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ON δ-PERFECT FUNCTIONS

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Abstract. δ-continuous [6] and δ-perfect [5] functions are both introduced by T. Noiri in the similar fashion as continuous and perfect functions. The purpose of the present paper is to investigate several properties of δ-perfect functions and also to determine some topological properties which are preserved by δ-continuous δ-perfect functions.

1. Introduction

T. Noiri initiated the concepts of δ-perfect [5] and δ-continuous [6] functions. The purpose of this paper is to investigate certain properties of δ-perfect functions specially, in addition, when the function is also δ-continuous. We start this discussion with a new characterization of δ-perfect functions. A new class of functions under the terminology N-compact function are defined and investigated w.r.t. their relationship with δ-perfect functions; further, we have established that for a δ-continuous function, the concepts of δ-perfectness and N-compactness are identical when the range space is locally nearly compact and Hausdorff. Preservation of certain topological properties by δ-perfect δ-continuous functions are also investigated.

Throughout this paper, by X or Y we shall mean topological spaces. A set A is called regular open if $A = \operatorname{int}(\operatorname{cl} A)$ and regular closed if $A = \operatorname{cl}(\operatorname{int} A)$. The collection of all regular open sets containing the point x of X is denoted by $RO(x)$. A point x is said to be in the δ-closure [12] of a subset A of X, denoted by δ-cl A, if for every $U \in RO(x)$, $U \cap A \neq \Phi$. A is δ-closed if $A = \delta - \operatorname{cl} A$. The complement of δ-closed set is called δ-open. A subset A of X is said to be an NC-set [1] if every regular open cover of A has a finite subcover. If $A = X$ and A is an NC-set, then X is called a nearly compact space [10]. A space X is said to be locally nearly compact [1] if for each point x of X, there exists a neighbourhood U of x such that $\operatorname{cl} U$ is an NC-set in X. A function $f : X \to Y$ is said to be δ-continuous [6] if for each $x \in X$ and each $V \in RO(f(x))$, there exist a $U \in RO(x)$ such that $f(U) \subset V$. A function $f : X \to Y$ is said to be δ-perfect [5] if for every filter base $\Im$ in $f(X)$ δ-converging to $y \in Y$, $f^{-1}(\Im)$ is δ-directed towards $f^{-1}(y)$. Equivalently f is δ-perfect iff point inverses are NC-sets in X and f is δ-closed i.e. images of every δ-closed sets in X is δ-closed in Y [5]. A space X is said to be almost regular

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[9] if for each regular closed set $F \subset X$ and each $x \notin F$, there exist disjoint open sets U and V in X such that $x \in U$ and $F \subset V$. For a space (X, T), the collection of all regular open sets of (X, T) forms a basis for the topology T_s. The space (X, T_s) is called the semiregularization space of (X, T).

2. δ-Continuous δ-Perfect Functions

Theorem 2.1. *For a function $F : X \to Y$, where Y is Hausdorff, the following are equivalent:*

(i) *f is δ-perfect,*

(ii) *for each $y \in Y$, $f^{-1}(y)$ is a δ-closed subset of X, and if $\mathcal{U}$ is a δ-open cover of X that is closed under finite unions, then $\{Y - f[X - U] : U \in \mathcal{U}\}$ is a δ-open cover of Y.*

Proof. The proof is similar to the proof of Theorem 1.8 (c) [7] and is thus omitted.

Lemma 2.2. [5] *If $f : X \to Y$ is a δ-perfect function, then $f^{-1}(K)$ is an NC-set in X for every NC-set K of Y.*

Theorem 2.3. *A composition of δ-perfect functions is δ-perfect.*

Proof. Since the composition of δ-closed function is δ-closed, the proof follows from the Lemma 2.2.

Definition 2.4. Let $f : X \to Y$ and $g : X \to Z$ be two functions. The function $F : X \to Y \times Z$ defined by $F(x) = (f(x), g(x))$ for each $x \in X$, is called the Diagonal product of f and g.

