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ON SOME INEQUALITIES OF CAUCHY-BUNYAKOVSKY-SCHWARZ TYPE AND APPLICATIONS

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Abstract. Some discrete inequalities of Cauchy-Bunyakovsky-Schwarz type for complex numbers with applications for the maximal deviation of a sequence from its weighted mean are given.

1. Introduction

The following result for complex numbers a_k, b_k, $k \in \{1,\dots,n\}$ is well known in the literature as the *Cauchy-Bunyakovsky-Schwarz* (*CBS*) *inequality*:

$$\Big|\sum_{k=1}^{n} a_k b_k\Big|^2 \le \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2, \tag{1.1}$$

with equality if and only if there is a complex number $c \in \mathbb{C}$ such that $a_k = c\overline{b_k}$ for each $k \in \{1,\dots,n\}$, and $\overline{b_k}$ is the complex conjugate of b_k.

A simple proof of this statement can be achieved by utilising the following *Lagrange identity* for complex numbers (see [2, p. 3])

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \Big|\sum_{k=1}^{n} a_k b_k\Big|^2 = \frac{1}{2}\sum_{k,l=1}^{n} \Big|\overline{a_k} b_l - \overline{a_l} b_k\Big|^2.$$

If p_k, $k \in \{1,\dots,n\}$ are positive weights, then the weighted version of (1.1) can be stated as

$$\Big|\sum_{k=1}^{n} p_k a_k b_k\Big|^2 \le \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2. \tag{1.2}$$

In [4], the following result connecting the unweighted version of the (CBS) inequality with the weighted one has been established (see also [2, p. 67-69]):

$$\Big(\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2\Big)^{\frac{1}{2}} - \Big|\sum_{k=1}^{n} x_k y_k\Big|$$

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$$= \sup_{\mathbf{p}\in S_n(\mathbf{1})} \left\{ \sum_{k=1}^{n} p_k|x_k|^2 \sum_{k=1}^{n} p_k|y_k|^2 - \left|\sum_{k=1}^{n} p_k x_k y_k\right| \right\}, \tag{1.3}$$

where $S_n(\mathbf{1}) = \{\mathbf{p} = (p_1,\dots,p_n) | 0 \le p_k \le 1 \text{ for each } k \in \{1,\dots,n\}\}$.

In the same paper the authors also established the following result concerning the length of summation in the CBS inequality:

$$\begin{aligned} &\left(\sum_{k=1}^{n} p_k|x_k|^2 \sum_{k=1}^{n} p_k|y_k|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_k x_k y_k\right| \\ &\quad = \sup_{I\subseteq\{1,\dots,n\}} \left[\left(\sum_{k\in I} p_k|x_k|^2 \sum_{k\in I} p_k|y_k|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_k x_k y_k\right|\right] \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &\left(\sum_{k=1}^{n} p_k|x_k|^2 \sum_{k=1}^{n} p_k|y_k|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_k x_k y_k\right| \\ &\quad \ge \max_{1\le k<l\le n} \left\{\left[p_k|x_k|^2 + p_l|x_l|^2\right]^{\frac{1}{2}} \left[p_k|y_k|^2 + p_l|y_l|^2\right]^{\frac{1}{2}} - |p_k x_k y_k + p_l x_l y_l|\right\}, \end{aligned} \tag{1.5}$$

for any $x_k, y_k \in \mathbb{C}$, $k \in \{1,\dots,n\}$.

For some historical facts on CBS inequality, see [9] and [2]. Refinements of this inequality are provided in [1], [6], [8] and in the Chapter 2 of [2]. Other results related to CBS inequality may be found in [5] and [7].

The aim of the present paper is to establish some inequalities of CBS type under the supplementary assumption that either $\sum_{k=1}^{n} x_k y_k = 0$ or $\sum_{k=1}^{n} p_k x_k y_k = 0$, when the weighted version is considered. Applications that provide upper bounds for the maximal deviation of a sequence x_k from the weighted mean $\sum_{j=1}^{n} p_j x_j$, namely, for the quantity

$$\max_{k\in\{1,\dots,n\}} \left| x_k - \sum_{j=1}^{n} p_j x_j \right|, \tag{1.6}$$

where $x_k \in \mathbb{C}$, $p_k \ge 0$, $k \in \{1,\dots,n\}$, $\sum_{k=1}^{n} p_k = 1$, are also given.

