TAMKANG JOURNAL OF MATHEMATICS
Volume 35, Number 3, Autumn 2004

A RANDOM VERSION OF SCHAEFER'S FIXED POINT THEOREM WITH APPLICATIONS TO FUNCTIONAL RANDOM INTEGRAL EQUATIONS

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Abstract. In this paper a random version of a fixed-point theorem of Schaefer is obtained and it is further applied to a certain nonlinear functional random integral equation for proving the existence result under Caratheodory conditions.

1. Introduction

Let $(\Omega, \mathcal{A})$ be a measurable space and let X be a Banach space with a Borel σ-algebra β_X. A mapping $x : \Omega \times X \to X$ is called random variable if for a $B \in \beta_{X'}$, $x^{-1}(B) \in \mathcal{A}$. A mapping $T : \Omega \times X \to X$ is called random operator if $T(.,x)$ is measurable for each $x \in X$, and is generally expressed as $T(\omega, x) := T(\omega)x$. A random variable $\xi : \Omega \to X$ is called random fixed point of the random operator $T(\omega) : \Omega \times X \to X$ if $T(\omega)\xi(\omega) = \xi(\omega)$ for each $\omega \in \Omega$. A random operator $T : \Omega \times X \to X$ is called continuous if $T(\omega)(.)$ is continuous for each $\omega \in \Omega$, $T(\omega)$ is called totally bounded if for any bounded set B in X, $T(\omega)(B)$ is a totally bounded subset of X for each $\omega \in \Omega$. Similarly a random operator $T(\omega)$ is called completely continuous on X if it is continuous and totally bounded random operator on X. Again the random operator $T : \Omega \times X \to X$ is called compact if $\overline{T(\omega)(X)}$ is a compact subset of X for each $\omega \in \Omega$. Note that every compact random operator is totally bounded, but the reverse implication may not hold. However, two notions are equivalent on a bounded subset of a Banach space X. Finally the random operator $T : \Omega \times X \to X$ is called contraction if for each $\omega \in \Omega$,

$$\|T(\omega)x - T(\omega)y\| \le k(\omega)\|x - y\| \tag{1.1}$$

for all $x, y \in X$, where $0 \le k(\omega) < 1$.

A **Kuratowskii measure** α of noncompactness of a bounded set A in X is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf\left\{ r > 0 : A = \bigcup_{i=1}^{n} A_i ; \operatorname{diam}(A_i) \le r, \forall\ i \right\}. \tag{1.2}$$

Received December 4, 2002; revised February 10, 2003.
2000 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Random fixed point theorem, random integral equation.

A random operator $T(\omega)$ is called α-condensing if for any bounded set A in X, $T(\omega)(A)$ is bounded and $\alpha(T(\omega)(A)) < \alpha(A)$ if $\alpha(A) > 0$, for each $\omega \in \Omega$.

It is known that contraction and compact random operators are α-condensing but the converse may not be true. We shall obtain a random version of the following generalization of Schaefer's nonlinear alternative due to Martelli [9].

Theorem A. [10] *Let X be a Banach space and let $T : X \to X$ be a continuous and α-condensing map. Then either*
(a) *T has a fixed point, or*
(b) *the set $\mathcal{E} = \{x \in X | x = \lambda Tx, \lambda \in (0,1)\}$ is unbounded.*

2. Random Fixed Point Theory

Theorem 2.1. *Let X be a separable Banach space and let $T : \Omega \times X \to X$ be a random operator satisfying for each $\omega \in \Omega$,*
(a) *$T(\omega)$ is continuous, and α-condensing, and*
(b) *the set $\mathcal{E} = \{x \in X \mid x = \lambda(\omega)T(\omega)x\}$ is bounded for any measurable function $\lambda : \Omega \to \boldsymbol{R}$ with $0 < \lambda(\omega) < 1$.*
Then $T(\omega)$ has a random fixed point.

Proof. Let $\omega \in \Omega$ be fixed. Then by a theorem of Martelli [9], $T(\omega)$ has a fixed point. We denote
$$F(\omega) = \{x \in X \mid T(\omega)x = x\}. \tag{2.1}$$

Obviously $F(\omega)$ is non-empty and compact for each $\omega \in \Omega$, since $T(\omega)$ is α-condensing on X. To finish, it is enough to prove that the set-map $F : \Omega \to K(X)$ is measurable and has closed values. Let C be a closed subset of X and let $\{x_n\}$ be a dense subset of C. Define
$$L(C) = \bigcap_{n=1}^{\infty} \bigcup_{x_i \in C_n} \left\{\omega \in \Omega \mid \|x_i - \omega x_i\| < \frac{1}{n}\right\} \tag{2.2}$$
for all $C_n = \{x \in X | d(x, C) < \frac{1}{n}\}$, where $d(x, C) = \inf\{d(x, c) | c \in C\}$.