Theorem 2.5. *Let $f : X \to Y$ and $g : X \to Z$ (where Z is Hausdorff) be δ-perfect and δ-continuous functions respectively and also let both be surjective. Then the set $\{(f(x), g(x)) : x \in X\}$ is δ-closed in $Y \times Z$.*

Proof. Let $(y, z) \notin \{(f(x), g(x)) : x \in X\} = F(X)$ (say) where $y \in Y$ and $z \in Z$ i.e. $f^{-1}(y) \cap g^{-1}(z) = \phi$. This implies that $z \notin gf^{-1}(y)$. Since $gf^{-1}(y)$ is an NC-set in the Hausdorff space Z, there exist disjoint regular open sets U and V in Z such that $z \in U$ and $gf^{-1}(y) \subset V$. Since f is δ-closed and $f^{-1}(y) \subset g^{-1}(V)$, there exists a regular open set V_y in Y containing y such that $f^{-1}(V_y) \subset g^{-1}(V)$. So $gf^{-1}(V_y) \subset V$. Therefore $U \cap gf^{-1}(V_y) = \phi$ i.e. $g^{-1}(U) \cap f^{-1}(V_y) = \phi$. Now $V_y \times U$ is a regular open set in $Y \times Z$ containing the point (y, z) disjoint from $F(X)$.

Theorem 2.6. *If $f : X \to Y$ is δ-perfect and $g : X \to Z$ is δ-continuous, where X, Z are Hausdorff spaces, then the diagonal product of f and g is δ-perfect.*

Proof. Let $(y, z) \in F(X)$, where $F : X \to Y \times Z$ is the diagonal product of f and g. We have $F^{-1}(y, z) = f^{-1}(y) \cap g^{-1}(z)$. Since Z is Hausdorff, it is clear that every one pointic set is δ-closed and since g is δ-continuous, $g^{-1}(z)$ is δ-closed in X. As f is δ-perfect, $f^{-1}(y)$ is an NC-set in the Hausdorff space X and hence it is δ-closed in X. Therefore $F^{-1}(y,z) = f^{-1}(y) \cap g^{-1}(z)$ is δ-closed in X and is contained in the NC-set $f^{-1}(y)$. So $F^{-1}(y,z)$ is an NC-set. Next we shall show that $F : X \to F(X)$ is a δ-closed function. Let A be any δ-closed subset of X. To show $F(A)$ is δ-closed it is sufficient to show that for any point $x^* \notin A$, either (1) $F(x^*) \in F(A)$ or (2) there is a regular open set in $Y \times Z$ of the point $F(x^*)$ which does not meet $F(A)$. Let $y^* = f(x^*)$ and $z^* = g(x^*)$ and $D = f^{-1}(y^*)$, $E = D \cap A$ and $G = g(E)$. If $g(x^*) \in G$, then $g(x^*) = g(x_1)$ for some $x_1 \in E$. Then $F(x^*) = (f(x^*), g(x^*)) = (f(x_1), g(x_1)) = F(x_1) \in F(A)$. Thus (1) is valid. Now suppose that $g(x^*) \notin G$. Since g is δ-continuous and E is an NC-set, by Lemma 5.7 of T. Noiri [6], $g(E) = G$ is an NC-set in the Hausdorff space Z. There exist disjoint regular open sets V^* and U^* in Z such that $g(x^*) \in V^*$ and $G \subset U^*$. The set $U = g^{-1}(U^*) \cup (X - A)$ is δ-open in X and $f^{-1}(y^*) \subset U$. Since f is δ-closed function, there exists a regular open set V_y^* in Y containing y^* such that $f^{-1}(V_y^*) \subset U$. The set $V_y^* \times V^*$ is a regular open set in $Y \times Z$ containing $F(x^*) = (f(x^*), g(x^*)) = (y^*, g(x^*))$. We claim that $(V_y^* \times V^*) \cap F(A) = \phi$. In fact, if for some $x \in A$, $F(x) \cap (V_y^* \times V^*) \neq \phi$ then $f(x) \in V_y^*$ and $g(x) \in V^*$. $f(x) \in V_y^*$ implies $x \in U$ and $g(x) \in V^*$ implies $x \notin U$ — a contradiction. Hence the proof.