2. The Results

The following result holds:

Theorem 1. *Let $a_k, b_k \in \mathbb{C}$, $k \in \{1,\dots,n\}$, $n \ge 2$ with the property that*

$$\sum_{k=1}^{n} a_k b_k = 0. \tag{2.1}$$

Then

$$\max_{i\in\{1,\dots,n\}} \{|a_i b_i|\} \le \frac{1}{2}\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}}. \tag{2.2}$$

The constant $\frac{1}{2}$ in (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. For any $i \in \{1,\dots,n\}$, we have

$$a_i b_i = -\sum_{\substack{k=1\\k\neq i}}^{n} a_k b_k. \tag{2.3}$$

Taking the modulus in (2.3) we have

$$\begin{aligned} |a_i b_i| &= \left|\sum_{\substack{k=1\\k\neq i}}^{n} a_k b_k\right| \leq \left(\sum_{\substack{k=1\\k\neq i}}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{k=1\\k\neq i}}^{n} |b_k|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^{n} |a_k|^2 - |a_i|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2 - |b_i|^2\right)^{\frac{1}{2}}, \end{aligned} \tag{2.4}$$

for any $i \in \{1,\dots,n\}$, where we used the Cauchy-Bunyakovsky-Schwarz inequality to state the required inequality in (2.4).

Utilising the elementary inequality for real numbers

$$(\alpha^2 - \beta^2)^{\frac{1}{2}} (\gamma^2 - \delta^2)^{\frac{1}{2}} \leq \alpha\gamma - \beta\delta,$$

provided $\alpha,\beta,\gamma,\delta > 0$ and $\alpha \geq \beta$, $\gamma \geq \delta$, we have

$$\begin{aligned} &\left(\sum_{k=1}^{n} |a_k|^2 - |a_i|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2 - |b_i|^2\right)^{\frac{1}{2}} \\ &\quad = \left\{\left[\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}}\right]^2 - |a_i|^2\right\}^{\frac{1}{2}} \left\{\left[\left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}}\right]^2 - |b_i|^2\right\}^{\frac{1}{2}} \\ &\quad \leq \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}} - |a_i b_i|, \end{aligned} \tag{2.5}$$

for each $i \in \{1,\dots,n\}$.

Now, on making use of (2.4) and (2.5) we get the desired inequality (2.2).

To prove the sharpness of the constant, we assume that the inequality (2.2) holds true for a constant $C > 0$, i.e.,

$$\max_{i\in\{1,\dots,n\}} |a_i b_i| \leq C\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}}, \tag{2.6}$$

provided a_k, b_k, $k \in \{1,\dots,n\}$ $(n \geq 2)$ are complex numbers such that $\sum_{k=1}^{n} a_k b_k = 0$.

Now, for $n = 2$, choose $a_1 = a$, $a_2 = -b$, $b_1 = b$, $b_2 = -a$ with $a, b > 0$. Then $a_1 b_1 + a_2 b_2 = 0$, $|a_1 b_1| = |a_2 b_2| = ab$ and by (2.6) we get

$$ab \leq C(a^2 + b^2) \quad \text{for } a, b > 0. \tag{2.7}$$

Choosing in (2.7) $a = b = 1$, we deduce $C \geq \frac{1}{2}$ and the proof is complete.

The following corollary is of interest.

Corollary 1. *Let $x_k \in \mathbb{C}$, $k \in \{1,\ldots,n\}$ and p_k, $k \in \{1,\ldots,n\}$ be a probability sequence, i.e., $p_k \geq 0$, $k \in \{1,\ldots,n\}$ and $\sum_{k=1}^n p_k = 1$. Then we have the inequality:*

$$\max_{i\in\{1,\ldots,n\}} \left\{ p_i \left| x_i - \sum_{j=1}^n p_j x_j \right| \right\} \leq \frac{1}{2} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \left| x_k - \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left\{ \sum_{k=1}^n |x_k|^2 + n \left| \sum_{j=1}^n p_j x_j \right|^2 - 2\mathrm{Re}\left[\left(\sum_{k=1}^n x_k \right) \left(\sum_{j=1}^n p_j \overline{x_j} \right) \right] \right\}^{\frac{1}{2}}. \tag{2.8}$$

Proof. If we choose $a_k = p_k$, $b_k := x_k - \sum_{j=1}^n p_j x_j$, then

$$\sum_{k=1}^n a_k b_k = \sum_{k=1}^n p_k \left(x_k - \sum_{j=1}^n p_j x_j \right) = 0$$

and the condition (2.1) is satisfied.