Clearly $L(C) \in \mathcal{A}$. Now it is shown as in Itoh [7] that $L(C) = F^{-1}(C)$. Hence $F(\omega)$ is measurable. Let $\{x_n\}$ be a sequence in $F(\omega)$ such that $x_n \to x$. Since $T(\omega)$ is continuous, $x_n = T(\omega)x_n$ implies $x = T(\omega)x$ which yields that $x \in F(\omega)$. Hence $F(\omega)$ is closed for each $\omega \in \Omega$, i.e. the set-map $F : \Omega \to K(X)$ has closed values. Now by an application of a selection theorem of Kuratowskii and Nardzewaski [8], the set-valued map F has a measurable selection $\xi : \Omega \to X$ such that $\xi(\omega) \in F(\omega)$ for all $\omega \in \Omega$. This completes the proof.

Corollary 2.1. *Let X be a separable Banach space and let $T : \Omega \times X \to X$ be a random operator satisfying for each $\omega \in \Omega$,*
(a) *$T(\omega)$ is completely continuous,*
(b) *the set $\mathcal{E} = \{x \in X | T(\omega)x = \alpha(\omega)x\}$ is bounded, for any measurable function $\alpha : \Omega \to \boldsymbol{R}^+$ with $\alpha(\omega) > 1$.*

Then $T(\omega)$ has a random fixed point.

Before proving the next fixed point result, we give a useful definition.

Definition 2.1. A random operator $T : \Omega \times X \to X$ is called D-Lipschitzician if there exists a continuous nondecreasing function $\phi : \Omega \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ satisfying for each $\omega \in \Omega$,

$$\|T(\omega)x - T(\omega)y\| \leq \phi_\omega(\|x - y\|) \tag{2.3}$$

for all $x,\ y \in X$, where $\phi_\omega(r) = \phi_\omega(\omega, r)$ with $\phi(\omega, 0) = 0$. The special case when $\phi_\omega(r) = \alpha(\omega)r$, $\alpha(\omega) > 0$ for all $\omega \in \Omega$, $T(\omega)$ is called Lipschtzician with Lipschitz constant $\alpha(\omega)$, $\omega \in \Omega$. In particular if $\alpha(\omega) < 1$ for all $\omega \in \Omega$, then $T(\omega)$ is called a contraction with contraction constant $\alpha(\omega)$. Again if $\phi_\omega(r) < r$, $r > 0$ for each $\omega \in \Omega$, then $T(\omega)$ is called a nonlinear contraction.

Our next results are random versions of the fixed point result of Krasnoselskii [6] and Dhage [3] in the framework of Schaefer fixed point theorem [10].

Theorem 3.2. *Let $S(\omega)$, $T(\omega) : X \to X$, X a separable Banach space, be two random operators satisfying for each $\omega \in \Omega$,*
(a) *$s(\omega)$ is nonlinear contraction,*
(b) *$T(\omega)$ is completely continuous, and*
(c) *the set $\mathcal{E} = \{x \in X | S(\omega)x + T(\omega)x = \alpha(\omega)x\}$ is bounded, for any measurable function $\alpha : \Omega \to \boldsymbol{R}^+$ with $\alpha(\omega) > 1$.*

Then the random equation

$$S(\omega)x + T(\omega)x = x \tag{2.4}$$

has a random solution.

Proof. Define a random operator $Q(\omega) : \Omega \times X \to X$ by

$$Q(\omega)x = S(\omega)x + T(\omega)x. \tag{2.5}$$

Obviously $Q(\omega)$ is a continuous random operator on X. Now for any $x,\ y \in X$, one has

$$\begin{aligned}\|Q(\omega)x - Q(\omega)y\| &\leq \|S(\omega)x - S(\omega)y\| + \|T(\omega)x - T(\omega)y\| \\ &\leq \phi_\omega(\|x - y\|) + \|T(\omega)x - T(\omega)y\|.\end{aligned} \tag{2.6}$$

Then for a fixed $\omega \in \Omega$, we have from (2.6),

$$\alpha(T(\omega))(B) \leq \phi_\omega(\alpha(B)) < \alpha(B) \quad \text{if } \alpha(B) > 0,$$

for a bounded subset B of X, and so $T(\omega)$ is a α-condensing random operator on X. Now an application of Theorem 2.1 yields that $T(\omega)$ has a random fixed point and consequently the random equation (2.4) has a random solution. The proof is complete.

