}\Big[\mathscr{F}\Big\{f^\gamma\Big(\frac{x}{\alpha + \beta}\Big)\Big\}\mathscr{F}\Big\{f^{2-\gamma}\Big(\frac{x}{\alpha - \beta}\Big)\Big\}\Big].$$

Proof. Let $f(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise-continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then the Fourier transform of $f(x)$, denoted by $\mathscr{F}\{f(x); p\}$ or $\mathscr{F}(p)$, is defined as

$$\mathscr{F}\{f(x); p\} = \mathscr{F}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}f(x)dx, \qquad (-\infty < p < \infty) \tag{8}$$

Hence, if $g(x) = f(x)$, $h(x) = e^{ipx}$ and $[a, b] \to (-\infty, \infty)$, are considered in inequality (3) then we get

$$\Big(\int_{-\infty}^{\infty} e^{i\alpha px}f(x)dx\Big)^2 \le \Big(\int_{-\infty}^{\infty} e^{i(\alpha + \beta)px}f^\gamma(x)dx\Big)\Big(\int_{-\infty}^{\infty} e^{i(\alpha - \beta)px}f^{2-\gamma}(x)dx\Big)$$
$$\Rightarrow \Big(\int_{-\infty}^{\infty}\frac{1}{\alpha}e^{ipu}f\Big(\frac{u}{\alpha}\Big)du\Big)^2 \le \Big(\int_{-\infty}^{\infty}\frac{1}{(\alpha + \beta)}e^{ipv}f^\gamma\Big(\frac{v}{\alpha + \beta}\Big)dv\Big)\Big(\int_{-\infty}^{\infty}\frac{1}{(\alpha - \beta)}e^{ipw}f^{2-\gamma}\Big(\frac{w}{\alpha - \beta}\Big)dw\Big)$$

By (8), above inequality is equivalent to

$$\frac{1}{\alpha^2}\mathscr{F}^2\Big\{f\Big(\frac{x}{\alpha}\Big)\Big\} \leq \frac{1}{\alpha^2-\beta^2}\Big[\mathscr{F}\Big\{f^\gamma\Big(\frac{x}{\alpha+\beta}\Big)\Big\}\mathscr{F}\Big\{f^{2-\gamma}\Big(\frac{x}{\alpha-\beta}\Big)\Big\}\Big]$$

$$\Rightarrow \mathscr{F}^2\Big\{f\Big(\frac{x}{\alpha}\Big)\Big\} \leq \frac{\alpha^2}{\alpha^2-\beta^2}\Big[\mathscr{F}\Big\{f^\gamma\Big(\frac{x}{\alpha+\beta}\Big)\Big\}\mathscr{F}\Big\{f^{2-\gamma}\Big(\frac{x}{\alpha-\beta}\Big)\Big\}\Big], \ \alpha > 0, \ \alpha > |\beta| \text{ and } \gamma \in \mathbb{R}, \quad (9)$$

if and only if the corresponding integrals exist.

Corollary 2. *Let $\mathscr{F}\{f(x);p\}$ or $\mathscr{F}(p)$ be the Fourier transform of a function $f(x)$ defined on $(-\infty,\infty)$ with parameter p. Then, for every real numbers α, β and γ such that $\alpha > 0$ and $\alpha > |\beta|$, we have*

$$\mathscr{F}^2\Big\{f\Big(\frac{x}{\alpha}\Big)\Big\} \leq \frac{\alpha^2}{\alpha^2-\beta^2} \ \frac{1}{\sqrt{2\pi}}\mathscr{F}\Big\{\int_{-\infty}^\infty f^\gamma\Big(\frac{u}{\alpha+\beta}\Big)f^{2-\gamma}\Big(\frac{x-u}{\alpha-\beta}\Big)du\Big\}$$

Proof. According to the convolution theorem of Fourier transform, i.e.