Applying the inequality (2.2), we obtain

$$\max_{i\in\{1,\ldots,n\}} \left\{ p_i \left| x_i - \sum_{j=1}^n p_j x_j \right| \right\} \leq \frac{1}{2} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \left| x_k - \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |x_k|^2 - 2\mathrm{Re}\left(\sum_{k=1}^n x_k \cdot \sum_{j=1}^n p_j \overline{x_j} \right) + n \left| \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}}$$

and the inequality (2.8) is obtained.

Remark 1. If $\min_{i\in\{1,\ldots,n\}} p_i = p_m > 0$, then from (2.8) we can obtain a coarser and perhaps more useful inequality, providing some upper bounds for the maximal deviation of x_k from the weighted mean $\sum_{j=1}^n p_j x_j$, namely,

$$\max_{k\in\{1,\ldots,n\}} \left| x_k - \sum_{j=1}^n p_j x_j \right| \leq \frac{1}{2p_m} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \left| x_k - \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}}. \tag{2.9}$$

The following weighted version of Theorem 1 may be stated as well:

Theorem 2. *Let $x_k, y_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$ and p_k, $k \in \{1, \dots, n\}$ be a probability sequence with the property that*

$$\sum_{k=1}^{n} p_k x_k y_k = 0. \tag{2.10}$$

Then

$$\max_{i\in\{1,\dots,n\}} \{p_i |x_i y_i|\} \le \frac{1}{2}\left(\sum_{k=1}^{n} p_k |x_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} p_k |y_k|^2\right)^{\frac{1}{2}}. \tag{2.11}$$

The constant $\frac{1}{2}$ in (2.11) is best possible in (2.11).

Proof. It follows from Theorem 1 on choosing $a_k = \sqrt{p_k} x_k$, $b_k = \sqrt{p_k} y_k$.

Remark 2. One should notice that Theorem 1 and Theorem 2 are equivalent in the sense that one implies the other.

The above result provides the opportunity to obtain a different bound for the maximal deviation of x_k from the weighted mean.

Corollary 2. *With the assumptions in Corollary 1, we have the inequality:*

$$\begin{aligned} \max_{i\in\{1,\dots,n\}} \left\{p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right\} &\le \frac{1}{2}\left(\sum_{k=1}^{n} p_k \left| x_k - \sum_{j=1}^{n} p_j x_j \right|^2\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\left[\sum_{k=1}^{n} p_k |x_k|^2 - \left|\sum_{j=1}^{n} p_j x_j\right|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{2.12}$$

Proof. Follows by Theorem 2 on choosing $y_k = 1$, $k \in \{1, \dots, n\}$.

Remark 3. If $\min_{i\in\{1,\dots,n\}} p_i = p_m > 0$, then

$$\max_{i\in\{1,\dots,n\}} \left| x_k - \sum_{j=1}^{n} p_j x_j \right| \le \frac{1}{2p_m}\left(\sum_{k=1}^{n} p_k \left| x_k - \sum_{j=1}^{n} p_j x_j \right|^2\right)^{\frac{1}{2}}. \tag{2.13}$$

Remark 4. It is natural to ask which of the bounds for the maximal deviation

$$\max_{i\in\{1,\dots,n\}} \left\{p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right\}$$

provided by (2.8) and (2.12) are better and when, respectively?