Corollary 2.2. *Let X be a separable Banach space and let $S,\ T : Q \times X \to X$ be two random operators satisfying for each $\omega \in \Omega$,*

(a) *$S(\omega)$ is contraction,*
(b) *$T(\omega)$ is completely continuous, and*
(c) *the set $\xi = \{x \in X \mid S(\omega)x + T(\omega)x = \alpha(\omega)x\}$ is bounded, for any measurable $\alpha : \Omega \to \boldsymbol{R}^+$ with $\alpha(\omega) > 1$.*

Then the random equation (2.4) *has a random solution.*

Theorem 2.3. *Let X a separable Banach algebra and let $S, T : \Omega \times X \to X$ be two random operators satisfying for each $\omega \in \Omega$,*
(a) *$s(\omega)$ is D-Lipschitzician,*
(b) *$T(\omega)$ is continuous and compact and*
(c) *the set $\mathcal{E} = \{x \in X | S(\omega)xT(\omega)x = \alpha(\omega)x\}$ is bounded, for any measurable $\alpha : \Omega \to \boldsymbol{R}^+$ with $\alpha(\omega) > 1$.*

Then the random equation

$$S(\omega)xT(\omega)x = x \tag{2.7}$$

has a random solution whenever $M(\omega)\phi_\omega(r) < r$, $r > 0$ for each $\omega \in \Omega$, where

$$M(\omega) = \|T(\omega)(X)\| = \sup\{\|x(\omega)\| : x \in T(\omega)(X)\}.$$

Proof. Let $\omega \in \Omega$ be fixed and define a mapping $Q : \Omega \times X \to X$ by

$$Q(\omega) = S(\omega)xT(\omega)x. \tag{2.8}$$

Clearly $Q(.)x$ is measurable for each $x \in X$, and hence $Q(\omega)$ is a random operator. Let B be a bounded subset of X and let $x, y \in B$ be any two points. Then by (2.8),

$$\begin{aligned} \|Q(\omega)x - Q(\omega)y\| &= \|S(\omega)xT(\omega)x - S(\omega)yT(\omega)y\| \\ &\le \|T(\omega)x\| \, \|S(\omega)x - S(\omega)y\| + \|S(\omega)x\| \, \|T(\omega)x - T(\omega)y\| \\ &\le \|T(\omega)(B)\| \, \|S(\omega)x - S(\omega)y\| + \|S(\omega)(B)\| \, \|T(\omega)x - T(\omega)y\| \\ &\le \|T(\omega)(X)\|\phi_\omega(\|x-y\|) + \|S(\omega)(B)\| \, \|T(\omega)x - T(\omega)y\| \end{aligned} \tag{2.9}$$

Now for a fixed $x_0 \in B$,

$$\begin{aligned} \|S(\omega)x\| &\le \|S(\omega)x_0\| + \|S(\omega)x - S(\omega)x_0\| \\ &\le \|S(\omega)x_0\| + \phi_\omega(\|x - x_0\|) \\ &\le \|S(\omega)x_0\| + \phi_\omega(\operatorname{diam} B) \end{aligned}$$

and therefore,

$$\begin{aligned} \|S(\omega)(B)\| &= \sup\{\|S(\omega)x\| : x \in B\} \\ &\le \|S(\omega)x_0\| + \phi_\omega(\operatorname{diam} B) \\ &< \infty. \end{aligned}$$

Hence from (2.9) it follows that

$$\|Q(\omega)x - Q(\omega)y\| \leq M(\omega)\phi_\omega(\|x-y\|) + \beta(\omega)\|T(\omega)x - T(\omega)y\| \tag{2.10}$$

for all $x, y \in B$, where $\beta(\omega) = \|S(\omega)x_0\| + \phi_\omega(\operatorname{diam} B) < \infty$.