$$\mathscr{F}\{f(x)\} \ \mathscr{F}\{g(x)\} = \mathscr{F}\Big\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(u)g(x-u)du\Big\}. \tag{10}$$

Using (10), inequality (9) is in fact equivalent to:

$$\mathscr{F}^2\Big\{f\Big(\frac{x}{\alpha}\Big)\Big\} \leq \frac{\alpha^2}{\alpha^2-\beta^2} \frac{1}{\sqrt{2\pi}}\mathscr{F}\Big\{\int_{-\infty}^\infty f^\gamma\Big(\frac{u}{\alpha+\beta}\Big)f^{2-\gamma}\Big(\frac{x-u}{\alpha-\beta}\Big)du\Big\}, \ \alpha > 0, \ \alpha > |\beta| \text{ and } \gamma \in \mathbb{R}. \quad (11)$$

2.3. An inequality for the Fourier cosine transform

Theorem 3. *Let $\mathscr{F}_C\{f(x);p\}$ or $\mathscr{F}_C(p)$ be the Fourier cosine transform of a function $f(x)$ defined for $x > 0$ with parameter p such that $p > 0$. Then, for every real numbers α and β such that $\alpha \neq \beta$, we have*

$$\mathscr{F}_C^2\{f^\alpha(x)\} \leq \mathscr{F}_C\{f^{\alpha+\beta}(x)\}\mathscr{F}_C\{f^{\alpha-\beta}(x)\}.$$

Proof. Let $f(x)$ be a function defined for $x > 0$ and be piecewise-continuously differentiable and absolutely integrable in $(0,\infty)$, then the Fourier cosine transform of $f(x)$, denoted by $\mathscr{F}_C\{f(x);p\}$ or $\mathscr{F}_C(p)$, is defined as

$$\mathscr{F}_C\{f(x);p\} = \mathscr{F}_C(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos px\,dx, \qquad (p > 0). \tag{12}$$

Hence, if $g(x) = \cos px$, $h(x) = f(x)$, $[a,b] \to [0,\infty)$ and $\gamma = 1$ are considered in inequality (3) then we get

$$\Big(\int_0^\infty f^\alpha(x)\cos px\,dx\Big)^2 \leq \Big(\int_0^\infty f^{\alpha+\beta}(x)\cos px\,dx\Big)\Big(\int_0^\infty f^{\alpha-\beta}(x)\cos px\,dx\Big).$$

By applying (12) in above inequality, the following result will eventually be obtained

$$\mathscr{F}_C^2\{f^\alpha(x)\} \leq \mathscr{F}_C\{f^{\alpha+\beta}(x)\}\mathscr{F}_C\{f^{\alpha-\beta}(x)\}, \qquad \alpha \neq \beta. \tag{13}$$

2.4. An inequality for the Fourier sine transform

Theorem 4. *Let $\mathscr{F}_S\{f(x); p\}$ or $\mathscr{F}_S(p)$ be the Fourier sine transform of a function $f(x)$ defined for $x > 0$ with parameter p such that $p > 0$. Then, for every real numbers α and β such that $\alpha \neq \beta$, we have*

$$\mathscr{F}_S^2\{f^\alpha(x)\} \leq \mathscr{F}_S\{f^{\alpha+\beta}(x)\}\mathscr{F}_S\{f^{\alpha-\beta}(x)\}$$

Proof. Let $f(x)$ be a function defined for $x > 0$ and be piecewise-continuously differentiable and absolutely integrable in $(0, \infty)$, then the Fourier sine transform of $f(x)$, denoted by $\mathscr{F}_S\{f(x); p\}$ or $\mathscr{F}_S(p)$, is defined as

$$\mathscr{F}_S\{f(x); p\} = \mathscr{F}_S(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px\, dx, \qquad (p > 0). \tag{14}$$