For $n = 2$, let $p_1 = p$, $p_2 = 1 - p$, $p \in [0,1]$, $x_1 = x$, $x_2 = y$, then we have the specific case of

$$\begin{aligned} B_1(p,x,y) &:= \frac{1}{2}\left[p^2 + (1-p)^2\right]^{\frac{1}{2}} \left[(x - px - (1-p)y)^2 + (y - px - (1-p)y)^2\right]^{\frac{1}{2}} \\ &= \frac{1}{2}\left[p^2 + (1-p)^2\right]^{\frac{1}{2}} \left[(1-p)^2(x-y)^2 + p^2(x-y)^2\right]^{\frac{1}{2}} \end{aligned}$$

$$= \frac{1}{2} \cdot \left[p^2 + (1-p)^2 \right] |x-y|$$

and

$$\begin{aligned} B_2(p,x,y) &:= \frac{1}{2} \left[p(x-px-(1-p)y)^2 + (1-p)(y-px-(1-p)y)^2 \right]^{\frac{1}{2}} \\ &= \frac{1}{2} \left[p(1-p)^2(x-y)^2 + (1-p)p^2(x-y)^2 \right]^{\frac{1}{2}} \\ &= \frac{1}{2} \cdot \sqrt{p(1-p)} |x-y|. \end{aligned}$$

Since $p^2 + (1-p)^2 \geq \sqrt{p(1-p)}$ for $p \in [0,1]$, we have that the bound (2.12) is always better than (2.8) for $n = 2$.

Remark 5. For $n = 3$, $p_1 = p$, $p_2 = q$, $p_3 = r$, $x_1 = x$, $x_2 = y$, $x_3 = z$, we should compare the bounds

$$\begin{aligned} B_1(p,q,r,x,y,z) &= \frac{1}{2}(p^2+q^2+r^2)^{\frac{1}{2}} \times \Big[p(x-px-qy-rz)^2 \\ &\quad + q(y-px-qy-rz)^2 + r(z-px-qy-rz)^2 \Big]^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} B_2(p,q,r,x,y,z) &= \frac{1}{2} \Big[p(x-px-qy-rz)^2 + q(y-px-qy-rz)^2 \\ &\quad + r(z-px-qy-rz)^2 \Big]^{\frac{1}{2}}. \end{aligned}$$

The plot of the function

$$\Delta(0.1,0.5,0.4,x,y,-4) = B_1(0.1,0.5,0.4,x,y,-4) - B_2(0.1,0.5,0.4,x,y,-4)$$

on the box $[0,6] \times [8,10]$ shows that one bound is not always better the other (see Figure 1):

Remark 6. In the case of uniform distribution, i.e., when $p_i = \frac{1}{n}$, $i \in \{1,\ldots,n\}$, we obtain from both inequalities (2.8) and (2.12) the same result:

$$\begin{aligned} \max_{k\in\{1,\ldots,n\}} \left| x_k - \frac{1}{n}\sum_{j=1}^n x_j \right| &\leq \frac{1}{2}\sqrt{n} \sum_{k=1}^n \left| x_k - \frac{1}{n}\sum_{j=1}^n x_j \right|^2 \\ &= \frac{1}{2}\left[n\sum_{k=1}^n |x_k|^2 - \left| \sum_{k=1}^n x_k \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{2.14}$$

3. Related Results

The following result may be stated as well.

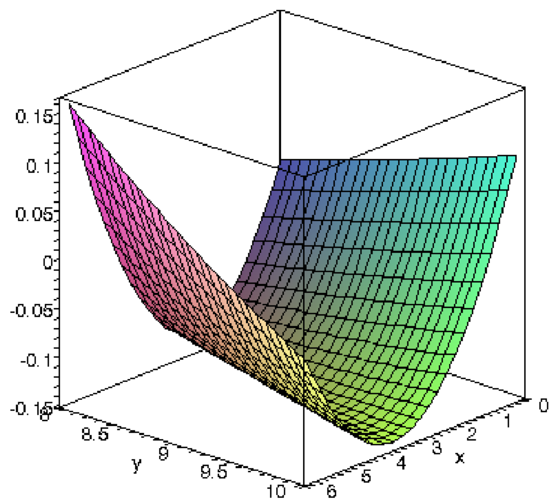

Figure 1. Plot of the difference $\Delta(0.1, 0.5, 0.4, x, y, -4)$ showing a transition from positive to negative.