Now proceeding as in Dhage [3], it is proved that

$$\alpha(Q(\omega)(B)) \leq M(\omega)\phi_\omega(\alpha(B)) < \alpha(B), \quad \alpha(B) > 0.$$

This shows that $Q(\omega)$ is a α-condensing random operator on X. To prove the continuity of $Q(\omega)$, let $\{x_n\}$ be a sequence is X converging to point x in X. Then we have

$$\begin{aligned}\|Q(\omega)x_n - Q(\omega)x\| &\leq \|T(\omega)x_n\|\,\|S(\omega)x_n - S(\omega)x\| + \|S(\omega)x\|\,\|T(\omega)x_n - T(\omega)x\| \\ &\leq M(\omega)\phi_\omega(\|x_n - x\|) + \|S(\omega)x\|\,\|T(\omega)x_n - T(\omega)x\| \\ &\to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which shows that $Q(\omega)$ is a continuous random operator on X. Now an application of Theorem 2.1 yields that $Q(\omega)$ has a random fixed point and consequently the random equation (2.7) has a random solution. This completes the proof.

Corollary 2.3. *Let X be a separable Banach algebra and let $S, T : \Omega \times X \to X$ be two random operators satisfying for each $\omega \in \Omega$,*
(a) *$S(\omega)$ is Lipschitzician with Lipschitz constant $\alpha(\omega)$,*
(b) *$T(\omega)$ is continuous and compact, and*
(c) *the set $\mathcal{E} = \{x \in X \mid S(\omega)xT(\omega)x = \lambda(\omega)x\}$ is bounded, for any measurable $\lambda : \Omega \to \boldsymbol{R}^+$ with $\lambda(\omega) > 1$.*

Then the operator equation (2.8) *has a random solution whenever $\alpha(\omega)M(\omega) < 1$ for each $\omega \in \Omega$, where $M(\omega) = \|T(\omega)(X)\|$.*

3. Random Integral Equations

In this section we shall apply the results of previous section to nonlinear functional random integral equations involving the Caratheodory functions for proving the existence of the random solution.

Given a closed and bounded interval $J = [0,1]$ in $\boldsymbol{R}$, the set of all real numbers, consider the nonlinear functional random integral equation (in short RIE),

$$x(t,\omega) = q(t,\omega) + \int_0^{\sigma(t)} f(s, x(\eta(s),\omega)),\omega)ds, \quad t \in J, \tag{3.1}$$

here $q : J \times \Omega \to \boldsymbol{R}$, $f : J \times \boldsymbol{R} \times \Omega \to R$ and $\sigma, \eta : J \to J$.

Let $C(J,\boldsymbol{R})$ and $BM(J,\boldsymbol{R})$ denote respectively the spaces of all continuous and bounded and measurable real-valued functions on J. We define a norm $\|.\|_C$ in $C(J,\boldsymbol{R})$ by $\|x\|_C = \sup_{t\in J}|x(t)|$ and a norm $\|.\|_B$ in $BM(J,\boldsymbol{R})$ by $\|x\|_B = \max_{t\in J}|x(t)|$.

Obviously $C(J,\boldsymbol{R}) \subset BM(J,\boldsymbol{R})$. We shall seek the random solution of the RIE (3.1) in the space $BM(J,\boldsymbol{R})$ under suitable conditions. We need the following definition in the sequel.

Definition 3.1. A function $\beta : J \times \boldsymbol{R} \times \Omega \to \boldsymbol{R}$ is called L^1_ω Caratheodory if for each $\omega \in \Omega$,

(i) $t \mapsto f(t,x,\omega)$ is measurable for all $x \in \boldsymbol{R}$,

(ii) $x \mapsto f(t,x,\omega)$ is almost everywhere continuous for $t \in J$, and

(iii) for given real number $k > 0$, there exists a function $h_k : \Omega \to L^1(J,\boldsymbol{R})$ such that

$$|f(t,x,\omega)| \le h_k(t,\omega), \quad \text{a.e. } t \in J$$

for all $x \in \boldsymbol{R}$ with $|x| \le k$.

We consider the following assumptions:

($\mathbf{H}_0$) The functions σ, $\eta : J \to J$ are continuous with $\sigma(t) \le t$ and $\eta(t) \le t$ for all $t \in J$,

($\mathbf{H}_1$) $q : \Omega \to C(J,\boldsymbol{R})$ is measurable,

($\mathbf{H}_2$) $f(t,x,\omega)$ is L^1_ω-Caratheodory,

($\mathbf{H}_3$) $\omega \mapsto f(t,x,\omega)$ is measurable for all $t \in J$ and $x \in \boldsymbol{R}$.

($\mathbf{H}_4$) There exists a function $\phi : \Omega \to L^1(J,\boldsymbol{R})$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$|f(t,x,\omega)| \le \phi(t,\omega)\psi(|x|), \quad \text{a.e. } t \in J$$

for all $x \in \boldsymbol{R}$ and $\omega \in \Omega$.