Definition 2.7. A function $f : X \to Y$ is said to be N-compact if $f^{-1}(K)$ is an NC-set in X whenever K is an NC-set in Y.

Remark 2.8. Clearly by Lemma 2.2, every δ-perfect function is N-compact but that the converse is not true follows from the following example.

Example 2.9. Consider the identity function $i : (N, T_1) \to (N, T_2)$, where N is the set of naturals, T_1 is the discrete topology and T_2 is the topology generated by the collection $\{\{1,2\}, \{3,4\}, \ldots\}$. Only finite sets are NC-sets in (N, T_2) and as such i is N-compact but $\{1\}$ is δ-closed in (N, T_1) but is not so in (N, T_2).

It is therefore natural under what conditions an N-compact function would be a δ-perfect function. The following theorem establishes one such condition.

Theorem 2.10. *If $f : X \to Y$ is δ-continuous N-compact function from a Hausdorff space X into a locally nearly compact Hausdorff space Y then f is δ-perfect.*

Proof. Since f is N-compact function, the point inverses are NC-sets. Let A be a δ-closed subset of X and let $y \notin f(A)$ be in the δ-closure of $f(A)$. Since Y is locally nearly compact, there exists a regular open set U in Y such that $y \in U$ and $\operatorname{cl} U$ is an NC-set in Y. Now $f(A) \cap \operatorname{cl} U$ can not be an NC-set. In fact, if it is an NC-set, there exist disjoint regular open sets V_1 and V_2 in Y such that $y \in V_1$ and $f(A) \cap \operatorname{cl} U \subset V_2$ (since Y is Hausdorff). Then $U \cap V_1 \cap f(A) \subset V_1 \cap \operatorname{cl} U \cap f(A) = \phi$ — which contradicts the fact that y is in the δ-closure of $f(A)$. As $\operatorname{cl} U$ is an NC-set and f is N-compact, $f^{-1}(\operatorname{cl} U)$ is an NC-set in X. Therefore $A \cap f^{-1}(\operatorname{cl} U)$ is an NC-set in the Hausdorff space X and so $f[A \cap f^{-1}(\operatorname{cl} U)] = f(A) \cap \operatorname{cl} U$ is an NC-set — a contradiction. So $y \in f(A)$.

Corollary 2.11. *A δ-continuous function $f : X \to Y$, where X is Hausdorff and Y is locally nearly compact Hausdorff is N-compact iff it is δ-perfect.*

In the above discussion δ-continuity plays a very crucial role. It is of interest under what conditions on the domain and co-domain spaces, the other restrictions on f may imply that f is δ-continuous.

Theorem 2.12. *If $f : X \to Y$ is a surjective function from a almost regular space X onto a nearly compact space Y with the property that f is δ-closed and point inverses are δ-closed sets, then f is δ-continuous.*

Proof. Let f be not δ-continuous. Then by Theorem 2.2 of T. Noiri [6], there exist a point $x \in X$ and a $V \in RO(f(x))$ such that for every $U \in RO(x)$, $f(U) \cap (Y - V) \neq \phi$. Since f is δ-closed $f(\operatorname{cl} U) \cap (Y - V)$ is a δ-closed set in Y. The collection $\{f(\operatorname{cl} U) \cap (Y - V) : U \in RO(x)\}$ has the finite inter-section property. If not i.e. if there exist $U_1, U_2, \ldots, U_n \in RO(x)$ such that $\cap_{i=1}^{n}[f(\operatorname{cl} U_i) \cap (Y - V)] = \varnothing$, then it can be easily shown that $f(\cap_{i=1}^{n} U_i) \cap (Y - V) = \varnothing$, which shows that f is δ-continuous — a contradiction. As Y is nearly compact, $\cap_{U \in RO(x)}[f(\operatorname{cl} U) \cap (Y - V)] \neq \varnothing$. Let y^* belong to the intersection, then clearly $f(x) \neq y^*$. So $x \notin f^{-1}(y^*)$. By the almost regularity of X, there exist disjoint regualr open sets U_1^* and U_2^* in X such that $x \in U_1^*$ and $f^{-1}(y^*) \subset U_2^*$. So $y^* \notin f(\operatorname{cl} U_1^*)$. But $y^* \in f(\operatorname{cl} U_1^*)$ — a contradiction. So f is δ-continuous.