Theorem 3. *Let $a_k, b_k \in \mathbb{C}\setminus\{0\}$, $k \in \{1,\dots,n\}$ so that $\sum_{k=1}^{n} a_k b_k = 0$. Then for any probability sequence p_k, $k \in \{1,\dots,n\}$, we have:*

$$\frac{\sum_{j=1}^{n} p_j |a_j|^2}{\sum_{k=1}^{n} |a_k|^2} + \frac{\sum_{j=1}^{n} p_j |b_j|^2}{\sum_{k=1}^{n} |b_k|^2} \le 1. \tag{3.1}$$

Proof. We know, from the proof of Theorem 1, that

$$\begin{aligned} |a_i b_i|^2 &\le \Big(\sum_{k=1}^{n} |a_k|^2 - |a_i|^2\Big)\Big(\sum_{k=1}^{n} |b_k|^2 - |b_i|^2\Big) \\ &= \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 + |a_i|^2 |b_i|^2 - |a_i|^2 \sum_{k=1}^{n} |b_k|^2 - |b_i|^2 \sum_{k=1}^{n} |a_k|^2, \end{aligned}$$

which is clearly equivalent with

$$|a_i|^2 \sum_{k=1}^{n} |b_k|^2 + |b_i|^2 \sum_{k=1}^{n} |a_k|^2 \le \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \tag{3.2}$$

for each $i \in \{1,\dots,n\}$.

Now, if we multiply (3.2) by $p_i \ge 0$ and sum over $i \in \{1,\dots,n\}$, we deduce:

$$\sum_{i=1}^{n} p_i |a_i|^2 \sum_{k=1}^{n} |b_k|^2 + \sum_{i=1}^{n} p_i |b_i|^2 \sum_{k=1}^{n} |a_k|^2 \le \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \tag{3.3}$$

which is clearly equivalent with (3.1).

Corollary 3. *With the assumptions of the above theorem, we have:*

$$\sum_{i=1}^{n} p_i|a_i|^2 \sum_{i=1}^{n} p_i|b_i|^2 \leq \frac{1}{4}\sum_{k=1}^{n}|a_k|^2 \sum_{k=1}^{n}|b_k|^2. \tag{3.4}$$

The constant $\frac{1}{4}$ *is best possible in* (3.4).

Proof. On utilising the inequality $\alpha^2+\beta^2 \geq 2\alpha\beta$, $\alpha,\beta \in \mathbb{R}_+$, we have

$$\begin{aligned}&\sum_{j=1}^{n} p_j|a_j|^2 \sum_{k=1}^{n}|b_k|^2 + \sum_{j=1}^{n} p_j|b_j|^2 \sum_{k=1}^{n}|a_k|^2\\ &\quad \geq 2\left(\sum_{j=1}^{n} p_j|a_j|^2 \sum_{j=1}^{n} p_j|b_j|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}|a_k|^2 \sum_{k=1}^{n}|b_k|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.5}$$

Now, by (3.3) and (3.5) we deduce the desired inequality (3.4).

To prove the sharpness of the constant, we assume that (3.4) holds true with a $D>0$, i.e.,

$$\sum_{j=1}^{n} p_j|a_j|^2 \sum_{j=1}^{n} p_j|b_j|^2 \leq D\sum_{k=1}^{n}|a_k|^2 \sum_{k=1}^{n}|b_k|^2,$$

provided $\sum_{k=1}^{n} a_k b_k = 0$, $n \geq 2$.

For $n=2$, we choose $a_1=a$, $a_2=-b$, $b_1=b$, $b_2=-a$ and $p_1=p$, $p_2=1-p$ to get:

$$\left[pa^2+(1-p)b^2\right]\left[pb^2+(1-p)a^2\right] \leq D\left[a^2+b^2\right]^2. \tag{3.6}$$

If in (3.6) we choose $p=\frac{1}{2}$, then we get

$$\frac{1}{4}(a^2+b^2)^2 \leq D(a^2+b^2)^2,$$

which shows that $D \geq \frac{1}{4}$.