Theorem 3.1. *Suppose that the assumptions* ($\mathbf{H}_0$)-($\mathbf{H}_4$) *hold. Further if for each* $\omega \in \Omega$,

$$\int_{\|q(\omega)\|_C}^{\infty} \frac{ds}{\psi(s)} > \|\phi(\omega)\|_{L^1}$$

then the RIE (3.1) *has a random solution on* J.

Proof. It is known that $BM(J,\boldsymbol{R})$ is a separable Banach space. Let $\omega \in \Omega$ be fixed. Define an operator $T : \Omega \times BM(J,\boldsymbol{R}) \to BM(J,\boldsymbol{R})$ by

$$T(\omega)x(t) = q(t,\omega) + \int_0^{\sigma(t)} f(s,x(\eta(s),\omega),\omega)ds, \quad t \in J. \tag{3.2}$$

By ($\mathbf{H}_1$), $q(t,\omega)$ is measurable in ω for all $t \in J$. Now $\int_0^{\sigma(t)} f(s,x(\eta(s),\omega),\omega)ds$ is the limit of a finite sum of measurable functions, so $\omega \mapsto \int_0^{\sigma(t)} f(s,x(\eta(s),\omega),\omega)ds$, is measurable. Again the sum of two measurable functions is again measurable, and therefore $T(\omega)$ defines a random operator $T : \Omega \times BM(J,\boldsymbol{R}) \to BM(J,\boldsymbol{R})$. We shall show that $T(\omega)$ satisfies the conditions (a) and (b) of Theorem 2.1.

Step I: First we shall show that the random operator $T(\omega)$ is completely continuous on $BM(J,\boldsymbol{R})$. Since $f(t,x,\omega)$ is L^1-Caratheodory, using the standard arguments and the dominated convergence theorem it is proved that $T(\omega)$ is continuous random operator on $BM(J, \boldsymbol{R})$. Now let $S \subset BM(J, \boldsymbol{R})$ be bounded set with bound k. Then for each $\omega \in \Omega$, by $(\mathbf{H}_3)$,

$$\begin{aligned} T(\omega)x(t) &= q(t,\omega) + \int_0^{\sigma(t)} |f(s, x(\eta(s),\omega),\omega)|ds \\ &\leq \|q(\omega)\|_C + \int_0^{\sigma(t)} h_k(s,\omega)ds \\ &\leq \|q(\omega)\|_C + \|h_k(\omega)\|_{L^1} \end{aligned}$$

for all $x \in S$, showing that $\{(T(\omega)(S)\}$ is a uniformly bounded set in $BM(J, \boldsymbol{R})$ for each $\omega \in \Omega$. Now let $t, \tau \in J$. Then for any $x \in S$,

$$\begin{aligned} |T(\omega)x(t) - T(\omega)x(\tau)| &\leq |q(t,\omega) - q(\tau,\omega)| \\ &\quad + \left| \int_0^{\sigma(t)} f(s, x(\eta(s),\omega),\omega)ds - \int_0^{\sigma(t)} f(s, x(\eta(s),\omega),\omega)ds \right| \\ &\leq |q(t,\omega) - q(\tau,\omega)| + |p(t,\omega) - p(\tau,\omega)| \end{aligned} \tag{3.3}$$

for each $\omega \in \Omega$, where $p(t,\omega) = \int_0^{\sigma(t)} h_k(s,\omega)ds$.

Since $t \mapsto q(t,\omega)$ is continuous on compact interval J, it is uniformly continous. Also $t \mapsto p(t,\omega)$ is a uniformly continuous, it follows from (3.3) that $\{T(\omega)(S)\}$ is compact set in $BM(J, \boldsymbol{R})$ by Aezela – Ascoli theorem, for each $\omega \in \Omega$. Therefore T is a completely continuous random operator on $\Omega \times BM(J, \boldsymbol{R})$.

Step II: Now for any solution x to $T(\omega)x = \alpha(\omega)x$, $\alpha(\omega) > 1$, one has

$$\begin{aligned} |x(t,\omega)| &\leq |T(\omega)x(t,\omega)| \\ &\leq |q(t,\omega)| + \int_0^{\sigma(t)} |f(s, x(\eta(s),\omega),\omega)|ds \\ &\leq |q(t,\omega)| + \int_0^{t} \phi(s,\omega)\psi(|x(\eta(s),\omega)|)ds \\ &\leq \|q(\omega)\| + \int_0^{t} \phi(s,\omega)\psi(|x(\eta(s),\omega)|)ds \end{aligned} \tag{3.4}$$

for each $\omega \in \Omega$.