Next we shall show that the product of two δ-perfect functions is δ-perfect.

Lemma 2.13. *Let X_1 and X_2 be two topological spaces and let K_i be NC-sets in X_i for $i = 1, 2$. If V be a regular open set of $X_1 \times X_2$ containing $K_1 \times K_2$, there exist δ-open sets U_i of X_i containing K_i such that $K_1 \times K_2 \subseteq U_1 \times U_2 \subseteq V$.*

Proof. We fix $x \in K_1$ and then for each $y \in K_2$, $(x, y) \in V$. So there exist open sets W_i of X_i such that $x \in W_1^y$ and $y \in W_2^y$ and $(x, y) \in W_1^y \times W_2^y \subset \operatorname{int}\operatorname{cl} W_1^y \times \operatorname{int}\operatorname{cl} W_2^y \subset \operatorname{int}\operatorname{cl} V = V$. Then the collection $\{\operatorname{int}\operatorname{cl} W_2^y : y \in K_2\}$ covers K_2. Since K_2 is an NC-set, there exist $y_1, \ldots, y_n \in K_2$ such that $K_2 \subseteq \cup_{i=1}^{n} \operatorname{int}\operatorname{cl} W_2^{y_i} = W_x$ (say). Let $U_x = \cap_{i=1}^{n} \operatorname{int}\operatorname{cl} W_1^{y_i}$. Clearly $\{x\} \times K_2 \subseteq U_x \times W_x \subseteq V$. Since K_1 is an NC-set and the collection $\{U_x : x \in K_1\}$ is a regular open cover of K_1, then there exist $x_1, \ldots, x_m \in K_1$ such that $K_1 \subseteq \cup_{i=1}^{m} U_{x_i} = U_1$ (say) and $U_2 = \cap_{i=1}^{m} W_{x_i}$. Then $K_1 \times K_2 \subseteq U_1 \times U_2 \subseteq V$.

Theorem 2.14. *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be two δ-perfect functions, then the function $f = f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is δ-perfect.*

Proof. Let $y = (y_1, y_2) \in Y_1 \times Y_2$. Then $f^{-1}(y) = (f_1 \times f_2)^{-1}(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2)$, which is an NC-set in $X_1 \times X_2$. Let P be any δ-closed set in $X_1 \times X_2$. Let $(y_1, y_2) \notin f(P)$. Then $(f_1 \times f_2)^{-1}(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2) \subset X_1 \times X_2 - P$. By Lemma 2.13, there exist δ-open sets U_i containing $f_i^{-1}(y_i)$ such that $(f_1 \times f_2)^{-1}(y_1, y_2) \subseteq$

$U_1 \times U_2 \subseteq X_1 \times X_2 - P$. Clearly $y_i \in Y_i - f_i(X_i - U_i) = V_i$ and each V_i is δ-open in Y_i, also $f_i^{-1}(V_i) \subset U_i$. So $(y_1, y_2) \in V_1 \times V_2$. We shall show that $V_1 \times V_2 \cap f(P) = \varnothing$. Let $(y_1^*, y_2^*) \in V_1 \times V_2$. Then $f_i^{-1}(y_1^*) \subset U_i$ and so $(f_1 \times f_2)^{-1}(y_1^*, y_2^*) \subset U_1 \times U_2 \subset X_1 \times X_2 - P$. Therefore $(y_1^*, y_2^*) \in Y - f(P)$. Hence $f = f_1 \times f_2$ is δ-closed. Therefore $f = f_1 \times f_2$ is δ-perfect.