Corollary 4. *Let* $x_k \in \mathbb{C}$, $k \in \{1,\dots,n\}$ *and* p_k, $k \in \{1,\dots,n\}$ *be a probability sequence. Then:*

$$\begin{aligned}\sum_{k=1}^{n} p_k|x_k|^2 - \left|\sum_{k=1}^{n} p_k x_k\right|^2 &= \sum_{j=1}^{n} p_j\left|x_j - \sum_{l=1}^{n} p_l x_l\right|^2\\ &\leq \frac{1}{4}\cdot\frac{\sum_{k=1}^{n} p_k^2}{\sum_{k=1}^{n} p_k^3}\sum_{k=1}^{n}\left|x_k - \sum_{l=1}^{n} p_l x_l\right|^2.\end{aligned}$$

Proof. It is obvious by (3.4) on choosing $a_k = p_k$ and $b_k = x_k - \sum_{l=1}^{n} p_l x_l$, $k \in \{1,\dots,n\}$.

The following result that provides a refinement of Theorem 2 should be noted.

Theorem 4. *Let $x_k, y_k \in \mathbb{C}$, $k \in \{1,\dots,n\}$ and p_k, $k \in \{1,\dots,n\}$ be a probability sequence with the property that*

$$\sum_{k=1}^{n} p_k x_k y_k = 0. \tag{3.7}$$

Then

$$\begin{aligned}
&\max_{i\in\{1,\dots,n\}}\{p_i|x_i y_i|\}\\
&\quad\le \frac{1}{2}\cdot\frac{\max_{i\in\{1,\dots,n\}}\left[p_i|x_i|^2\sum_{k=1}^{n}p_k|y_k|^2+p_i|y_i|^2\sum_{k=1}^{n}p_k|x_k|^2\right]}{\left(\sum_{k=1}^{n}p_k|x_k|^2\sum_{k=1}^{n}p_k|y_k|^2\right)^{\frac{1}{2}}}\\
&\quad\le \frac{1}{2}\cdot\left(\sum_{k=1}^{n}p_k|x_k|^2\sum_{k=1}^{n}p_k|y_k|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.8}$$

Proof. As in the proof of Theorem 1, we have

$$p_i|x_i y_i| \le \left(\sum_{k=1}^{n}p_k|x_k|^2 - p_i|x_i|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}p_k|y_k|^2 - p_i|y_i|^2\right)^{\frac{1}{2}},$$

which gives

$$\begin{aligned}
p_i^2|x_i y_i|^2 &\le \left(\sum_{k=1}^{n}p_k|x_k|^2 - p_i|x_i|^2\right)\left(\sum_{k=1}^{n}p_k|y_k|^2 - p_i|y_i|^2\right)\\
&= \sum_{k=1}^{n}p_k|x_k|^2\sum_{k=1}^{n}p_k|y_k|^2 + p_i^2|x_i|^2|y_i|^2 - p_i|x_i|^2\sum_{k=1}^{n}p_k|y_k|^2 - p_i|y_i|^2\sum_{k=1}^{n}p_k|x_k|^2,
\end{aligned}$$

i.e.,

$$p_i|x_i|^2\sum_{k=1}^{n}p_k|y_k|^2 + p_i|y_i|^2\sum_{k=1}^{n}p_k|x_k|^2 \le \sum_{k=1}^{n}p_k|x_k|^2\sum_{k=1}^{n}p_k|y_k|^2 \tag{3.9}$$

for each $i \in \{1,\dots,n\}$.

Taking the maximum in (3.9) over $i \in \{1,\dots,n\}$, we get the second inequality in (3.8).

The first inequality follows by the elementary fact that

$$\begin{aligned}
&p_i|x_i|^2\sum_{k=1}^{n}p_k|y_k|^2 + p_i|y_i|^2\sum_{k=1}^{n}p_k|x_k|^2\\
&\qquad\qquad \ge 2p_i|x_i||y_i|\left(\sum_{k=1}^{n}p_k|x_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}p_k|y_k|^2\right)^{\frac{1}{2}},
\end{aligned}$$

for each $i \in \{1,\dots,n\}$.

Remark 7. The inequality (3.8) is obviously a refinement of the inequality (2.11) in Theorem 2. However, the inequality (3.8) is not apparently useful in deriving upper bounds for the maximal deviation of x_k from its weighted mean $\sum_{j=1}^{n} p_j x_j$, as the inequality (2.11).

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