For a fixed $\omega \in \Omega$, define $\mu(t,\omega) = \sup_{\tau\in[0,t]} |x(\tau,\omega)|$. Since $x : \Omega \to BM(J, \boldsymbol{R})$, there exists a $t^* \in [0,t]$ such that $\mu(t,\omega) = |x(t^*,\omega)|$. Hence from the inequality (3.4),

$$\begin{aligned} \mu(t,\omega) = |x(t^*,\omega)| &\leq \|q(\omega)\|_C + \int_0^{\sigma(t^*)} \phi(s,\omega)\psi(|x(\eta(s),\omega)|)ds \\ &\leq \|q(\omega)\|_C + \int_0^{t} \phi(s,\omega)\psi(\mu(t,\omega))ds \end{aligned}$$

Let $\mu(t,\omega) = \|q(\omega)\|_C + \int_0^t \phi(s,\omega)\psi(\mu(t,\omega))ds$.
Then $\mu(t,\omega) \le u(t,\omega)$, and

$$\begin{aligned}\frac{du(t,\omega)}{dt} &= \phi(t,\omega)\psi(\mu(t,\omega)) \le \phi(t,\omega)\psi(u(t,\omega))\\ u(0,\omega) &= \|q(\omega)\|_C\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{u'(t,\omega)}{\psi(u(t,\omega))} &= \phi(t,\omega)\\ u(0,\omega) &= \|q(\omega)\|_C\,.\end{aligned}$$

Integrating over 0 to t,

$$\int_0^t \frac{du(t,\omega)}{\psi(u(t,\omega))}dt \le \int_0^t \phi(s,\omega)ds \le \int_0^1 \phi(s,\omega)ds.$$

By the change of the variable formula, we obtain

$$\int_{\|q(\omega)\|_C}^{\mu(t,\omega)} \frac{ds}{\psi(s)} \le \|\phi(\omega)\|_{L^1} < \int_{\|q(\omega)\|_C}^{\infty} \frac{ds}{\psi(s)}. \tag{3.5}$$

The inequality shows that there exists a constant $K > 0$ such that for each $\omega \in \Omega$, $u(t,\omega) \le K$ for all $t \in J$ and hence $\mu(t,\omega) \le K$ for all $t \in J$. Since $|x(t,\omega)| \le \mu(t,\omega)$ for every $t \in J$, we have $\|x(\omega)\|_B \le k$ for each $\omega \in \Omega$. Thus the condition (b) of Theorem 2.1 is satisfied. Now an application of Corollary 2.1 yields that the RIE (3.1) has a random solution on J. This completes the proof.

As an application of Theorem 3.1, we consider the nonlinear functional random differential equation (in short RDE)

$$\left.\begin{aligned}\tfrac{du(t,\omega)}{dt} &= f(t,x(\eta(t),\omega),\omega) \quad \text{a.e. } t \in J\\ x(0,\omega) &= q(\omega)\end{aligned}\right\} \tag{3.6}$$

where $q : \Omega \to \boldsymbol{R}$ is a real-valued random variable, $f : J \times R \times \Omega \to \boldsymbol{R}$ and $\eta : J \to J$ is continuous.

By the random solution to the RDE (3.6), we mean a measurable function $x : \Omega \to AC(J,\boldsymbol{R})$ that satisfies the equations (3.6), there $AC(J,\boldsymbol{R})$ is a space of all absolutely continuous real-valued functions on J.

Theorem 3.2. *Assume that the hypotheses* $(\mathbf{H}_2)$, $(\mathbf{H}_4)$ *hold. Further if* $\eta(t) \le t$ *for all* $t \in J$ *and if for each* $\omega \in \Omega$,

$$\int_{\|q(\omega)\|_C}^{\infty} \frac{ds}{\psi(s)} > \|\phi(\omega)\|_{L^1}$$

holds, then the RDE (3.6) *has a random solution on* J.

Proof. The RDE (3.6) is equivalent to the random integral equation

$$x(t, \omega) = q(\omega) + \int_0^t f(s, x(\eta(s), \omega), \omega) ds, \quad t \in J. \tag{3.7}$$

Now the desired conclusion follows by an application of Theorem 3.1 with $q(t, \omega) = q(\omega)$ and $\sigma(t) = t$ for all $t \in J$. In this case $AC(J, \boldsymbol{R}) \subset BM(J, \boldsymbol{R})$. This completes the proof.

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