Now, applying inequality (3) for $g(x) = \sin px$, $h(x) = f(x)$, $[a, b] \to [0, \infty)$ and $\gamma = 1$ results in:

$$\left(\int_0^\infty f^\alpha(x) \sin px\, dx\right)^2 \leq \left(\int_0^\infty f^{\alpha+\beta}(x) \sin px\, dx\right)\left(\int_0^\infty f^{\alpha-\beta}(x) \sin px\, dx\right).$$

By (14), which is equivalent to

$$\mathscr{F}_S^2\{f^\alpha(x)\} \leq \mathscr{F}_S\{f^{\alpha+\beta}(x)\}\mathscr{F}_S\{f^{\alpha-\beta}(x)\}, \qquad \alpha \neq \beta. \tag{15}$$

2.5. An inequality for the Mellin transform

Theorem 5. *Let $M\{f(x); p\}$ or $M(p)$ be the Mellin transform of a function $f(x)$ defined for $x > 0$ with parameter p. Then, for every real numbers α and β such that $\alpha \neq \beta$, we have*

$$M^2\{f^\alpha(x)\} \leq M\{f^{\alpha+\beta}(x)\}\, M\{f^{\alpha-\beta}(x)\}.$$

Proof. Let a function $f(x)$ be defined for $0 < x < \infty$, then the Mellin transform of $f(x)$, denoted by $M\{f(x); p\}$ or $M(p)$, is defined as

$$M\{f(x); p\} = M(p) = \int_0^\infty x^{p-1} f(x)\, dx. \tag{16}$$

Hence, if $g(x) = x^{p-1}$, $h(x) = f(x)$, $[a, b] \to [0, \infty)$ and $\gamma = 1$ are considered in inequality (3), then we get

$$\left(\int_0^\infty x^{p-1} f^\alpha(x)\, dx\right)^2 \leq \left(\int_0^\infty x^{p-1} f^{\alpha+\beta}(x)\, dx\right)\left(\int_0^\infty x^{p-1} f^{\alpha-\beta}(x)\, dx\right).$$

By (16), this can be written as

$$M^2\{f^\alpha(x)\} \leq M\{f^{\alpha+\beta}(x)\}M\{f^{\alpha-\beta}(x)\} \qquad \alpha \neq \beta. \tag{17}$$

This completes the proof.

2.6. An inequality for the Hankel transform

Theorem 6. *Let $H_\nu\{f(x); p\}$ or $H_\nu(p)$ be the Hankel transform of a function $f(x)$ defined for $x > 0$ with order $\nu \geq 0$ and parameter p. Then, for every real numbers α and β such that $\alpha \neq \beta$, we have*

$$H_\nu^2\{f^\alpha(x)\} \leq H_\nu\{f^{\alpha+\beta}(x)\} \; H_\nu\{f^{\alpha-\beta}(x)\}.$$

Proof. Let $f(x)$ be a function defined for all positive values of the variable x, i.e., for $x > 0$, Then the Hankel transform of $f(x)$, denoted by $H_\nu\{f(x); p\}$ or $H_\nu(p)$, is defined as

$$H_\nu\{f(x); p\} = H_\nu(p) = \int_0^\infty x J_\nu(px) f(x) dx \tag{18}$$

where $J_\nu(px)$ is the Bessel function of the first kind of order $\nu \geq 0$.

By (3) with $g(x) = x J_\nu(px)$, $h(x) = f(x)$, $[a, b] \to [0, \infty)$ and $\gamma = 1$, we get

$$\left(\int_0^\infty x J_\nu(px) f^\alpha(x) dx \right)^2 \leq \left(\int_0^\infty x J_\nu(px) f^{\alpha+\beta}(x) dx \right) \left(\int_0^\infty x J_\nu(px) f^{\alpha-\beta}(x) dx \right).$$

By (18), this is equivalent to

$$H_\nu^2\{f^\alpha(x)\} \leq H_\nu\{f^{\alpha+\beta}(x)\} \; H_\nu\{f^{\alpha-\beta}(x)\} \qquad \alpha \neq \beta. \tag{19}$$

This completes the proof.

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