Lemma 2.15. [5] *A space X is nearly compact iff for any space Y, the projection mapping $\pi_Y : X \times Y \to Y$ is δ-perfect.*

Theorem 2.16. *Let $\{X_\alpha : \alpha \in I\}$ be a collection of spaces with the product X. Then $\pi\{X_\alpha : \alpha \in I - \{j\}\}$ is nearly compact iff $\pi_j : X \to X_j$ is δ-perfect.*

Proof. Immediate from Lemma 2.15.

Perfect continuous functions preserve, in both directions, certain topological properties. Here, we shall investigate certain topological properties which are preserved by δ-perfect δ-continuous functions.

Definition 2.17. [8] A space X is said to be nearly paracompact if every regular open cover of X has an open locally finite refinement.

Theorem 2.18. *If $f : (X, T) \to (Y, \sigma)$ is a surjective δ-perfect δ-continuous function, then the following are true:*

i) *(X, T) is almost regular iff (Y, σ) is almost regular.*
ii) *(X, T) is Hausdorff iff (Y, σ) is Hausdorff.*
iii) *(X, T) is nearly compact iff (Y, σ) is nearly compact.*
iv) *(X, T) is locally nearly compact Hausdorff iff (Y, σ) is locally nearly compact Hausdorff.*

Proof. $f : (X, T) \to (Y, \sigma)$ is δ-continuous iff $f : (X, T_s) \to (Y, \sigma_s)$ is continuous [6] and $f : (X, T) \to (Y, \sigma)$ is δ-perfect iff $f : (X, T_s) \to (Y, \sigma_s)$ is perfect [5], where (X, T_s) and (Y, σ_s) are semiregularizations of (X, T) and (Y, σ) respectively. Also a space (X, T) is almost regular (resp. Hausdorff, nearly compact, locally nearly compact Hausdorff and nearly paracompact iff (X, T_s) is regular (resp. Hausdorff, compact, locally compact Hausdorff [3] and paracompact [4]). Since regularity, Hausdorffness, compactness and local compactness (in presence of Hausdorffness) are preserved in both directions by perfect continuous functions, all the results are immediate.

Definition 2.19. A space X is said to be weakly T_2 if every point is the intersection of regular closed sets of X.

Theorem 2.20. *Let $f : X \to Y$ be a δ-perfect surjuctive function. Then we have the following:*

i) *If X is weakly T_2 then Y is also weakly T_2.*
ii) *If f is δ-continuous and Y is nearly paracompact, then X is also nearly paracompact.*

Proof. i) Let X be weakly T_2. Then every point in X is the intersection of regular closed sets of X. Therefore every point in X is δ-closed. Let $y \in Y$. Then for every $x \in f^{-1}(y)$, $f(x) = y$. Since f is δ-perfect and hence δ-closed, $\{y\}$ is δ-closed i.e. intersection of regular closed sets of Y. Therefore Y is weakly T_2.

ii) It is immediate from the argument given in the proof of Theorem 2.18.

Lemma 2.21. [6] *If $f : X \to Y$ is θ-continuous* [2] *and almost open* [11] *then f is δ-continuous.*

Theorem 2.22. *Let X be nearly compact and Y be nearly paracompact then $X \times Y$ is nearly paracompact.*

Proof. Since $\pi_Y : X \times Y \to Y$ is continuous open and hence θ-continuous almost open, by Lemma 2.21, π_Y is δ-continuous. Since X is nearly compact by Lemma 2.15, $\pi_Y : X \times Y \to Y$ is δ-perfect. As Y is nearly paracompact and π_Y is δ-perfect δ-continuous surjection, by Theorem 2.20, $X \times Y$ is nearly paracompact.

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